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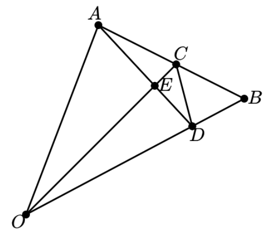
Exercise (Vectors)

Last updated: 17/02/2025

1. In the figure below, OAB is a triangle. C and D are points on AB and OB respectively such that $AC : CB = 8 : 7$ and $OD : DB = 16 : 5$. OC and AD intersect at a point E. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



   (a) Express $\overrightarrow{OC}$ and $\overrightarrow{AD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (b) Let $\overrightarrow{OE} = r\overrightarrow{OC}$ and $\overrightarrow{AE} = k\overrightarrow{AD}$.

      i. Express $\overrightarrow{OE}$ in terms of r, $\mathbf{a}$ and $\mathbf{b}$.

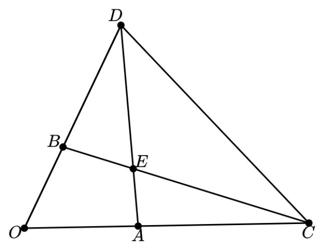
      ii. Express $\overrightarrow{OE}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.
      Hence show that $r = \dfrac{6}{7}$ and $k = \dfrac{3}{5}$.

   (c) It is given that $EC : ED = 1 : 2$.

      i. Using (b), or otherwise, find $EA : EO$.

      ii. Explain why $OACD$ is a cyclic quadrilateral.

2. The figure below shows a triangle OCD. A and B are points on OC and OD respectively such that $OA : AC = OB : BD = 1 : h$, where $h > 0$. AD and BC intersect at E such that $AE : ED = \mu : (1 - \mu)$ and $BE : EC = \lambda : (1 - \lambda)$, where $0 < \mu < 1$ and $0 < \lambda < 1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



   (a) By considering $\overrightarrow{OE}$, show that $\mu = \lambda$.

   (b) F is a point on CD such that O, E and F are collinear. Show that OF is a median of ΔOCD.

   (c) Using the above results, show that in a triangle, the centroid divides every median in $2 : 1$.

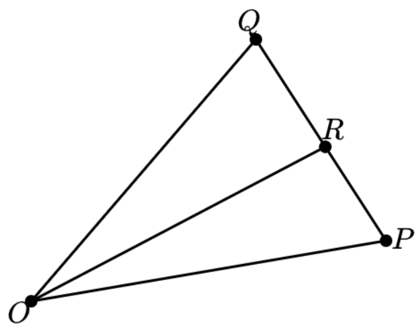
3. Given $\overrightarrow{OA} = 5\mathbf{i} - \mathbf{j}$, $\overrightarrow{OB} = -3\mathbf{i} + 5\mathbf{j}$ and APB is a straight line.

   (a) Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

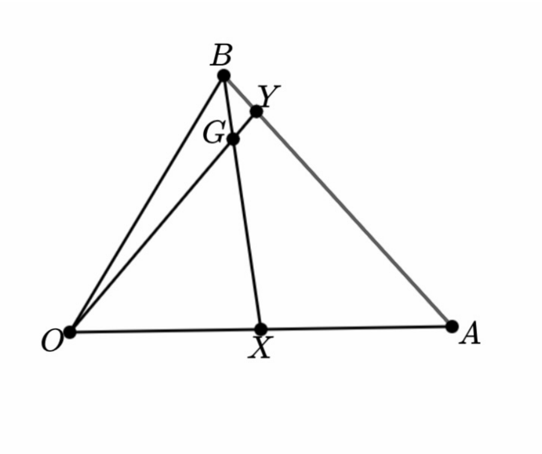
   (b) If $|\overrightarrow{AP}| = 4$, find $\overrightarrow{AP}$.

4. (a) In the figure below, OPQ is a triangle. R is a point on PQ such that $PR : RQ = r : s$. Express $\overrightarrow{OR}$ in terms of $r, s, \overrightarrow{OP}$ and $\overrightarrow{OQ}$.

   Hence show that if $\overrightarrow{OR} = m\overrightarrow{OP} + n\overrightarrow{OQ}$, then $m + n = 1$.



   (b) In the figure below, OAB is a triangle. X is the mid-point of OA and Y is a point on AB. BX and OY intersect at point G where $BG : GX = 1 : 3$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



   i. Express $\overrightarrow{OG}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

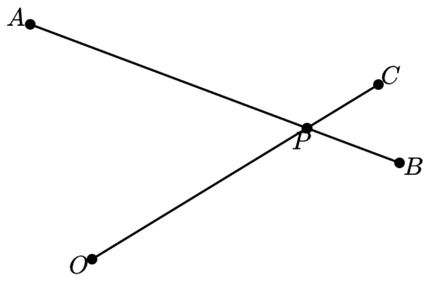
   ii. Using (a), express $\overrightarrow{OY}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
   (Hint: Put $\overrightarrow{OY} = k\overrightarrow{OG}$)

   iii. Moreover, AG is produced to a point Z on OB. Let $\overrightarrow{OZ} = h\overrightarrow{OB}$.

      A. Find the value of h.

      B. Explain whether ZY is parallel to OA or not.

5. In the figure below, the point P divides both line segments AB and OC in the same ratio $3:1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Hence show that OA is parallel to BC.

$\boxed{\textbf{Solutions}}$

1. (a) $\overrightarrow{OC} = \frac{7}{15}\mathbf{a} + \frac{8}{15}\mathbf{b}$, $\overrightarrow{AD} = -\mathbf{a} + \frac{16}{21}\mathbf{b}$.

   (b) i. $\overrightarrow{OE} = \frac{7r}{15}\mathbf{a} + \frac{8r}{15}\mathbf{b}$

   ii. $\overrightarrow{OE} = \mathbf{a} + k(-\mathbf{a} + \frac{16}{21}\mathbf{b}) = (1-k)\mathbf{a} + \frac{16k}{21}\mathbf{b}$

   $$\begin{cases} \frac{7r}{15} = 1-k \\ \frac{8r}{15} = \frac{16k}{21} \end{cases}$$

   By solving the equations, $r = \frac{6}{7}$ and $k = \frac{3}{5}$.

   (c) i. $2:1 = ED:EC = \frac{2}{5}AD : \frac{1}{7}OC$, so $EA:EO = \frac{3}{5}AD : \frac{6}{7}OC = \frac{1}{4}(ED:EC) = 1:2$.

   ii. $EC:ED = EA:EO$, so $EC \cdot EO = EA \cdot ED$, which implies $OACD$ is a cyclic quadrilateral.

2. (a) $\overrightarrow{OE} = \lambda(1+h)\mathbf{a} + (1-\lambda)\mathbf{b} = (1-\mu)\mathbf{a} + \mu(1+h)\mathbf{b}$, hence $\lambda = \mu = \frac{1}{h+2}$.

   (b) Suppose $\overrightarrow{OF} = t\overrightarrow{OE}$, then $t\lambda(h+1) + t(1-\lambda) = 1+h$, $t = \frac{h+2}{2}$, $\overrightarrow{OG} = \frac{1+h}{2}(\mathbf{a}+\mathbf{b}) = \frac{\overrightarrow{OC} + \overrightarrow{OD}}{2}$, so F is the middle point of CD.

   (c) Let $h = 1$, $\lambda = \frac{1}{h+2} = 1/3$, then $AE:ED = BE:EC = 1:2$.

3. (a) $\overrightarrow{AB} = -8\mathbf{i} + 6\mathbf{j}$, $\left|\overrightarrow{AB}\right| = 10$.

   (b) $\overrightarrow{AP} = \pm\frac{2}{5}\overrightarrow{AB} = \pm(-\frac{16}{5}\mathbf{i} + \frac{12}{5}\mathbf{j})$

4. (a) $\overrightarrow{OR} = \frac{s}{r+s}\overrightarrow{OP} + \frac{r}{r+s}\overrightarrow{OQ}$, where $\frac{s}{r+s} + \frac{r}{r+s} = 1$.

   (b) i. $\overrightarrow{OG} = \frac{1}{4}\overrightarrow{OX} + \frac{3}{4}\overrightarrow{OB} = \frac{1}{8}\mathbf{a} + \frac{3}{4}\mathbf{b}$

   ii. $\overrightarrow{OY} = k\overrightarrow{OG} = \frac{k}{8}\mathbf{a} + \frac{3k}{4}\mathbf{b}$, by (a), $\frac{k}{8} + \frac{3k}{4} = 1$, $k = \frac{8}{7}$, $\overrightarrow{OY} = \frac{1}{7}\mathbf{a} + \frac{6}{7}\mathbf{b}$.

   iii. A. $\overrightarrow{AG} = -\frac{7}{8}\mathbf{a} + \frac{3}{4}\mathbf{b}$, suppose $\overrightarrow{AZ} = t\overrightarrow{AG}$, then $-\frac{7t}{8}\mathbf{a} + \frac{3t}{4}\mathbf{b} + \mathbf{a} = h\mathbf{b}$, $t = \frac{8}{7}$, $h = \frac{6}{7}$.

   B. $ZY = \frac{1}{7}\mathbf{a}$, hence ZY is parallel to OA.

5. (a) $\overrightarrow{OP} = \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

   (b) $\overrightarrow{OC} = \frac{4}{3}\overrightarrow{OP} = \frac{1}{3}\mathbf{a} + \mathbf{b}$
   $\overrightarrow{BC} = \frac{1}{3}\mathbf{a}$, hence OA is parallel to BC.