Inhomogeneous Diophantine approximation with general error functions

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I. Classic results

- α an irrational real number.
- $\|\cdot\|$ the distance to the nearest integer.

Minkowski (1907) : If $y \notin \mathbb{Z} + \alpha\mathbb{Z}$, then for infinitely many $n \in \mathbb{Z}$,

$$\|n\alpha - y\| < \frac{1}{4|n|}.$$

Khintchine (1926) : For any real number y, there exist infinitely many $n \in \mathbb{N}$ satisfying the Diophantine inequalities :

$$\|n\alpha - y\| < \frac{1}{\sqrt{5}n}.$$

Cassels (1950) : The following set $E(\alpha, c)$ is of full measure for any constant $c > 0$:

$$E(\alpha, c) := \left\{ y \in \mathbb{R} : \|n\alpha - y\| < \frac{c}{n} \text{ for infinitely many } n \right\}.$$

II. About Hausdorff dimension

Define

$$\omega(\alpha) := \sup\{\theta \geq 1 : \liminf_{n\to\infty} n^{\theta}\|n\alpha\| = 0\}.$$

Remark that α is a Liouville number if $\omega(\alpha) = \infty$.

Bernik-Dodson 1999 : the Hausdorff dimension of the set

$$E_{\gamma}(\alpha) := \left\{ y \in \mathbb{R} : \|n\alpha - y\| < \frac{1}{n^{\gamma}} \text{ for infinitely many } n \right\} \quad (\gamma \geq 1),$$

satisfies

$$\frac{1}{\omega(\alpha)\cdot\gamma} \leq \dim_H E_{\gamma}(\alpha) \leq \frac{1}{\gamma}.$$

Bugeaud/Schmeling-Troubetzkoy 2003 : for any irrational α,

$$\dim_H E_{\gamma}(\alpha) = \frac{1}{\gamma}.$$

III. With a general error function

Let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a function decreasing to zero. Consider the set

$$E_\varphi(\alpha) := \{y \in \mathbb{R} : \|n\alpha - y\| < \varphi(n) \text{ for infinitely many } n\}.$$

For an increasing function $\psi : \mathbb{N} \to \mathbb{R}^+$, define the lower and upper orders at infinity by

$$\lambda(\psi) := \liminf_{n\to\infty} \frac{\log \psi(n)}{\log n} \qquad \text{and} \qquad \kappa(\psi) := \limsup_{n\to\infty} \frac{\log \psi(n)}{\log n}.$$

Denote

$$u_\varphi := \frac{1}{\lambda(1/\varphi)} \qquad l_\varphi := \frac{1}{\kappa(1/\varphi)}.$$

The results of Bugeaud and Schmeling, Troubetzkoy imply the inequality

$$l_\varphi \le \dim_H(E_\varphi(\alpha)) \le u_\varphi.$$

IV. Some known results

A.-H. Fan, J. Wu 2006 :

(1) If α is of bounded type

$$\dim_H(E_\varphi(\alpha)) = u_\varphi$$

(2) There exists a Liouville number α and an error function φ such that

$$\dim_H E_\varphi(\alpha) = l_\varphi < u_\varphi.$$

J. Xu preprint :

(1) For any α,

$$\limsup_{n\to\infty} \frac{\log q_n}{-\log \varphi(q_n)} \le \dim_H(E_\varphi(\alpha)) \le u_\varphi,$$

where q_n denotes the denominator of the n-th convergent of the continued fraction of α.

(2) For any irrational number α with $\omega(\alpha) = 1$,

$$\dim_H(E_\varphi(\alpha)) = u_\varphi.$$

V. Our results

Theorem (L-Rams 2012)

For any α with $\omega(\alpha) = w \in [1, \infty]$, we have

$$\min\left\{u_\varphi,\ \max\left\{l_\varphi, \frac{1+u_\varphi}{1+w}\right\}\right\} \le \dim_H(E_\varphi(\alpha)) \le u_\varphi.$$

Corollary : If $w \le 1/u_\varphi$, then

$$\dim_H(E_\varphi(\alpha)) = u_\varphi.$$

Example : Take $w = 2$, $u = 1/2$ and $l = 1/3$. Construct α such that for all n, $q_n^2 \le q_{n+1} \le 2q_n^2$. Define

$$\varphi(n) = \max\left\{n^{-1/l}, q_k^{-1/l}\right\} \quad \text{if} \quad q_{k-1}^{u/l} < n \le q_k^{u/l}.$$

By Corollary,

$$\lim_{n\to\infty} \frac{\log q_n}{-\log \varphi(q_n)} = l < u = \dim_H(E_\varphi(\alpha)).$$

Thus the lower bound of Xu is not optimal.

VI. Our results-continued

Theorem (L-Rams)

For any irrational α *and for any* $0 \le l < u \le 1$, *with* $u > 1/w$, *there exists a decreasing function* $\varphi : \mathbb{N} \to \mathbb{R}^+$, *with* $l_\varphi = l$ *and* $u_\varphi = u$, *such that*

$$\dim_H(E_\varphi(\alpha)) = \max\left\{l, \frac{1+u}{1+w}\right\} < u.$$

Theorem (L-Rams)

Suppose $0 \le l < u \le 1$. *There exists a decreasing function* $\varphi : \mathbb{N} \to \mathbb{R}^+$, *with* $l_\varphi = l$ *and* $u_\varphi = u$, *such that for any* α *which is not a Liouville number,*

$$\dim_H(E_\varphi(\alpha)) = u.$$

VII. Method

$\rightarrow$ Upper bound is OK. We treat the lower bound.
Let $B \geq 1$ and suppose there exists $\{n_i\}$ such that

$$\frac{\log q_{n_i+1}}{\log q_{n_i}} \to B.$$

Let $\{m_i\}$ be such that $q_{n_i} < m_i \leq q_{n_i+1}$. By passing to subsequences, we suppose the limit

$$N := \lim_{i\to\infty} \frac{\log m_i}{\log q_{n_i}}$$

exists. (Obviously, $1 \leq N \leq B$.)
Let $K > 1$. Denote

$$E_i := \left\{ y \in \mathbb{R} : ||n\alpha - y|| < \frac{1}{2} q_{n_i}^{-K} \ \text{ for some } n \in (m_{i-1}, m_i] \right\}.$$

Let

$$E := \bigcap_{i=1}^{\infty} E_i \qquad \text{and} \qquad F := \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_i.$$

VIII. Method-continued

Proposition (L-Rams)

If $\{n_i\}$ is increasing sufficiently fast then

$$\dim_H E = \dim_H F = \min\left(\frac{N}{K},\ \max\left(\frac{1}{K}, \frac{1}{1+B-N}\right)\right).$$

Choose a sequence m_i of natural numbers such that

$$\lim_{i\to\infty} \frac{\log m_i}{-\log \varphi(m_i)} = u_\varphi,$$

and choose n_i such that $q_{n_i} < m_i \leq q_{n_i+1}$.
Take $K = N/u_\varphi$. We have $E \subset E_\varphi$ and then

$$\dim_H E_\varphi \geq \min\left(u_\varphi, \max\left(\frac{u_\varphi}{N}, \frac{1}{1+B-N}\right)\right).$$

Optimize the above value for $B \in [1, w], N \in [1, B]$.