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Exercise (Quadratic Polynomial)

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Part A: Basic Questions

1. Let $f(x) = ax^2 + bx - 1$, where a and b are constants. Suppose $f(x)$ is divisible by $x - 1$. Also, when divided by $x + 1$, $f(x)$ leaves a remainder of 4. Find the values of a and b.

2. Let $f(x) = 2x^2 + ax + b$.
   (a) Given that if $f(x)$ is divided by $x - 1$, the remainder is -5. Also, if $f(x)$ is divided by $x + 2$, the remainder is 4. Find the values of a and b.
   (b) Hence, find the value of x such that $f(x) = 0$.

3. Consider the function $f(x) = x^2 + bx - 15$, where b is a constant. It is given that the graph of $y = f(x)$ passes through the point $(4\,,\,9)$.
   (a) Find b. Hence, or otherwise, find the two x-intercepts of the graph of $y = f(x)$.
   (b) Let k be a constant. If the equation $f(x) = k$ has two distinct real roots, find the range of values of k.
   (c) Write down the equation of a straight line which intersects the graph of $y = f(x)$ at only one point.

4. Let $\mathrm{p}(x) = 4x^2 + 12x + c$, where c is a constant. The equation $\mathrm{p}(x) = 0$ has equal roots. Find
   (a) c,
   (b) the x-intercept(s) of the graph of $y = \mathrm{p}(x) - 169$.

5. Let $f(x) = x^2 + 2x - 2$ and $g(x) = -2x^2 - 12x - 23$.
   (a) Express $g(x)$ in the form $a(x + b)^2 + c$, where a, b and c are real constants. Hence show that $g(x) < 0$ for all real values of x.
   (b) Let k_1 and k_2 $(k_1 > k_2)$ be the two values of k such that the equation $f(x) + kg(x) = 0$ has equal roots. Find k_1 and k_2.

Part B: Advanced Questions

6. Given that the graph of the quadratic function $y = ax^2 + bx + c$ has a vertex on the y-axis, $c - b = 2$, and passes through the point $(2\,,\,8)$ (as shown in Figure 1), find the analytical expression of this quadratic function.

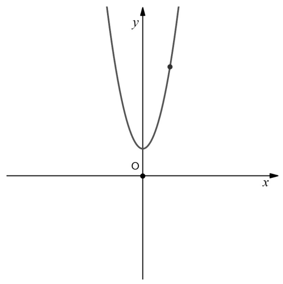

Figure 1: Question 6

7. The x-coordinate of the vertex of a parabola $y = ax^2 + bx + c$ is 1, and the parabola passes through points $A(1\,,\,5)$ and $B(3\,,\,1)$. Find the equation of this parabola.

8. Given that the graph of the quadratic function $y = ax^2+bx+c$ passes through the point $(-1\,,\,18)$, has a distance of 3 between its two x-intercepts, and satisfies that $b^2 - 4ac = 9$, with the vertex in the fourth quadrant, find the values of b and c.

9. Assuming that the quadratic equation of x, $2ax^2 - 2x - 3a - 2 = 0$ has one root greater than 1 and another root less than 1, find the range of a.

10. Assuming that the quadratic function $y = ax^2 + bx + c$ attains a maximum value of 3 at $x = 1$, and the length of the segment intercepted by its graph on the x-axis is 4, find the coefficients a, b, and c of the quadratic function.

11. Suppose a and b are two real numbers that satisfy

$$\sqrt{a^2 - 2a + 1} + \sqrt{36 - 12a + a^2} = 10 - |b + 3| - |b - 2|\,.$$

Find the maximum value of $a^2 + b^2$.

12. Given that the two real roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are α and β, find the relationship between m and k such that α and β lie between the two real roots of the equation $x^2 - 2mx + k = 0$.

$\boxed{\textbf{Solutions}}$

1. $a = 3$, $b = -2$.

2. (a) $a = 1$, $b = -6$.

   (b) $x = -\dfrac{3}{2}$ or $x = 2$.

3. (a) $b = 2$, and the two x-intercepts are -5 and 3.

   (b) $k > -16$.

   (c) $y = -16$.

4. (a) $c = 9$.

   (b) The x-intercepts of the graph are -8 and 5.

5. (a) $g(x) = -2(x+3)^2 - 5$.

   (b) $k_1 = 1$ and $k_2 = -\dfrac{3}{10}$.

6. Because $y = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$, and the vertex of the graph is on the y-axis, we can obtain that

$$-\frac{b}{2a} = 0\,,$$

   equivalently, $b = 0$. Furthermore, by $c - b = 2$ we obtain $c = 2$. Since the graph passes through $(2\,,\,8)$, then $8 = a \times 2^2 + 0 \times 2 + 2$, $a = \dfrac{3}{2}$. Therefore, the analytical expression is

$$y = \frac{3}{2}x^2 + 2.$$

7. $y = -x^2 + 2x + 4$.

8. $b = -7$, $c = 10$.

9. According to the problem statement, the graph of $y = f(x) = 2ax^2 - 2x - 3a - 2$ should have x-intercepts on both sides of $(1\,,\,0)$. We obtain from the graph that

$$\begin{cases} a < 0, \\ f(1) > 0 \end{cases} \quad \text{or} \quad \begin{cases} a > 0, \\ f(1) < 0. \end{cases}$$

   Equivalently

$$\begin{cases} a < 0, \\ 2a - 2 - 3a - 2 > 0 \end{cases} \quad \text{or} \quad \begin{cases} a > 0, \\ 2a - 2 - 3a - 2 < 0. \end{cases}$$

   Solving which yields $a > 0$ or $a < -4$.

10. Since the function reaches a maximum of 3 at $x = 1$, it can be expressed as

$$y = a(x-1)^2 + 3 \quad (a < 0).$$

   Also because its graph intersects with the x-axis, with segment length of 4, and its axis of symmetry being $x = 1$, it is bound to pass through the points $A(-1\,,\,0)$ and $B(3\,,\,0)$. Therefore, $0 = a(-1-1)^2 + 3$, $a = -\dfrac{3}{4}$.

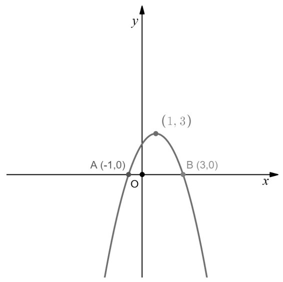

Figure 2: Answer of Question 10

Furthermore,

$$y = -\frac{3}{4}(x-1)^2 + 3 = -\frac{3}{4}x^2 + \frac{3}{2}x + \frac{9}{4}.$$

Thus $a = -\frac{3}{4}$, $b = \frac{2}{3}$, $c = \frac{9}{4}$

11. $|a-1| + |a-6| + |b+3| + |b-2| = 10$.

    Note that we have $|b-2| + |b+3| \geq 5$ and $|a-1| + |a-6| \geq 5$ and their sum is less than 10.

    Therefore, a can only take value from 1 to 6 while b can only take value from -3 to 2.

    Then, it is easy to see that the maximum of $a^2 + b^2$ is $6^2 + (-3)^2 = 45$.

12. $k < m^2 - 1$