Isodiametric Problem w.r.t. Hausdorff Measure

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Isodiametric problem in $\mathbb{R}^n$ asks for convex domains $C \subset \mathbb{R}^n$ of diameter $|C| = 1$ that have maximum volume.

(Bieberbach, 1915) proved that for any compact domain $\Omega \subset \mathbb{R}^n$

$$\text{Volume}(\Omega) \leq \text{Volume(Ball of diameter 1)} \left( \frac{|\Omega|}{2} \right)^n$$

and that equality holds if and only if $\Omega$ is a ball.

Therefore, the ball $B \subset \mathbb{R}^n$ of diameter one is the unique solution of classical isodiametric problem in $\mathbb{R}^n$.

This is “isodiametric problem with respect to Lebesgue measure”. How about replacing Lebesgue measure by Hausdorff measure restricted to a self-similar set with OSC?
For a self-similar set $E \subset \mathbb{R}^n$ with OSC, it is known that
\[
\sup \left\{ \frac{\mathcal{H}^s(X \cap E)}{|X|^s} : |X| > 0 \right\} = 1.
\]

Isodiametric Problem on $E$ then asks for compact convex domain $\Omega \subset \mathbb{R}^n$ with
\[
\frac{\mathcal{H}^s(\Omega \cap E)}{|\Omega|^s} = 1.
\]

We call such a domain $\Omega$ an extremal set.

**AIM** to find extremal sets $\Omega$ for specific $E$ and to study structure of $\Omega$.

1. “Shape” of $\Omega$
2. “Relative location” of $\Omega$ in $E$
3. Diameter $|\Omega|$ of $\Omega$
When Hausdorff dimension $s$ of $E$ is smaller than or equal to 1, there are many examples of $E$ for which an extremal set $\Omega$ is found.

When Hausdorff dimension $s$ of $E$ is strictly greater than 1, every known example of $E$ s.t. \textit{an extremal set $\Omega$ has been found} satisfies the following 2 properties:

(1) $s \in \mathbb{Z}$;

(2) $\mathcal{H}^s|_E$ and Lebesgue measure on $E$ differ by a constant.

We will consider concrete self-similar sets $E$ of \textit{dimension $s$ in $(1, \infty) \setminus \mathbb{Z}$} and try to find extremal sets $\Omega$.

Current Talk is about a family of self-similar fractals $E$ on $\mathbb{R}^2$. We can determine the \textit{shape} and \textit{location} of extremal sets $\Omega$. 
Let \( F_\lambda = E_\lambda \times \mathbb{R} \) for \( 0 < \lambda < \frac{1}{2} \), where \( E_\lambda \) denotes the middle \((1 - 2\lambda)\) Cantor set.

By Marstrand’s formula the set \( F_\lambda \) has Hausdorff dimension
\[
s = 1 - \frac{\ln 2}{\ln \lambda} \in (1, 2).
\]

**Observations:**

1. \( \mathcal{H}^s(X \cap F_\lambda) \leq |X|^s \) for any compact set \( X \).

2. For \( 0 < \lambda < \frac{1}{2} \), there is an extremal set \( \Omega \) for IP on \( F_\lambda \).

**Theorem**

If \( \lambda \leq \frac{1}{5} \), \( \Omega \) is a copy of some extremal set \( \Omega_\lambda \) with \([0, 1] \subset \text{proj}_1(\Omega_\lambda)\) such that \( \mathcal{M} \circ \mathcal{S}(\Omega_\lambda) \cap F_\lambda = \text{Disk} \cap F_\lambda \).

\[
\mathcal{H}^s(X \cap F_\lambda) = \mathcal{H}^s(E_\lambda \times [0, 1]) \mathcal{H}^{s-1} \times \mathcal{H}^1(X) \text{ for compact } X \subset \mathbb{R}^2.
\]
Theorem

If $\lambda \leq \frac{1}{5}$ diameter $t_\lambda$ of $\Omega_\lambda$ ($> \frac{2}{\sqrt{3}}$) is determined by

$$2 \int_0^\lambda \frac{tdF_\lambda(x)}{\sqrt{t^2 - (1 - 2x)^2}} = \frac{sf(\lambda, t)}{t}$$

where $D_t$ is a disk of diameter $t \geq \frac{2}{\sqrt{3}}$ centered on the line $x = \frac{1}{2}$,

$F_\lambda(x) = \mathcal{H}^{s-1}(E_\lambda \cap [0, x])$, \hspace{1cm} $f(\lambda, t) = \mathcal{H}^{s-1} \times \mathcal{H}^1(D_t \cap F_\lambda) =$

$$4 \int_0^\lambda \sqrt{\frac{t^2}{4} - \left(\frac{1}{2} - x\right)^2} \ dF_\lambda(x) = 2 \int_0^\lambda \sqrt{t^2 - (1 - 2x)^2} \ dF_\lambda(x)$$

and

$$\varphi_t(\lambda, t) = \frac{f_t(\lambda, t)}{t^s} - \frac{sf(\lambda, t)}{t^{s+1}} = t^{-s} \left[ 2 \int_0^\lambda \frac{tdF_\lambda(x)}{\sqrt{t^2 - (1 - 2x)^2}} - \frac{sf(\lambda, t)}{t} \right]$$
\[
\begin{align*}
\text{lower bound: } f_L(t, \lambda, n) &= \frac{4}{2^n} \sum_{x \in A_n} \sqrt{\frac{t^2}{4} - \left(\frac{1}{2} - x\right)^2} \\
\text{upper bound: } f_U(t, \lambda, n) &= \frac{4}{2^n} \sum_{x \in A_n} \sqrt{\frac{t^2}{4} - \left(\frac{1}{2} - x - \lambda^n\right)^2}
\end{align*}
\]

Key fact: \(|f_U - f_L| \leq 3\lambda^n\). Considering the case \(n = 4\), we have

<table>
<thead>
<tr>
<th>Value of (\lambda)</th>
<th>Upper Bound of (\varphi(\lambda, t_\lambda))</th>
<th>Lower Bound of (\varphi(\lambda, t_\lambda))</th>
<th>Interval containing (\mathcal{H}^s(E_\lambda \times [0, 1]))</th>
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</thead>
<tbody>
<tr>
<td>(\frac{1}{5})</td>
<td>0.702626</td>
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<td>((1.3800165, 1.38010795))</td>
</tr>
</tbody>
</table>
Thank you!