Some progresses on Lipschitz equivalence of self-similar sets

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(With Hui Rao, Yang Wang and Li-Feng Xi)

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Chinenes University of Hong Kong – Dec 10-14, 2012
Part I. Lipschitz equivalence of dust-like self-similar sets
Definition

Let $E, F$ be compact sets in $\mathbb{R}^d$. We say that $E$ and $F$ are **Lipschitz equivalent**, and denote it by $E \sim F$, if there exists a bijection $g : E \to F$ which is bi-Lipschitz, i.e. there exists a constant $C > 0$ such that for all $x, y \in E$,

$$C^{-1}|x - y| \leq |g(x) - g(y)| \leq C|x - y|.$$
Question
Under what conditions, two self-similar sets are Lipschitz equivalent?

- Necessary condition: same Hausdorff dimension.
- The condition is not sufficient even for dust-like case. (The generating IFS satisfies the strong separation condition.)

Example
Let $E$ be the Cantor middle-third set. Let $s = \log 2 / \log 3$ and $3 \cdot r^s = 1$. Let $F$ be the dust-like self-similar set generated as the following figure. Then $E \not\sim F$. 
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\[ \begin{array}{c}
\text{1/3} \\
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\end{array} \quad \begin{array}{c}
r \\
r \\
r \\
\end{array} \]
Let $E, F$ be dust-like self-similar sets generated by the IFS 
$\{\Phi_j\}_{j=1}^{n}, \{\Psi_j\}_{j=1}^{m}$ on $\mathbb{R}^d$, respectively.

$\rho_j$ (resp. $\tau_j$) is the contraction ratio of $\Phi_j$ (resp. $\Psi_j$).

$\mathbb{Q}(a_1, \ldots, a_m)$: subfield of $\mathbb{R}$ generated by $\mathbb{Q}$ and $a_1, \ldots, a_m$.

$\text{sgp}(a_1, \ldots, a_m)$: subsemigroup of $(\mathbb{R}^+, \times)$ generated by $a_1, \ldots, a_m$.

**Theorem (Falconer-Marsh, 1992)**

Assume that $E \sim F$. Let $s = \dim_H E = \dim_H F$. Then

1. \(\mathbb{Q}(\rho_1^s, \ldots, \rho_m^s) = \mathbb{Q}(\tau_1^s, \ldots, \tau_n^s)\);
2. \(\exists p, q \in \mathbb{Z}^+, \text{ s.t. } \text{sgp}(\rho_1^p, \ldots, \rho_m^p) \subset \text{sgp}(\tau_1, \ldots, \tau_n) \text{ and } \text{sgp}(\tau_1^q, \ldots, \tau_n^q) \subset \text{sgp}(\rho_1, \ldots, \rho_m)\).

Using (2), we can show that $E \not\sim F$ in the above example.
Let $E$, $F$ be dust-like self-similar sets generated by the IFS $\{\Phi_j\}_{j=1}^n$, $\{\Psi_j\}_{j=1}^m$ on $\mathbb{R}^d$, respectively.

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- \( \rho_j \) (resp. \( \tau_j \)) is the contraction ratio of \( \Phi_j \) (resp. \( \Psi_j \)).
- \( \mathbb{Q}(a_1, \ldots, a_m) \): subfield of \( \mathbb{R} \) generated by \( \mathbb{Q} \) and \( a_1, \ldots, a_m \).
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What’s the necessary and sufficient condition? How about for two branches case?

\[ \rho_1 \quad \rho_2 \]

\[ \tau_1 \quad \tau_2 \]

- WLOG, we may assume that \( \rho_1 \leq \rho_2, \tau_1 \leq \tau_2 \) and \( \rho_1 \leq \tau_1 \).
- Conjecture. Lipschitz equivalent iff \( (\rho_1, \rho_2) = (\tau_1, \tau_2) \).
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Some Notations

- $K$: self-similar set determined by the IFS $\{\mathbb{R}^d; f_1, \ldots, f_m\}$.
- $\rho_j$: contraction ratio of $f_j$, $\forall j$.
- $(\rho_1, \ldots, \rho_m)$ is called a contraction vector (c.v.) of $K$.
- For any c.v. $\rho = (\rho_1, \ldots, \rho_m)$ with $\sum \rho_j^d < 1$, we define $\mathcal{D}(\rho)$ to be all dust-like self-similar sets with c.v. $\rho$ in $\mathbb{R}^d$.
- Throughout the talk, the dimension $d$ will be implicit.
- Define $\dim_H \mathcal{D}(\rho) = \dim_H E$, for some (then for all) $E \in \mathcal{D}(\rho)$.
- $E \sim F$ for any $E, F \in \mathcal{D}(\rho)$.
- Define $\mathcal{D}(\rho) \sim \mathcal{D}(\tau)$ if $E \sim F$ for some $E \in \mathcal{D}(\rho)$ and $F \in \mathcal{D}(\tau)$.
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Step 1 to solve the Question on two branches case

Assume that $\mathcal{D}(\rho_1, \rho_2) \sim \mathcal{D}(\tau_1, \tau_2)$. By FM’ theorem, one of followings must happen:

(1). $\log \rho_1 / \log \rho_2 \notin \mathbb{Q}$.

(2). $\exists \lambda \in (0, 1)$, and $p_1, q_1, p_2, q_2 \in \mathbb{Z}^+$ such that

$$\rho_1 = \lambda^{p_1}, \quad \rho_2 = \lambda^{p_2}, \quad \tau_1 = \lambda^{q_1}, \quad \tau_2 = \lambda^{q_2}.$$
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Let’s study case (2) first.

From \( s = \dim_H D(\rho_1, \rho_2) = \dim_H D(\tau_1, \tau_2) \), we have

\[
(\lambda^{p_1})^s + (\lambda^{p_2})^s = (\lambda^{q_1})^s + (\lambda^{q_2}) = 1.
\]

Denote \( x = \lambda^s \), then

\[
x^{p_1} + x^{p_2} = x^{q_1} + x^{q_2} = 1.
\]

That is,

\[
x^{p_1} + x^{p_2} - 1 = 0 \quad \text{and} \quad x^{q_1} + x^{q_2} - 1 = 0
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have same root in \((0, 1)\), where \( p_1 \geq p_2, q_1 \geq q_2, p_1 \geq q_1 \).

Using Ljunggren’s result on the irreducibility of trinomials \( x^n \pm x^m \pm 1 \), we proved that the above happens iff

- \((p_1, p_2) = (q_1, q_2)\) or
- \((p_1, p_2, q_1, q_2) = \gamma(5, 1, 3, 2)\) for some \( \gamma \in \mathbb{Z}^+ \).
Step 2 to solve the Question

Let’s study case (2) first.

- From \( s = \dim_H \mathcal{D}(\rho_1, \rho_2) = \dim_H \mathcal{D}(\tau_1, \tau_2) \), we have
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Using Ljunggren’s result on the irreducibility of trinomials \( x^n \pm x^m \pm 1 \), we proved that the above happens iff

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2. \((p_1, p_2, q_1, q_2) = \gamma(5, 1, 3, 2)\) for some \( \gamma \in \mathbb{Z}^+ \).
Thus, Case (2) holds will imply \((\rho_1, \rho_2) = (\tau_1, \tau_2)\) or there exists \(\lambda \in (0, 1)\), s.t.

\[
(\rho_1, \rho_2, \tau_1, \tau_2) = (\lambda^5, \lambda, \lambda^3, \lambda^2).
\]  (1)

We can check that \(D(\lambda^5, \lambda) \sim D(\lambda^3, \lambda^2)\) as following figure.
Step 2 to solve the Question

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Let's study Case (1) now.

Given a c.v. $\vec{\rho} = (\rho_1, \ldots, \rho_m)$. Define

$$\langle \vec{\rho} \rangle := \{ \rho_1^{\alpha_1} \cdots \rho_m^{\alpha_m} : \alpha_1, \ldots, \alpha_m \in \mathbb{Z} \}.$$ 

- $\langle \vec{\rho} \rangle$ is an abelian group and has a nonempty basis.
- Define $\text{rank}(\langle \vec{\rho} \rangle)$ to be the cardinality of the basis.
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Assume that both $\vec{\rho}$ and $\vec{\tau}$ have full rank $m$. Then $D(\vec{\rho}) \sim D(\vec{\tau})$ iff $\vec{\rho}$ is a permutation of $\vec{\tau}$.
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Part II. Lipschitz equivalence of self-similar sets with touching structures
David and Semmes conjectured that $M \not\sim M'$.

Rao, R and Xi (2006) obtained that $M \sim M'$. 

Figure: Initial construction of $M$ and $M'$
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Figure: Initial construction of $M$ and $M'$
Generalized $\{1,3,5\}-\{1,4,5\}$ problem

\[\begin{array}{cccc}
\mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 \\
\mathcal{Q}_1 & \mathcal{Q}_2 & \mathcal{Q}_3 \\
\end{array}\]

**Figure:** Initial construction of $M_\rho$ and $M'_\rho$

Xi and R (2007): $M_\rho \sim M'_\rho$ iff $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$. 

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Some progresses on Lipschitz equivalence of self-similar sets
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Figure: Initial construction of $D$ and $T$, where $n = 6$

- $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_n)$ is a c.v. in $\mathbb{R}$ with $n \geq 3$.
- $D \in \mathcal{D}(\overrightarrow{\rho})$.
- $T$: attractor of IFS $\{\psi_j(x) = \rho_j x + t_j\}_{j=1}^n$ satisfying:
  - The subintervals $\psi_1([0, 1]), \ldots, \psi_n([0, 1])$ are spaced from left to right without overlapping.
  - Left endpoint of $\psi_1([0, 1])$ is 0; right endpoint of $\psi_n([0, 1])$ is 1.
  - $\exists j \in \{1, 2, \ldots, n - 1\}$, such that the intervals $\psi_j([0, 1])$ and $\psi_{j+1}([0, 1])$ are touching, i.e. $\psi_j(1) = \psi_{j+1}(0)$.

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Huo-Jun Ruan (With Hui Rao, Yang Wang and Li-Feng Xi) Some progresses on Lipschitz equivalence of self-similar sets
General Case

\[ \mathcal{D}_1 \mid \mathcal{D}_2 \mid \mathcal{D}_3 \mid \mathcal{D}_4 \mid \mathcal{D}_5 \mid \mathcal{D}_6 \]

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Huo-Jun Ruan (With Hui Rao, Yang Wang and Li-Feng Xi) Some progresses on Lipschitz equivalence of self-similar sets
Theorem (R-Wang-Xi, Preprint)

Assume that $D \sim T$. Then $\log \rho_1 / \log \rho_n \in \mathbb{Q}$.

- A letter $j \in \{1, \ldots, n\}$ is a (left) touching letter if $\Psi_j([0, 1])$ and $\Psi_{j+1}([0, 1])$ are touching, i.e. $\Psi_j(1) = \Psi_{j+1}(0)$.
- $\Sigma_T$: the set of all (left) touching letters.

Theorem (R-Wang-Xi, Preprint)

Let $n = 4$, $\rho_1 = \rho_4$, and $\Sigma_T = \{2\}$. Assume that $D \sim T$. Let $s = \dim_H D = \dim_H T$ and $\mu_j = \rho_j^s$ for $1 \leq j \leq 4$. Then $\mu_2$ and $\mu_3$ must be algebraically dependent, namely there exists a nonzero rational polynomial $P(x, y)$ such that $P(\mu_2, \mu_3) = 0$. Some progresses on Lipschitz equivalence of self-similar sets
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Assume that $\log \rho_1 / \log \rho_n \in \mathbb{Q}$. Then, $D \sim T$ if every touching letter for $T$ is substitutable.

Corollary

Let $M \xrightarrow{\rho}$ and $M' \xrightarrow{\rho}$ be sets defined in generalized $\{1,3,5\}$-$\{1,4,5\}$ problem. Then $M \xrightarrow{\rho} \sim M' \xrightarrow{\rho}$ iff $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$.

Note: If $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$, the unique touching letter $\{2\}$ is substitutable.

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