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Exercise (Functions)

Last updated: 17/02/2025

Part A: Basic Questions

1. Consider the function $f(x) = |x| - x^2$. For $x = 0, 1, -2, 3, -4, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}$, find the corresponding values of $f(x)$.

2. It is known that $y = ax^2 + b$ and when $x = 1$, $y = 10$; when $x = \frac{1}{2}$, $y = -8$. Find the value of a and b.

3. In which of the following questions is y a function of x? Write the expression of the function $y = f(x)$.

   (a) x is the length of one side of the square and y is the area of this square.

   (b) x is the length of one side of the rectangle, the length of the other side of the rectangle is a constant value a, and y is the area of this rectangle.

   (c) One side of the triangle is x, the height on this side is y, and the area of the triangle is a constant S.

4. A car leaves the station and after 45 minutes it arrives at A, 28 kilometers from the station, and thereafter the car has a constant speed of 40 kilometers per hour, find the relationship between the distance s kilometers from the station after this car has a constant speed and the time t hours from the station, and find the domain of t.

5. Knowing that $f(x + 1) = x^2 - 3x + 2$, find $f(x)$.

6. Given that $a < b < c$, find the minimum of the function

$$y = |x - a| + |x - b| + |x - c|$$

7. Find the domain of definition of the following functions

   (a) $y = \sqrt{4 - x^2} + \dfrac{1}{x}$.

   (b) $y = \sqrt{|x| - 3} + \dfrac{1}{x^2 - 4x + 5}$.

8. Find the domain of the following functions.

   (a) $y = \dfrac{2}{x + |x|}$.

   (b) $y = \sqrt{x^2 - 3x + 2} + \dfrac{1}{x^2 + 2x - 8}$.

Part B: Advanced Questions

9. Given the function $f(x)$ satisfying
$$f(x+\frac{1}{x}) = x^2 + \frac{1}{x^2},$$
find the value of $f(3)$.

10. Suppose $f(x) = ax^2 + bx + c$, prove that $f(x+3) - 3f(x+2) + 3f(x+1) - f(x) = 0$.

11. Knowing that $f(x)$ satisfying $f(\frac{1}{x}) = x + \sqrt{1+x^2}$, find $f(x)$.

12. Suppose $f(1) = 0$, $f(2) = 1$, and
$$a^2 f(n+2) = b^2 f(n)$$
($a > 0$, $b > 0$, n is a positive integer), find $f(n)$.

13. Let the perimeter of the rectangle be a constant l and the length of one side be x. Find the maximum value of the area y of the rectangle.

14. Suppose $S(t) = \frac{2t}{1+t^2}$, $C(t) = \frac{1-t^2}{1+t^2}$, prove that:
    (a) $S(t) = S(\frac{1}{t})$.
    (b) $S^2(t) + C^2(t) = 1$.

15. Knowing that $f(\frac{2x+1}{x}) = x^2 - 3x + 7$, find $f(x)$.

16. Suppose $y = ax + \frac{1}{a}(2-x)$, where $a > 0$. Find the minimum of y when $0 \le x \le 1$.

$\boxed{\textbf{Solutions}}$

1. When $x = 0$, $f(x) = 0$;

   When $x = 1$, $f(x) = 0$;

   When $x = -2$, $f(x) = -2$;

   When $x = 3$, $f(x) = -6$;

   When $x = -4$, $f(x) = -12$;

   When $x = -\frac{1}{2}$, $f(x) = \frac{1}{4}$;

   When $x = \frac{1}{2}$, $f(x) = \frac{1}{4}$;

   When $x = -\frac{1}{3}$, $f(x) = \frac{2}{9}$.

2. From the question $\begin{cases} a + b = 10, \\ \frac{a}{4} + b = -8. \end{cases}$ Hence, we have $a = 24$, $b = -14$.

3. (a) $y = x^2$.

   (b) $y = xa$.

   (c) $y = \frac{2S}{x}$.

4. $s = 28 + 40(t - \frac{3}{4})$, where the domain is $t \geq \frac{3}{4}$.

5. Let $a = x + 1$.

   Then, $f(a) = (a-1)^2 - 3(a-1) + 2 = a^2 - 5a + 6$.

   Substituting a with x gives $f(x) = x^2 - 5x + 6$.

6. We will consider four cases.

   (a) $x \leq a$, then $y = a - x + b - x + c - x = a + b + c - 3x \geq b + c - 2a$.

   (b) $a \leq x \geq b$, then $y = x - a + b - x + c - x = b + c - a - x \geq c - a$.

   (c) $b \leq x \geq c$, then $y = x - a + x - b + c - x = c - a - b + x \geq c - a$.

   (d) $c \leq x$, then $y = x - a + x - b + x - c = 3x - a - b - c \geq 2c - a - b$.

   Among them, the smallest is $c - a$.

7. (a) From the question $\begin{cases} 4 - x^2 \geq 0, \\ x \neq 0, \end{cases}$, which means $\begin{cases} -2 \leq x \leq 2, \\ x \neq 0. \end{cases}$

   Hence, the domain is $-2 \leq x \leq 2$ and $x \neq 0$.

   (b) From the question $\begin{cases} |x| - 3 \leq 0, \\ x^2 - 4x + 5 \neq 0, \end{cases}$, which means $\begin{cases} x \geq 3 \text{ or } x \leq 3, \\ x \neq 5 \text{ or } x \neq -1. \end{cases}$

   Hence, the domain is $(-\infty, -3]$, $[3, 5)$, and $(5, +\infty)$.

8. (a) From the question, $x + |x| \neq 0$. Hence, $x > 0$ is the domain.

   (b) From the question, $\begin{cases} x^2 - 3x + 2 \geq 0, \\ x^2 + 2x - 8 \neq 0, \end{cases}$. Hence, the domain is $(-\inf, -4)$, $(-4, 1]$, and $(2, +\inf)$.

9. Note that $(x + \frac{1}{x})^2 = x^2 + \frac{1}{x^2} + 2$, we can conclude that $f(y) = y^2 - 2$. Hence, $f(3) = 7$.

10. By direct computation, $f(x+3)-3f(x+2)+3f(x+1)-f(x)$
$=[a(x+3)^2+b(x+3)+c]-3[a(x+2)^2+b(x+2)+c]+3[a(x+1)^2+b(x+1)+c]-[ax^2+bx+c]$
$=a[(x+3)^2-3(x+2)^2+3(x+1)^2-x^2]+b[(x+3)-3(x+2)+3(x+1)-x]+c(1-3+3-1)$
$=0.$

11. Take $a=\frac{1}{x}$.

    Then, $f(a)=\frac{1}{a}+\sqrt{a+\frac{1}{a^2}}$.

    Substituting a with x gives $f(x)=\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}$.

12. We will consider two cases.

    (i) When n is an odd number, let $n=2k+1$.
    Then $f(n)=f(2k+1)=(\frac{b^2}{a^2})f(2(k-1)+1)=(\frac{b^2}{a^2})^2f(2(k-2)+1)=\cdots=(\frac{b^2}{a^2})^kf(1)=0$

    (ii) When n is an even number, let $n=2k$.
    Then $f(n)=f(2k)=(\frac{b^2}{a^2})f(2(k-1))=(\frac{b^2}{a^2})^2f(2(k-2))=\cdots=(\frac{b^2}{a^2})^{k-1}f(2)=(\frac{b^2}{a^2})^{k-1}$

13. From the question, $y=(\frac{l}{2}-x)x$.

    The minimum value is given by Cauchy inequality, namely $y\leq(\frac{\frac{l}{2}-x+x}{2})^2=\frac{l^2}{16}$.

14. (a) $S(\frac{1}{t})=\frac{\frac{2}{t}}{1+\frac{1}{t^2}}=\frac{2t}{1+t^2}=S(t)$.

    (b) $S^2(t)+C^2(t)=\frac{(2t)^2+(1-t^2)^2}{(1+t^2)^2}=\frac{4t^4+4t^2+1}{4t^4+4t^2+1}=1$.

15. Let $y=\frac{2x+1}{x}$, then $x=\frac{1}{y-2}$.

    $f(y)=x^2-3x+7=(\frac{1}{y-2})^2-\frac{3}{y-2}+7$.

    Substituting y with x gives $f(x)=\frac{1}{(x-2)^2}-\frac{3}{x-2}+7$.

16. Note that $y=(a-\frac{1}{a})x+\frac{2}{a}$, we will need to consider whether the coefficient of x is positive or not.

    (i) If $a-\frac{1}{a}\geq 0$, which means $a\geq 1$, y will reach the minimum when $x=0$. The minimum is $\frac{2}{a}$.

    (ii) If $a-\frac{1}{a}<0$, which means $0<a<1$, y will reach the minimum when $x=1$. The minimum is $\frac{1}{a}+a$.