Hausdorff dimension of affine random covering sets in torus

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Joint work with E. Järvenpää, H. Koivusalo, B. Li and V. Suomala
Introduction

- Let \((l_n)\) be a sequence of positive numbers.
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Let \((\xi_n)\) be a sequence of independent random variables uniformly distributed on the circle \(S^1\).

Remark

Almost surely

\[
\mathbb{L}(E) = 1 \quad \text{provided that} \quad \sum_{n=1}^{\infty} l_n = \infty
\]

\[
\mathbb{L}(E) = 0 \quad \text{provided that} \quad \sum_{n=1}^{\infty} l_n < \infty
\]
• Let \((l_n)\) be a sequence of positive numbers.
• Let \((\xi_n)\) be a sequence of independent random variables uniformly distributed on the circle \(S^1\).
• Define the random covering set by

\[
E = \limsup_{n \to \infty} [\xi_n, \xi_n + l_n].
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Introduction: the case of full measure

Dvoretzky covering problem (1956)

What conditions on \((l_n)\) guarantee that \(E = S^1\) almost surely?

Theorem (Shepp 1972)

\[ E = S^1 \text{ almost surely if and only if } \sum_{n=1}^{\infty} \frac{1}{n^2} \exp(l_1 + \cdots + l_n) = \infty. \]


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Question

What is the dimension of $E$?

• Fan and Wu (2004): almost surely $\dim H_E = 1/\alpha$ in the case $l_n = a/n^{\alpha}$ for some $a > 0$ and $\alpha > 1$.

• Durand (2010): $\dim H_E = \inf \{0 < s < 1 | \sum_{n=1}^{\infty} l_{sn} < \infty\}$ almost surely.

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E = \limsup_{n \to \infty} (g_n + \xi_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (g_k + \xi_k).
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Almost surely (1) \(L(E) = 1\) provided that \(\sum_{n=1}^{\infty} L(g_n) = \infty\).

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1. \(\mathcal{L}(E) = 1\) provided that \(\sum_{n=1}^{\infty} \mathcal{L}(g_n) = \infty\)
2. \(\mathcal{L}(E) = 0\) provided that \(\sum_{n=1}^{\infty} \mathcal{L}(g_n) < \infty\).
For a ball $B = B(x, r) \subset \mathbb{R}^d$ and $0 < s < d$ write $B^s = B(x, r^s_d)$. 

Mass transference principle (Beresnevich and Velani)

Let $(B_n) \subset \mathbb{R}^d$ be a sequence of balls whose radii converge to zero. Suppose that for any ball $B \subset \mathbb{R}^d$\[H^d(B \cap \limsup_{n \to \infty} B_n) = H^d(B)\]Then for any ball $B \subset \mathbb{R}^d$\[H^s(B \cap \limsup_{n \to \infty} B_n) = \infty\]Maarit Järvenpää

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Higher dimensional case: uniformly ball like sets

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- Assume that \(B(x_n, r_n) \subset g_n\) and there is \(C < \infty\) such that \(\rho_n \leq Cr_n\) for all \(n\).
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**Proposition**

Almost surely

\[
\dim_H E = \min\{s_0, d\},
\]

where \(s_0 = \inf\{s \geq 0 \mid \sum_{n=1}^{\infty} \rho_n^s < \infty\}\).
Proof: the upper bound

For \( s > s_0 \) we obtain

\[
\mathcal{H}^s(E) \leq \lim \inf_{N \to \infty} \sum_{n=N}^{\infty} \rho_n^s = 0,
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Proof: the upper bound

For \( s > s_0 \) we obtain

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\mathcal{H}^s(E) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \rho_n^s = 0,
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giving \( \dim_H E \leq \min\{s_0, d\} \).
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Obviously, $E \supset \limsup_{n \to \infty} B_n$ where $B_n = B(x_n, r_n)$. 
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Consider $s < \min\{s_0, d\}$. 
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Obviously, \( E \supset \limsup_{n \to \infty} B_n \) where \( B_n = B(x_n, r_n) \).
Consider \( s < \min\{s_0, d\} \). Then

\[
\sum_{n=1}^{\infty} \mathcal{L}(B_n^s) = K \sum_{n=1}^{\infty} r_n^s \geq KC^{-1} \sum_{n=1}^{\infty} \rho_n^s = \infty.
\]

Hence \( \mathcal{L}(\limsup_{n \to \infty} B_n^s) = 1 \), implying \( \mathcal{L}(\limsup_{n \to \infty} B_n^s \cap B) = \mathcal{L}(B) \) for any ball \( B \subset T_d \).

The mass transference principle gives

\( \mathcal{H}s(\limsup_{n \to \infty} B_n^s) = \infty \),

which leads to \( \text{dim}_{\mathcal{H}} E \geq \min\{s_0, d\} \).
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Hence $\mathcal{L}(\limsup_{n \to \infty} B_n^s) = 1$, implying

$$\mathcal{L}(\limsup_{n \to \infty} B_n^s \cap B) = \mathcal{L}(B)$$

for any ball $B \subset \mathbb{T}^d$. 

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Consider \( s < \min\{s_0, d\} \). Then

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\sum_{n=1}^{\infty} L(B_n^s) = K \sum_{n=1}^{\infty} r_n^s \geq KC^{-1} \sum_{n=1}^{\infty} \rho_n^s = \infty.
\]

Hence \( L(\limsup_{n \to \infty} B_n^s) = 1 \), implying

\[
L(\limsup_{n \to \infty} B_n^s \cap B) = L(B)
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for any ball \( B \subset \mathbb{T}^d \).

The mass transference principle gives \( H^s(\limsup_{n \to \infty} B_n) = \infty \),
Proof: the lower bound

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The mass transference principle gives $\mathcal{H}^s(\limsup_{n \to \infty} B_n) = \infty$, which leads to $\dim_H E \geq \min\{s_0, d\}$. 

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Higher dimensional case: main theorem

- Given a contractive linear injection \( L : \mathbb{R}^d \to \mathbb{R}^d \), let \( 0 < \alpha_d(L) \leq \cdots \leq \alpha_1(L) < 1 \) be the singular values of \( L \).
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• Given a contractive linear injection $L : \mathbb{R}^d \to \mathbb{R}^d$, let $0 < \alpha_d(L) \leq \cdots \leq \alpha_1(L) < 1$ be the singular values of $L$.

• For $0 < s \leq d$, define the **singular value function** by

$$ \Phi^s(L) = \alpha_1(L) \cdots \alpha_{m-1}(L) \alpha_m(L)^{s-m+1}, $$

where $m$ is the integer such that $m - 1 < s \leq m$. 
Higher dimensional case: main theorem

- Assume that $g_n = \Pi(L_n(R))$ where $R \subset [0, 1]^d$ has non-empty interior and and $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ is the natural covering map.
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• Assume that $\alpha_i(L_n) \downarrow 0$ as $n \rightarrow \infty$. 
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- Assume that \( \alpha_i(L_n) \downarrow 0 \) as \( n \to \infty \).
- Define

\[
    s_0 = \inf\{0 < s \leq d \mid \sum_{n=1}^{\infty} \Phi^s(L_n) < \infty\},
\]

with the interpretation \( s_0 = d \) if \( \sum_{n=1}^{\infty} \Phi^d(L_n) = \infty \).

Theorem

Almost surely \( \dim H_E = s_0 \).
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**Theorem**

Almost surely $\dim_H E = s_0$. 
Outline of the proof: the upper bound

- Enough to consider the case where $g_n$ is a rectangular parallelepiped.
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- The verification of the upper bound: Falconer.
Outline of the proof: the lower bound

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- Consider $m - 1 < s < s_0(g_n) \leq m$ where $m$ is an integer.
- Construct an event $\Omega(\infty) \subset \Omega$, having positive probability, and a random Cantor set $C^\omega$ such that $C^\omega \subset E^\omega$ for all $\omega \in \Omega(\infty)$.
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- This gives \( P(\dim_H E^\omega \geq s) > 0 \).
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- Construct an event $\Omega(\infty) \subset \Omega$, having positive probability, and a random Cantor set $C^\omega$ such that $C^\omega \subset E^\omega$ for all $\omega \in \Omega(\infty)$.
- Using potential theoretical methods, verify that $\dim_H C^\omega \geq s$ almost surely conditioned on $\Omega(\infty)$.
- This gives $P(\dim_H E^\omega \geq s) > 0$.
- The Kolmogorov zero-one law implies that $P(\dim_H E \geq s) = 1$. 

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