Multifractal analysis: an example with two different Olsen's cutoff functions

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Besicovitch spaces

$(\mathbb{X}, \mathrm{d})$: a metric space having the *Besicovitch property:*

There exists an integer constant C_B such that one can extract C_B countable families $\left\{\{\mathrm{B}_{j,k}\}_k\right\}_{1 \le j \le C_B}$ from any collection $\mathscr{B}$ of balls so that

1. $\bigcup_{j,k} \mathrm{B}_{j,k}$ contains the centers of the elements of $\mathscr{B}$,
2. for any j and $k \neq k'$, $\mathrm{B}_{j,k} \cap \mathrm{B}_{j,k'} = \emptyset$.

$\mathrm{B}(x, r)$ stands for the open ball $\mathrm{B}(x, r) = \{y \in \mathbb{X}\ ;\ \mathrm{d}(x, y) < r\}$. The letter B with or without subscript will implicitly stand for such a ball. When dealing with a collection of balls $\{\mathrm{B}_i\}_{i \in I}$ the following notation will implicitly be assumed: $\mathrm{B}_i = \mathrm{B}(x_i, r_i)$.

Coverings and packings

δ-cover of $E \subset \mathbb{X}$: a collection of *balls* of radii not exceeding δ whose union contains E. A *centered cover* of E is a cover of E consisting in balls whose centers belong to E.

δ-packing of $E \subset \mathbb{X}$: a collection of disjoint balls of radii not exceeding δ centered in E.

Besicovitch δ-cover of $E \subset \mathbb{X}$: a centered δ-cover of E which can be decomposed into C_B packings.

Packing measures and dimension

$$\begin{aligned} \overline{\mathscr{P}}^t_\delta(E) &= \sup\left\{\sum r_j^t \ ; \ \{\mathrm{B}_j\} \ \delta\text{-packing of } E\right\}, \\ \overline{\mathscr{P}}^t(E) &= \lim_{\delta\searrow 0} \overline{\mathscr{P}}^t_\delta(E), \\ \mathscr{P}^t(E) &= \inf\left\{\sum \overline{\mathscr{P}}^t(E_j) \ ; \ E \subset \bigcup E_j\right\}, \end{aligned}$$

$$\begin{aligned} \Delta(E) &= \inf\{t\in\mathbb{R} \ ; \ \overline{\mathscr{P}}^t(E)=0\} = \sup\{t\in\mathbb{R} \ ; \ \overline{\mathscr{P}}^t(E)=\infty\} \\ \dim_P E &= \inf\{t\in\mathbb{R} \ ; \ \mathscr{P}^t(E)=0\} = \sup\{t\in\mathbb{R} \ ; \ \mathscr{P}^t(E)=\infty\} \end{aligned}$$

One has $\Delta(E) = \overline{\dim}_B E$.

Centered Hausdorff measures

$$\begin{aligned}
\overline{\mathscr{H}}^t_\delta(E) &= \inf\left\{\sum r_j^t \ ; \ \{\mathrm{B}_j\} \text{ centered } \delta\text{-cover of } E\right\}, \\
\overline{\mathscr{H}}^t(E) &= \lim_{\delta\searrow 0} \overline{\mathscr{H}}^t_\delta(E), \\
\mathscr{H}^t(E) &= \sup\left\{\overline{\mathscr{H}}^t(F) \ ; \ F \subset E\right\}.
\end{aligned}$$

$$\dim_H E = \inf\{t \in \mathbb{R} \ ; \ \mathscr{H}^t(E) = 0\} = \sup\{t \in \mathbb{R} \ ; \ \mathscr{H}^t(E) = \infty\}$$

Lower bounds for dimensions

ν: a non-negative function defined on the set of balls of $\mathbb{X}$.

$$\begin{aligned}
\overline{\nu}_\delta(E) &= \inf\left\{\sum \nu(\mathrm{B}_j)\ :\ \{\mathrm{B}_j\}\text{ centered }\delta\text{-cover of }E\right\}\\
\overline{\nu}(E) &= \lim_{\delta\searrow 0}\overline{\nu}_\delta(E)\\
\nu^\sharp(E) &= \sup_{F\subset E}\overline{\nu}(F)
\end{aligned}$$

Lemma

If $\nu^\sharp(E) > 0$, *then*

$$\dim_H E \geq \operatorname*{ess\,sup}_{x\in E,\,\nu^\sharp}\liminf_{r\searrow 0}\frac{\log\nu(\mathrm{B}(x,r))}{\log r}, \tag{1}$$

$$\dim_P E \geq \operatorname*{ess\,sup}_{x\in E,\,\nu^\sharp}\limsup_{r\searrow 0}\frac{\log\nu(\mathrm{B}(x,r))}{\log r}, \tag{2}$$

To prove (1), take $\gamma < \operatorname{ess\,sup}_{x\in E,\, \nu^\sharp} \liminf_{r\searrow 0} \frac{\log \nu\big(\mathrm{B}(x,r)\big)}{\log r}$ and consider the set $F = \left\{ x \in E \; ; \; \liminf_{r\searrow 0} \frac{\log \nu\big(\mathrm{B}(x,r)\big)}{\log r} > \gamma \right\}$. We have $\nu^\sharp(F) > 0$. For all $x \in F$, there exists $\delta > 0$ such that, for all $r \le \delta$, one has $\nu\big(\mathrm{B}(x,r)\big) \le r^\gamma$. Consider the set

$$F(n) = \left\{ x \in F \; ; \; \forall r \le 1/n, \; \nu\big(\mathrm{B}(x,r)\big) \le r^\gamma \right\}.$$

We have $F = \bigcup_{n\ge 1} F(n)$. Since $\nu^\sharp(F) > 0$, there exists n such that $\nu^\sharp\big(F(n)\big) > 0$, and therefore there is a subset G of $F(n)$ such that $\overline{\nu}(G) > 0$. Then for any centered δ-cover $\{\mathrm{B}_j\}$ of G, with $\delta \le 1/n$, one has

$$\overline{\nu}_\delta(G) \le \sum \nu(\mathrm{B}_j) \le \sum r_j^\gamma.$$

Therefore,

$$\overline{\nu}_\delta(G) \quad \le \quad \overline{\mathscr{H}}^\gamma_\delta(G),$$

and

$$0 < \overline{\nu}(G) \quad \le \quad \overline{\mathscr{H}}^\gamma(G) \le \mathscr{H}^\gamma(G),$$

which implies $\dim_H E \ge \dim_H G \ge \gamma$.

To prove (2), take $\gamma < \operatorname{ess\,sup}_{x \in E,\, \nu^\sharp} \limsup_{r \searrow 0} \frac{\log \nu\big(\mathrm{B}(x,r)\big)}{\log r}$ and consider the set $F = \left\{ x \in E \;;\; \limsup_{r \searrow 0} \frac{\log \nu\big(\mathrm{B}(x,r)\big)}{\log r} > \gamma \right\}$. We have $\nu^\sharp(F) > 0$, so there exists a subset F' of F such that $\overline{\nu}(F') > 0$. Let G be a subset of F'. Then, for all $x \in G$, for all $\delta > 0$, there exists $r \le \delta$ such that $\nu\big(\mathrm{B}(x,r)\big) \le r^\gamma$. Then for all δ, by using the Besicovitch property, there exists a collection $\big\{\{\mathrm{B}_{j,k}\}_j\big\}_{1 \le k \le C_B}$ of δ-packings of G which together cover G and such that $\nu(\mathrm{B}_{j,k}) \le r_{j,k}^\gamma$. Then one has

$$\overline{\nu}_\delta(G) \le \sum_{j,k} \nu(\mathrm{B}_{j,k}) \le \sum r_{j,k}^\gamma.$$

This implies that there exists k such that $\sum_j r_{j,k}^\gamma \ge \frac{1}{C_B}\, \overline{\nu}_\delta(G)$. So we have $\overline{\mathscr{P}}_\delta^\gamma(G) \ge \frac{1}{C_B}\, \overline{\nu}_\delta(G)$. This implies $\overline{\mathscr{P}}^\gamma(G) \ge \frac{1}{C_B}\, \overline{\nu}(G)$. So if $F' = \bigcup G_j$, one has

$$\sum \overline{\mathscr{P}}^\gamma(G_j) \ge \frac{1}{C_B} \sum \overline{\nu}(G_j) \ge \frac{1}{C_B} \overline{\nu}(F') > 0,$$

so $\mathscr{P}^\gamma(F') > 0$. Therefore, $\dim_P F \ge \gamma$.

Level sets of local Hölder exponents

μ : a non-negative function of balls of $\mathbb{X}$ such that

$$\mu(\mathsf{B}) = 0 \text{ and } \mathsf{B}' \subset \mathsf{B} \Longrightarrow \mu(\mathsf{B}') = 0.$$

S_μ, the *support* of μ, is the complement of $\bigcup_{\mu(\mathsf{B})=0} \mathsf{B}$.

$$\begin{aligned}
\overline{X}_\mu(\alpha) &= \left\{ x \in \mathsf{S}_\mu \ ; \ \limsup_{r \searrow 0} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} \le \alpha \right\}, \\
\underline{X}_\mu(\alpha) &= \left\{ x \in \mathsf{S}_\mu \ ; \ \liminf_{r \searrow 0} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} \ge \alpha \right\}, \\
X_\mu(\alpha, \beta) &= \underline{X}_\mu(\alpha) \cap \overline{X}_\mu(\beta),
\end{aligned}$$

and

$$X_\mu(\alpha) = \underline{X}_\mu(\alpha) \cap \overline{X}_\mu(\alpha).$$

Olsen's packing measures

$$\overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E) \quad = \quad \sup\left\{ \sum^{*} r_j^t \mu(\mathsf{B}_j)^q \ ; \ \{\mathsf{B}_j\} \ \delta\text{-packing of } E \right\},$$

where $*$ means that one only sums the terms for which $\mu(\mathsf{B}_j) \neq 0$,

$$\overline{\mathscr{P}}^{q,t}_{\mu}(E) \quad = \quad \lim_{\delta \searrow 0} \overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E),$$

$$\mathscr{P}^{q,t}_{\mu}(E) \quad = \quad \inf\left\{ \sum \overline{\mathscr{P}}^{q,t}_{\mu}(E_j) \ ; \ E \subset \bigcup E_j \right\},$$

$$\tau_\mu(q) \quad = \quad \inf\{t \in \mathbb{R} \ ; \ \overline{\mathscr{P}}^{q,t}_{\mu}(\mathsf{S}_\mu) = 0\} = \sup\{t \in \mathbb{R} \ ; \ \overline{\mathscr{P}}^{q,t}_{\mu}(\mathsf{S}_\mu) = \infty\}$$

$$B_\mu(q) \quad = \quad \inf\{t \in \mathbb{R} \ ; \ \mathscr{P}^{q,t}_{\mu}(\mathsf{S}_\mu) = 0\} = \sup\{t \in \mathbb{R} \ ; \ \mathscr{P}^{q,t}_{\mu}(\mathsf{S}_\mu) = \infty\}$$

τ_μ and B_μ are convex.

Alternate definition of τ_μ

Fix $\lambda < 1$ and define

$$\widetilde{\mathscr{P}}^{q,t}_{\mu,\delta}(E) = \sup \left\{ \sum^{*} r_j^t \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} \; ; \; \{\mathsf{B}_j\} \text{ packing of } E \text{ with } \lambda\delta < r_j \leq \delta \right\},$$

$$\widetilde{\mathscr{P}}^{q,t}_{\mu}(E) = \overline{\lim_{\delta \searrow 0}} \, \widetilde{\mathscr{P}}^{q,t}_{\mu,\delta}(E),$$

and

$$\widetilde{\tau}_{\mu,E}(q) = \sup \left\{ t \in \mathbb{R} \; ; \; \widetilde{\mathscr{P}}^{q,t}_{\mu}(E) = +\infty \right\}.$$

Proposition

For any $\lambda < 1$, one has $\widetilde{\tau}_{\mu,\mathsf{S}_\mu} = \tau_\mu$ and

$$\tau_\mu(q) =$$

$$\overline{\lim_{\delta \searrow 0}} \frac{-1}{\log \delta} \log \sup \left\{ \sum^{*} \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} \; ; \; \{\mathsf{B}_j\} \textit{ packing of } \mathsf{S}_\mu \textit{ with } \lambda\delta < r_j \leq \delta \right\}.$$

Olsen's Hausdorff measures

$$\begin{aligned}
\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E) &= \inf\left\{ \sum^{*} r_j^t \mu(\mathsf{B}_j)^q \ ; \ \{\mathsf{B}_j\} \text{ centered } \delta\text{-cover of } E \right\}, \\
\overline{\mathscr{H}}^{q,t}_{\mu}(E) &= \lim_{\delta \searrow 0} \overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E), \\
\mathscr{H}^{q,t}_{\mu}(E) &= \sup\left\{ \overline{\mathscr{H}}^{q,t}_{\mu}(F) \ ; \ F \subset E \right\}.
\end{aligned}$$

$$b_\mu(q) = \inf\{t \in \mathbb{R} \ ; \ \mathscr{H}^{q,t}_\mu(\mathsf{S}_\mu) = 0\} = \sup\{t \in \mathbb{R} \ ; \ \mathscr{H}^{q,t}_\mu(\mathsf{S}_\mu) = \infty\}$$

In general, b_μ is not convex. One always has

$$b_\mu \le B_\mu \le \tau_\mu.$$

Legendre transform: $f^*(y) = \inf_{x\in\mathbb{R}} xy + f(x)$.

Theorem (Olsen, Ben Nasr-Bhouri-Heurteaux)

1. $\dim_H X_\alpha \le b^*(\alpha)$.
2. $\dim_P X_\alpha \le B^*(\alpha)$.
3. *If* $-\alpha = B'(q)$ *exists and* $\dim_H X_\alpha = B^*(q)$, *then* $B(q) = b(q)$.
4. *If for some q,* $\mathscr{H}_\mu^{q,B(q)}(\mathsf{S}_\mu) > 0$ *and* $-\alpha = B'(q)$ *exists, then*

$$\dim_H X(\alpha) = \inf_{r\in\mathbb{R}} B(r) + \alpha r = B(q) - qB'(q).$$

Main lemma

$$\overline{\mathscr{Q}}^{q,t}_{\mu,\nu,\delta}(E) = \sup\left\{\sum^{*} r_j^t \mu(\mathsf{B}_j)^q \nu(\mathsf{B}_j) \; ; \; \{\mathsf{B}_j\} \ \delta\text{-packing of } E\right\},$$

$$\overline{\mathscr{Q}}^{q,t}_{\mu,\nu}(E) = \lim_{\delta\searrow 0} \overline{\mathscr{Q}}^{q,t}_{\mu,\nu,\delta}(E),$$

$$\mathscr{Q}_{\mu,\nu}(E) = \inf\left\{\sum \overline{\mathscr{Q}}_{\mu,\nu}(E_j) \; : \; E \subset \bigcup E_j\right\}.$$

$$\overline{\varphi}_{\mu,\nu}(q) = \inf\{t\in\mathbb{R} \; ; \; \overline{\mathscr{Q}}^{q,t}_{\mu,\nu}(\mathsf{S}_\mu) = 0\} = \sup\{t\in\mathbb{R} \; ; \; \overline{\mathscr{Q}}^{q,t}_{\mu,\nu}(\mathsf{S}_\mu) = \infty\}$$

$$\varphi_{\mu,\nu}(q) = \inf\{t\in\mathbb{R} \; ; \; \mathscr{Q}^{q,t}_{\mu,\nu}(\mathsf{S}_\mu) = 0\} = \sup\{t\in\mathbb{R} \; ; \; \mathscr{Q}^{q,t}_{\mu,\nu}(\mathsf{S}_\mu) = \infty\}$$

Lemma

Assume that $\varphi_{\mu,\nu}(0) = 0$ *and* $\nu^\sharp(\mathsf{S}_\mu) > 0$. *Then one has*

$$\nu^\sharp\left({}^{\mathsf{C}}X_\mu\left(-\varphi_r'(0), -\varphi_l'(0)\right)\right) = 0,$$

The same result holds with $\overline{\varphi}_{\mu,\nu}$.

Take $\gamma > -\varphi_l'(0)$, and choose γ' and $t > 0$ such that $\gamma > \gamma' > -\varphi_l'(0)$ and $\varphi(-t) < \gamma' t$. Then $\mathscr{P}_{(\mu,\nu)}^{(-t,1),\gamma' t}(\mathsf{S}_\mu) = 0$, so there exists a countable partition $\mathsf{S}_\mu = \bigcup E_j$ of S_μ such that

$$\sum_j \overline{\mathscr{P}}_{(\mu,\nu)}^{(-t,1),\gamma' t}(E_j) \le 1.$$

It results that $\overline{\mathscr{P}}_{(\mu,\nu)}^{(-t,1),\gamma t}(E_j) = 0$ for all j.
Consider the set

$$E(\gamma) = \left\{ x \in \mathsf{S}_\mu \; ; \; \limsup_{r \searrow 0} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} > \gamma \right\}.$$

If $x \in E(\gamma)$, for all $\delta > 0$, there exists $r \le \delta$ such that $\mu\big(\mathsf{B}(x,r)\big) \le r^\gamma$. Let F be a subset of $E(\gamma)$. Set $F_j = F \cap E_j$.
For $\delta > 0$, for all j, one can find a Besicovitch δ-cover $\{\mathsf{B}_{j,k}\}$ of F_j such that $\mu(\mathsf{B}_{j,k}) \le r_{j,k}^\gamma$.

We have,

$$\overline{\nu}_\delta(F_j) \le \sum_k \nu(\mathsf{B}_{j,k}) =$$

$$\sum_k \mu(\mathsf{B}_{j,k})^{-t}\mu(\mathsf{B}_{j,k})^t\nu(\mathsf{B}_{j,k}) \le \sum_k \mu(\mathsf{B}_{j,k})^{-t} r_{j,k}^{\gamma t}\nu(\mathsf{B}_{j,k}),$$

which, together with the Besicovitch property, implies

$$\overline{\nu}_\delta(F_j) \le C_B \overline{\mathscr{P}}^{(-t,1),\gamma t}_{(\mu,\nu),\delta}(E_j).$$

so

$$\overline{\nu}(F_j) \le C_B \overline{\mathscr{P}}^{(-t,1),\gamma t}_{(\mu,\nu)}(E_j) = 0.$$

This implies $\overline{\nu}(F) = 0$, and $\nu^\sharp(E(\gamma)) = 0$.
We conclude that

$$\nu^\sharp\left(\left\{x \in \mathsf{S}_\mu \ ; \ \limsup_{r \searrow 0} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} > -\varphi_l'(0)\right\}\right) = 0.$$

An example

Take $\mathbb{X} = \{0,1\}^{\mathbb{N}^*}$ endowed with the ultrametric which assigns diameter 2^{-n} to cylinders of order n.
We are given two numbers such that $0 < p < \tilde{p} \leq 1/2$ and a sequence of integers $1 = t_0 < t_1 < \cdots < t_n < \cdots$ such that $\lim_{n\to\infty} t_n/t_{n+1} = 0$.
We define a probability measure μ on $\{0,1\}^{\mathbb{N}^*}$: the measure assigned to the cylinder $[\varepsilon_1\varepsilon_2\ldots\varepsilon_n]$ is

$$\mu([\varepsilon_1\varepsilon_2\ldots\varepsilon_n]) = \prod_{j=1}^{n} \varpi_j(\varepsilon_j),$$

where

$$\varpi_j = \begin{cases} (p, 1-p) & \text{if } t_{2k-1} \leq j < t_{2k} \text{ for some } k, \\ (\tilde{p}, 1-\tilde{p}) & \text{if } t_{2k} \leq j < t_{2k+1} \text{ for some } k, \end{cases}$$

$$\mu([\varepsilon_1\varepsilon_2\ldots\varepsilon_n]) = \prod_{j=1}^{n} \varpi_j,$$

where

$$\varpi_j = \begin{cases} (p, 1-p) & \text{if } t_{2k-1} \le j < t_{2k} \text{ for some } k, \\ (\tilde{p}, 1-\tilde{p}) & \text{if } t_{2k} \le j < t_{2k+1} \text{ for some } k. \end{cases}$$

$$\sum_{j\in\{0,1\}} \mu([\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}j])^q = \mu([\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}])^q \times \begin{cases} (p^q + (1-p)^q) \\ (\tilde{p}^q + (1-\tilde{p})^q) \end{cases}$$

$$\sum \mu([\varepsilon_1\varepsilon_2\ldots\varepsilon_n])^q = (p^q + (1-p)^q)^{x_n} (\tilde{p}^q + (1-\tilde{p})^q)^{n-x_n}$$

$$0 \le \frac{x_n}{n} \le 1, \quad \liminf \frac{x_n}{n} = 0, \quad \limsup \frac{x_n}{n} = 1$$

τ, b. and B

Set

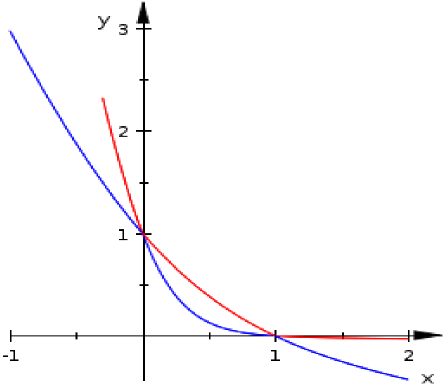
$$\begin{aligned}\theta(q) &= \log\big(p^q + (1-p)^q\big)\\ \tilde\theta(q) &= \log\big(\tilde p^q + (1-\tilde p)^q\big)\end{aligned}$$

Then

$$\begin{aligned}\limsup \frac{1}{n}\log\sum \mu\big([\varepsilon_1\varepsilon_2\ldots\varepsilon_n]\big)^q &= \max\{\theta(q),\tilde\theta(q)\}\\ \liminf \frac{1}{n}\log\sum \mu\big([\varepsilon_1\varepsilon_2\ldots\varepsilon_n]\big)^q &= \min\{\theta(q),\tilde\theta(q)\}\end{aligned}$$

It has been shown (Ben Nasr, Bhouri, and Heurteaux) that these are respectively $B_\mu(q)$ and $b_\mu(q)$.

$$
\begin{aligned}
b(q) &= \min\{\theta(q), \tilde{\theta}(q)\} \quad \text{blue curve} \\
B(q) &= \max\{\theta(q), \tilde{\theta}(q)\} \quad \text{red curve}
\end{aligned}
$$



Results

Theorem

1. *For* $\alpha \in \left(-\log_2(1-\tilde{p}), -\log_2 \tilde{p}\right)$, *we have*
$$\dim_H X_\mu(\alpha) = \inf_{q\in\mathbb{R}} b(q) + \alpha q.$$
2. *For* $\alpha \in \left(-\log_2(1-\tilde{p}), -\log_2 \tilde{p}\right) \setminus \left([-B_r'(0), -B_l'(0)] \cup [-B_r'(1), -B_l'(1)]\right)$, *we have*
$$\dim_P X_\mu(\alpha) = \inf_{q\in\mathbb{R}} B(q) + \alpha q.$$

We already know the upper bounds. Indeed, it is known that, if $\alpha = -B'(q)$, then

$$\dim_P X_\alpha \leq B^*(\alpha) = -q\, B'(q) + B(q) = \inf_t \alpha t + B(t).$$

It is also known that $\dim_H X_\alpha \leq \inf_t \alpha t + b(t)$. In particular, if α can be written as $-b'(q)$ then $\dim_H X_\alpha \leq -q\, b'(q) + b(q)$.

Proof

Given two numbers r and $\tilde{r}$ in the interval $(0,1)$, we perform the same construction as with p and $\tilde{p}$, but using the same sequence (t_j). We get a new measure ν.
We compute $\overline{\varphi}_{\mu,\nu}$:

$$\sum_{\varepsilon_1\ldots\varepsilon_n} \mu([\varepsilon_1\ldots\varepsilon_n])^t \nu([\varepsilon_1\ldots\varepsilon_n]) =$$

$$\left(r\,p^t + (1-r)\,(1-p)^t\right)^{x_n} \left(\tilde{r}\,\tilde{p}^t + (1-\tilde{r})\,(1-\tilde{p})^t\right)^{n-x_n}.$$

$$\overline{\varphi}_{\mu,\nu}(t) = \log_2 \max\{r\,p^t + (1-r)\,(1-p)^t, \tilde{p}^t + (1-\tilde{r})\,(1-\tilde{p})^t\}$$

If $r\log p + (1-r)\log(1-p) = \tilde{r}\log\tilde{p} + (1-\tilde{r})\log(1-\tilde{p})$,
then $\overline{\varphi}'_{\mu,\nu}(0)$ exists.

$$\alpha = -\varphi'_{\mu,\nu}(0) = r\log_2 p + (1-r)\log_2(1-p) = \tilde{r}\log_2\tilde{p} + (1-\tilde{r})\log_2(1-\tilde{p})$$

$r \log p + (1-r)\log(1-p) = \tilde{r}\log\tilde{p} + (1-\tilde{r})\log(1-\tilde{p})$ plus constraints $0 < r, \tilde{r} < 1$ imply that α can assume any value between $-\log_2(1-\tilde{p})$ and $-\log_2\tilde{p}$.
One has

$$-\frac{1}{n}\log_2\nu([\varepsilon_1\ldots\varepsilon_n]) = \frac{1}{n}\sum_{j=1}^{n}\log_2\varpi'_j(\varepsilon_j)$$

so, due to the strong law of large numbers, for n-almost t,

$$\begin{aligned} \liminf -\frac{1}{n}\log_2\nu(C_n(t)) &= \min\{\mathrm{h}(r), \mathrm{h}(\tilde{r})\} \\ \limsup -\frac{1}{n}\log_2\nu(C_n(t)) &= \max\{\mathrm{h}(r), \mathrm{h}(\tilde{r})\}, \end{aligned}$$

where $C_n(t)$ stands for the n-cylinder which contains t and
$\mathrm{h}(r) = -\log_2 r - \log_2(1-r)$.

it results from the preceding lemmas that

$$\dim_H X_\mu(\alpha) \quad \geq \quad \min\{\mathrm{h}(r), \mathrm{h}(\tilde r)\}$$

and

$$\dim_P X_\mu(\alpha) \quad \geq \quad \max\{\mathrm{h}(r), \mathrm{h}(\tilde r)\},$$

where r, $\tilde r$, and α are linked by relations
$\alpha = r\log_2 p + (1-r)\log_2(1-p) = \tilde r \log_2 \tilde p + (1-\tilde r)\log_2(1-\tilde p)$.

We have

$$\alpha = -\theta'(q) \quad \text{if} \quad q = \frac{\log \frac{1-r}{r}}{\log \frac{1-p}{p}} \quad \text{i.e,} \quad r = \frac{p^q}{p^q + (1-p)^q}$$

and

$$\alpha = -\tilde\theta'(\tilde q) \quad \text{if} \quad \tilde q = \frac{\log \frac{1-\tilde r}{\tilde r}}{\log \frac{1-\tilde p}{\tilde p}}, \quad \text{i.e,} \quad \tilde r = \frac{\tilde p^{\tilde q}}{\tilde p^{\tilde q} + (1-\tilde p)^{\tilde q}}$$

Now, fix q and $\tilde q$ as above. One can check that, for these values of q and $\tilde q$, one has

$$\theta(q) - q\,\theta'(q) = \mathrm{h}(r) \quad \text{and} \quad \tilde\theta(\tilde q) - \tilde q\,\tilde\theta'(\tilde q) = \mathrm{h}(\tilde r).$$

In order to have $\theta(q) = b(q)$, we must have $0 < q < 1$, which means

$$\log_2 \frac{1}{p^p(1-p)^{1-p}} < \alpha < \log_2 \frac{1}{\sqrt{p(1-p)}}. \tag{3}$$

In order to have $\tilde{\theta}(\tilde{q}) = b(\tilde{q})$, we must have $\tilde{q} < 0$ or $\tilde{q} > 1$, which means

$$\alpha \quad > \quad \log_2 \frac{1}{\sqrt{\tilde{p}(1-\tilde{p})}} \tag{4}$$

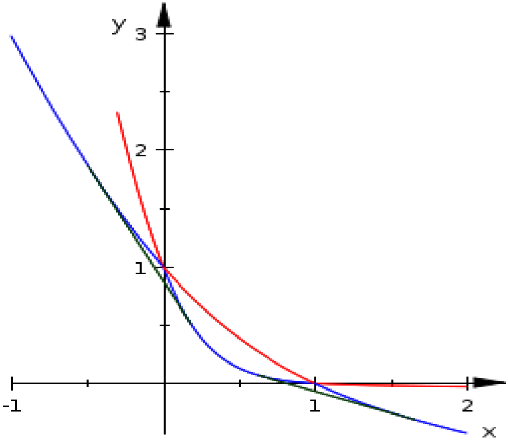
or

$$\alpha \quad < \quad \log_2 \frac{1}{\tilde{p}^{\tilde{p}}(1-\tilde{p})^{1-\tilde{p}}}. \tag{5}$$

One can check that at least one of the conditions (3), (4) and (5) is fulfilled.

But for any q such that $b'(q)$ exists, we have

$$\dim_H X_\mu\big(-b'(q)\big) \leq b(q) - q\, b'(q). \tag{6}$$


y
3
2
1
-1
1
2
x

The Gray code

w	: 0 1
$\varphi(w)$	: 0 1
w	: 00 01 10 11
$\varphi(w)$	: 00 01 11 10
w	: 000 001 010 011 100 101 110 111
$\varphi(w)$	: 000 001 011 010 110 111 101 100
w	: 0000 0001 0010 0011 0100 0101 0110 0111 1000 1001 1010 1011 $\cdots$
$\varphi(w)$	: 0000 0001 0011 0010 0110 0111 0101 0100 1100 1101 1111 1110 $\cdots$
...	: ....

Let ν be the image of the measure $[w] \longmapsto \mu[\varphi(w)]$ under the map $x_1 x_2 \cdots x_n \cdots \in \{0,1\}^{\mathbb{N}} \longmapsto \sum_{n \geq 1} x_n 2^{-n}$.

This is the measure considered by Ben Nasr, Bhouri, and Heurteaux. It is doubling and exhibits the same phenomenon as μ concerning b and B.

Recently, Shen Shuang proved that one gets the same result without composing with the Gray code.

Thank you!