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Exercise (Indices and Logarithms)

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Part A: Basic Questions

1. Which of the following is rational?
   A. $\sqrt{12^3}$
   B. $\sqrt{4} \times \sqrt{3}$
   C. $\sqrt{8} \div \sqrt{2}$
   D. $\sqrt{8} + \sqrt{8}$
   E. $\sqrt{3} - \sqrt{2}$

2. Express $\sqrt{25a} - \sqrt{4a}$ in terms of a.

3. (a) Simplify $\dfrac{a^{\frac{5}{4}} \sqrt[4]{a^3}}{a^{-2}}$.
   (b) Simplify $\dfrac{x^3y^2}{x^{-3}y}$ and express your answer with positive indices.
   (c) Simplify $\dfrac{a^3a^2}{b^{-2}}$ and express your answer with positive indices.

4. (a) Evaluate $\log_2 8 + \log_2 \dfrac{1}{16}$.
   (b) Simplify $\dfrac{\log a^3b^2 - \log ab^2}{\log \sqrt{a}}$.

5. Let a and b be constants. Denote the graph of $y = a + \log_b x$ by G. The x-intercept of G is 9 and G passes through the point $(243, 3)$. Express x in terms of y.

6. Solve $4^x = 10 - 4^{x+1}$.

7. Solve the following equations without using a calculator:
   (a) $3^x = \dfrac{1}{\sqrt{27}}$
   (b) $\log x + 2\log 4 = \log 48$

Part B: Advanced Questions

8. Let $x > y > 0$. If $\log(x+y) = a$ and $\log(x-y) = b$, then express $\log\sqrt{x^2-y^2}$ in terms of a and b.

9. If a and b are positive integers, then express $\log_{a^b} b^a$ in terms of a and b.

10. Let $b > 1$. If $a = \log_{12} b$, then express $\dfrac{1}{a}$ in terms of b.

11. If $\dfrac{3}{3\log x - 2} + 7 = \dfrac{2}{2\log x + 1}$, then $\log\dfrac{1}{x} =$?

12. If the roots of the equation $(\log_{\pi} x)^2 - 10\log_{\pi} x + 24 = \log_{\pi} x$ are α and β, then $\alpha\beta =$?

Solutions

1. Ans: C.
   A. $\sqrt{12^3} = \sqrt{2^6 3^3} = 24\sqrt{3}$ is irrational
   B. $\sqrt{4} \times \sqrt{3} = 2\sqrt{3}$ is irrational
   C. $\sqrt{8} \div \sqrt{2} = 2\sqrt{2} \div \sqrt{2} = 2$ is rational
   D. $2\sqrt{8} = 4\sqrt{2}$ is irrational
   E. Suppose $\sqrt{3} - \sqrt{2}$ was rational, then its square $5 - 2\sqrt{6}$ was rational, and thus $\sqrt{6}$ was rational, contradiction arises. Hence, $\sqrt{3} - \sqrt{2}$ is irrational.

2. Ans $= 5\sqrt{a} - 2\sqrt{a} = 3\sqrt{a}$

3. (a) a^4
   (b) $x^6 y$
   (c) $a^7 b^2$

4. (a) $3 + (-4) = -1$
   (b) $\dfrac{\log \dfrac{a^3 b^2}{ab^2}}{\log \sqrt{a}} = \dfrac{\log a^2}{\log \sqrt{a}} = \dfrac{4 \log \sqrt{a}}{\log \sqrt{a}} = 4$

5. We have
$$0 = a + \log_b 9 = a + 2\log_b 3$$
$$3 = a + \log_b 243 = a + 5\log_b 3$$
   Hence, $a = -2, \log_b 3 = 1$.
$$y + 2 = y - a = \log_b x = (\log_b 3)(\log_3 x) = \log_3 x$$
   Therefore, $x = 3^{y+2}$.

6. We have
$$4^x + 4^{x+1} = 10$$
$$5 \cdot 4^x = 10$$
$$4^x = 2$$
$$x = \frac{1}{2}$$

7. (a) We have
$$3^x = \frac{1}{\sqrt{27}} = \frac{1}{\sqrt{3^3}} = \frac{1}{3^{3/2}} = 3^{-3/2}$$
$$x = -\frac{3}{2}$$
   (b) We have
$$\log x = \log 48 - 2\log 4 = \log 48 - \log 4^2 = \log \frac{48}{4^2} = \log 3$$
   Hence $x = 3$.

8. Ans $= \log \sqrt{x^2 - y^2} = \dfrac{1}{2}\log(x^2 - y^2) = \dfrac{1}{2}[\log(x + y) + \log(x - y)] = \dfrac{a + b}{2}$

9. Ans $= b \log a + a \log b$

10. Ans $= \dfrac{1}{\log_{12} b} = \log_b 12$

11. Ans $= -\frac{1}{2}$ or $\frac{1}{3}$

Let $t = \log x$, then we have

$$\frac{3}{3t-2} + 7 = \frac{2}{2t+1}$$

$$t = \frac{1}{2} \text{ or } -\frac{1}{3}$$

$$\log \frac{1}{x} = -t = -\frac{1}{2} \text{ or } \frac{1}{3}$$

12. Ans $= \pi^{11}$

Let $\log_\pi x = t$, then we have

$$t^2 - 11t + 24 = 0$$

$$\log_\pi \alpha\beta = \log_\pi \alpha + \log_\pi \beta = t_1 + t_2 = 11$$

$$\alpha\beta = \pi^{11}$$