Localized Birkhoff average in beta dynamical systems

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Background
Let \((X, T, \mathcal{B}, \mu)\) be a measure-theoretic dynamical system.

**Birkhoff’s ergodic Thm**

- (I). For any \(f \in L^1(\mu)\),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \mathbb{E}(f|\mathcal{I})(x) \quad \mu - a.s.
\]

where \(\mathcal{I}\) is the \(\sigma\)-algebra of \(T\)-invariant sets.

- (II). If, furthermore, \(T\) is ergodic, then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f(x) d\mu \quad \mu - a.s.
\]
Multifractal analysis of Birkhoff average:

Classical case:

\[
\left\{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \alpha \right\}
\]

Remark: The Birkhoff average of a point \( x \) should depend on \( x \) itself.

Consider the size of the set

\[
\left\{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \psi(x) \right\}
\]

This is called as localized Birkhoff average: instead of a constant, the function \( \psi \) here varies with \( x \).
Known Results

- **Barral & Seuret 2011**: Localized Jarnik Thm:

  \[ \left\{ x : \delta(x) = f(x) \right\} \]

  where \( \delta(x) \) is the exact diophantine exponent of \( x \):

  \[ \delta(x) = \sup \left\{ \delta : |x - p/q| < q^{-\delta}, \text{ i.o. } q \in \mathbb{N} \right\} \]

- **Barral & Qu 2012**: Localized multifractal analysis:

  \[ E_f(\psi) := \left\{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \psi(x) \right\} \]

  where \((X, T)\) is a subshift of finite type.
Theorem (Barral & Qu, 2012)

Let $f, \psi$ be two continuous functions.

$$\dim_H E_f(\psi) = \sup \left\{ \frac{h_\mu}{\int \log |T'| d\mu} : \mu \in \mathcal{M}(T), \int f d\mu \in \mathcal{D}(\psi) \cap \mathcal{L}_f \right\}$$

where $\mathcal{M}(T)$ denotes the collection of all $T$-inv. probability measures and $h_\mu$ is the measure theoretic entropy of $\mu$. 

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Localized Birkhoff average in beta dynamical systems
Our concern

Question: Localized Birkhoff average in beta expansions

Intention:
- The beta-dynamical system for a general $\beta$ is not usually a subshift of finite type with mixing properties.
- When can we solve the question in beta-dynamical system by applying the method that approximating the system by subsystems of subshift of finite type?
Beta expansion
Notation

- **Algorithm.** For $\beta > 1$,

\[ Tx = \beta x - \lfloor \beta x \rfloor + 1, \]

- **Expansion.** Every $x \in (0, 1]$ can be expressed uniquely as an infinite series

\[ x = \frac{\epsilon_1(x, \beta)}{\beta} + \cdots + \frac{\epsilon_n(x, \beta)}{\beta^n} + \cdots \]

- **Expansion of 1.** Write

\[ 1 = \frac{\epsilon_1^*(\beta)}{\beta} + \cdots + \frac{\epsilon_n^*(\beta)}{\beta^n} + \cdots \]

- **Parry number.** Call $\beta$ a Parry number, if

\[ (\epsilon_1^*(\beta), \cdots, \epsilon_n^*(\beta), \cdots) \text{ eventually periodic.} \]

- **Admissible sequence $\Sigma_\beta : (\epsilon_1, \cdots, \epsilon_n)$ is called admissible if $\exists x$ s.t. $\epsilon_j(x, \beta) = \epsilon_j$ for $1 \leq j \leq n$.}
\( \Sigma^*_n : \beta\)-admissible of length \( n \);

Cylinder: for any \( (w_1, \cdots, w_n) \in \Sigma^*_\beta \),

\[
I_n(w) = \left\{ x : w_k(x, \beta) = w_k, 1 \leq k \leq n \right\}.
\]

Basic properties:

Characterization on admissible sequence (W. Parry): \( (w_1, \cdots, w_n) \) is \( \beta \)-admissible if

\[
w_k, \cdots, w_n \leq w^*_1(\beta), \cdots, w^*_n(\beta), \quad 1 \leq k \leq n.
\]

Monotonicity: if \( \beta < \beta' \)

\[
\Sigma^*_\beta \subset \Sigma^*_\beta', \quad n \geq 1.
\]

Cardinality of \( \Sigma^*_\beta \) (A. Renyi):

\[
\beta^n \leq \#\Sigma^*_\beta \leq \beta^{n+1}/(\beta + 1), \quad n \geq 1.
\]
A special feature: Lack of Markov properties

For any given $(\epsilon_1, \cdots, \epsilon_n) \in \Sigma^*_\beta$, the length of a cylinder

$$ I_n(\epsilon_1, \cdots, \epsilon_n) := \{ x \in [0, 1] : \epsilon_k(x) = \epsilon_k, 1 \leq k \leq n \} $$

satisfies that

$$ 0 < |I(\epsilon_1, \cdots, \epsilon_n)| \leq \beta^{-n}. $$

(1)

But when $\beta$ is a Parry number, there is a universal constant $C$ such that

$$ C\beta^{-n} \leq |I_n(\epsilon_1, \cdots, \epsilon_n)| \leq \beta^{-n}. $$

Remark: For a general $\beta > 1$, the lower bound in (1) cannot be improved, which constitutes one main difficulty in studying the metric theory related to beta expansions.

How to overcome this difficulty?
Approximation:

Step 1. Approximate $\beta$ by Parry numbers from below: let $\beta_M$ be the solution to the equation

$$1 = \frac{\epsilon_1^*(\beta)}{x} + \cdots + \frac{\epsilon_M^*(\beta)}{x^M}.$$ 

Then $\beta_M$ is a Parry number and $\beta_M \leq \beta$.

Step 2. Only concentrate on $x$ such that the digits sequence

$$(\epsilon_1(x, \beta), \cdots, \epsilon_n(x, \beta), \cdots) \in \Sigma_{\beta_M}.$$ 

Then for any such $x$,

$$\beta^{-(n+M)} \leq |I_n(x)| \leq \beta^{-n}.$$ 

Question: how many are lost?
Character on bad points

Lemma (Tan-W, 2011)

\[ w \in \Sigma_\beta \setminus \Sigma_{\beta^M} \text{ iff } (\epsilon_1^*(\beta), \cdots, \epsilon_M^*(\beta)) \text{ is a subword of } w. \]

Define map \( \pi \) on \( \Sigma_\beta \),

\[
\begin{align*}
\pi : w & \rightarrow w^* \\
\text{change } (\epsilon_1^*(\beta), \cdots, \epsilon_M^*(\beta)) & \rightarrow (\epsilon_1^*(\beta), \cdots, \epsilon_M^*(\beta) - 1)
\end{align*}
\]

Lemma (Tan-W, 2011)

For any \( w \in \Sigma_\beta^n \), write \( \pi(w) = w^* \). Then

\[ w^* \in \Sigma_{\beta^M}^n, \quad \#\pi^{-1}(w^*) \leq 2^{n/M}. \]
Since the digits in \( w \) and \( w^\ast \) differ only at very rare positions,

\[
\left| \sum_{j=0}^{n-1} f(T^j x) - \sum_{j=0}^{n-1} f(T^j x^\ast) \right| < n\epsilon
\]

when \( M \gg 1 \) for any \( x \in I_n(w), x^\ast \in I_n(w^\ast) \).

An application of above lemmas.

- Let \( \varphi \) be a continuous function on \([0, 1]\), then the pressure function

\[
P(\varphi, T_\beta) = \lim_{\beta' \to \beta} P(\varphi, T_{\beta'}).
\]
Let $f, \psi$ be two continuous functions. Write

$$
\mathcal{D}(\psi) = \{\psi(x) : x \in [0, 1]\}, \quad \mathcal{L}_f = \left\{ \int f \, d\mu : \mu \in \mathcal{M}(T) \right\}
$$

Define

$$
E_f(\psi) := \left\{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \psi(x) \right\}.
$$

**Theorem (Tan-W-Wu-Xu, 2012)**

$$
\dim_H E_\psi(f) = \sup \left\{ \frac{h_\mu}{\log \beta} : \mu \in \mathcal{M}(T) \text{ and } \int \psi \, d\mu \in \mathcal{D}(\psi) \cap \mathcal{L}_f \right\},
$$

where $\mathcal{M}(T)$ denotes the collection of all $T$-invariant probability measures and $h_\mu$ is the measure theoretic entropy of $\mu$. 
Sketch of Proof

Just consider the classic case:

\[ E_f(\alpha) := \left\{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \alpha \right\} \]

Upper bound:

Let \( \alpha \in \mathbb{R} \), \( f \) a continuous function, \( \epsilon > 0 \). Denote by

\[
\mathcal{F}_n(\beta, \epsilon) = \left\{ w \in \Sigma_\beta^n : \exists x \in I_n(w) \text{ s.t. } \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) - \alpha \right| < \epsilon \right\}.
\]

Then by an evident covering argument,

\[
\dim_H E_f(\alpha) \leq \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\log \# \mathcal{F}_n(\beta, \epsilon)}{n}.
\]
Lower bound: Define

\[ \mathcal{F}_n(\beta_M, \epsilon) = \left\{ w \in \sum_{\beta_M}^n : \exists x \in I_n(w) \text{ s.t. } \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) - \alpha \right| < \epsilon \right\}. \]

Claim:

For any \( w \in \mathcal{F}_n(\beta, \epsilon) \), we have \( w^* \in \mathcal{F}_n(\beta_M, 2\epsilon) \).

Lemma

\[ \#\mathcal{F}_n(\beta_M, \epsilon_n) \leq \#\mathcal{F}_n(\beta, \epsilon) \leq \#\mathcal{F}_n(\beta_M, 2\epsilon) \cdot 2^{n/M}. \]
Define a Cantor subset as

\[ C_M := \bigcap_{n=1}^{\infty} \bigcup_{I_n \in \mathcal{F}_n(\beta_M, \epsilon_n)} I_n \]

Simple observations:

- For every \( x \in C_M \), the lengths of the cylinders containing \( x \) are quite regular.
- For each level of the Cantor set \( C_M \), the number of cylinders in \( \mathcal{F}_n(\beta_M, \epsilon_n) \) is sufficiently large compared with \( \mathcal{F}_n(\beta, \epsilon_n) \).
Thanks for your attention!