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Exercise (Equation of Straight Lines)

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Part A: Basic Questions

1. If the lines $y = mx + b$ and $\dfrac{x}{a} + \dfrac{y}{b} = 1$ are perpendicular, find m in terms of a and b.

2. If the straight line $2x + y + k = 0$ passes through the point of intersection of the two straight lines $x + y - 3 = 0$ and $x - y + 1 = 0$, find the value of k.

3. The equation of the straight line L is $kx + 4y - 2k = 0$, where k is a constant. If L is perpendicular to the straight line $6x - 9y + 4 = 0$. Find the y-intercept of L.

4. The equation of the straight line L_1 is $4x + 3y - 36 = 0$. The straight line L_2 is perpendicular to L_1 and intersects L_1 at a point lying on the y-axis. Find the area of the region bounded by L_1, L_2 and the x-axis.

5. O is the origin. A and B are the points $(-2, 0)$ and $(4, 0)$ respectively. l is a straight line through A with slope 1. C is a point on l such that $CO = CB$.

   (a) Find the equation of l.

   (b) Find the coordinates of C.

   (c) Find the equation of the circle passing through O, B and C.

   (d) If the circle OBC cuts l again at D, find the coordinates of D.

6. If the straight lines $hx + ky + 15 = 0$ and $4x + 3y - 5 = 0$ are perpendicular to each other and intersect at a point on the x-axis, then find k.

Part B: Advanced Questions

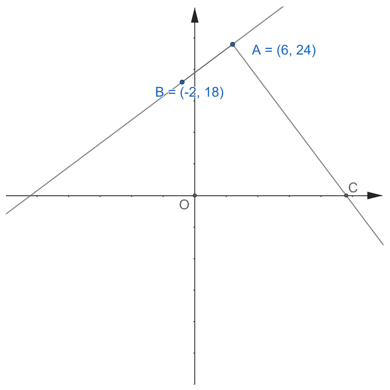
7. In the figure, the straight line passing through A and B is perpendicular to the straight line passing through A and C, where C is a point lying on the x-axis.

   (a) Find the equation of the straight line passing through A and B.

   (b) Find the coordinates of C.

   (c) Find the area of $\triangle ABC$.

   (d) A straight line passing through A cuts the line segment BC at D such that the area of $\triangle ABD$ is 90 square units. Let $BD : DC = r : 1$. Find the value of r.



8. The lines $3x - y - 8 = 0$ and $x - y - 2 = 0$ meet at a point P. L_1 and L_2 are lines passing through P and having slopes $\frac{1}{2}$ and 2 respectively. Find their equations.

9. Let J be the circle $x^2 + y^2 = r^2$, where $r > 0$.

   (a) Suppose that the straight line $L : y = mx + c$ is a tangent to J.

      i. Show that $c^2 = r^2(m^2 + 1)$.

      ii. If L passes through a point (h, k), show that $(k - mh)^2 = r^2(m^2 + 1)$.

   (b) J is inscribed in a triangle PQR. The coordinates of P and R are $(7, 4)$ and $(-5, 5)$ respectively.

      i. Find the radius of J.

      ii. Using (a)(ii), or otherwise, find the slope of PQ.

      iii. Find the coordinates of Q.

10. Two straight lines $L_1 : x - 2y + 3 = 0$ and $L_2 : 2x - y - 1 = 0$. Find the equation of the straight line passing through P and with equal positive intercepts, find the equation of L.

11. A and B are the points $(1, 2)$ and $(7, 4)$ respectively. P is a point on the line segment AB such that $\frac{AP}{PB} = k$.

   (a) Write down the coordinates of P in terms of k.

   (b) Hence find the ratio in which the line $7x - 3y - 28 = 0$ divides the line segment AB.

12. The coordinates of the points A and B are $(5, 7)$ and $(13, 1)$ respectively. Let P be a moving point in the rectangular coordinate plane such that P is equidistant from A and B. Denote the locus of P by Γ.

   (a) Find the equation of Γ.

   (b) Γ intersects the x-axis and the y-axis at H and K respectively. Denote the origin by O. Let C be the circle which passes through O, H and K. Someone claims that the circumference of C exceeds 30. Is the claim correct? Explain your answer.

Solutions

1. $\frac{x}{a} + \frac{y}{b} = 1$ can be changed into $y = -\frac{b}{a}x + b$.

   Since two lines are perpendicular, we have $m \cdot (-\frac{b}{a}) = -1$.

   Hence, $m = \frac{a}{b}$.

2. The intersection of the two straight lines is $(1, 2)$.

   Hence, $k = -4$.

3. Since two lines are perpendicular, we have $6k - 36 = 0$.

   Hence, $k = 6$.

   Then, the y-intercept of L is 3.

4. The intersection on y-axis must be $(0, 12)$.

   Note that L_2 is perpendicular to L_1, the equation of L_2 is given by $3x - 4y + 48 = 0$.

   The region bounded by L_1, L_2 and x-axis is a triangle with base $16 + 9 = 25$ and height 12.

   The desired area is 96.

5. (a) The equation of l is $y = x + 2$.

   (b) $C(2, 4)$.

   (c) The x-coordinate of the center of the circle is $\frac{0+4}{2} = 2$.
   Suppose the function is $(x-2)^2 + (y-b)^2 = r^2$ for some b and r.
   Take $O(0, 0)$ and $C(2, 4)$ and we will get $b = \frac{3}{2}$ and $r = \frac{5}{2}$.
   Therefore, the equation of the circle is $(x-2)^2 + (y - \frac{3}{2})^2 = \frac{25}{4}$.

   (d) Take $y = x + 2$ into the equation of the circle.
   We get $x = 2$ or $-\frac{1}{2}$.
   Hence, the coordinates of D is $(-\frac{1}{2}, \frac{3}{2})$.

6. $(\frac{5}{4}, 0)$ is a point on the line $4x + 3y - 5 = 0$ and also on x-axis.

   Hence, it is a point on $hx + ky + 15 = 0$. Therefore, $h = -12$.

   Since $hx + ky + 15 = 0$ and $4x + 3y - 5 = 0$ are perpendicular, we have $4h + 3k = 0$.

   Therefore, $k = 16$.

7. (a) The equation is $y = \frac{3}{4}x + \frac{39}{2}$.

   (b) The equation of AC is $y = -\frac{4}{3}x + 32$.
   Hence, the coordinates of C is $(24, 0)$.

   (c) Note that $|AB| = \sqrt{8^2 + 6^2} = 10$ and $|AC| = \sqrt{18^2 + 24^2} = 30$.
   Hence, the area is $\frac{10 \cdot 30}{2} = 150$.

   (d) $\frac{BD}{DC} = \frac{S_{\triangle ABD}}{S_{\triangle ADC}} = \frac{90}{150 - 90} = \frac{3}{2}$.
   Hence, $r = \frac{3}{2}$.

8. The coordinates of P is $(3,1)$.

   Hence, the equation of L_1 is given by $y=\frac{1}{2}(x-3)+1=\frac{1}{2}x-\frac{1}{2}$.

   The equation of L_2 is given by $y=2(x-3)+1=2x-5$.

9. (a) i. Combine the equation of L and J, we have $x^2+(mx+c)^2=r^2$.
   Hence, $(1+m^2)x^2+2mcx+c^2-r^2=0$.
   $\Delta=4m^2c^2-4(1+m^2)(c^2-r^2)=0$
   Therefore, $c^2=r^2(m^2+1)$.

   ii. Put (h,k) into L, we have $k=mh+c$.
   Hence, $(k-mh)^2=c^2=r^2(m^2+1)$.

   (b) i. The equation of PR is given by $\frac{y-4}{x-7}=\frac{-5-4}{-5-7}=\frac{3}{4}$, which is $3x-4y-5=0$.
   Therefore, x-intercept$=\frac{5}{3}$ and y-intercept$=\frac{-5}{4}$.
   Hence, we have

$$\frac{1}{2}r\sqrt{(\frac{5}{3})^2+(\frac{5}{4})^2}=\frac{1}{2}\cdot\frac{5}{3}\cdot\frac{5}{4}.$$

   Then, $r=1$.

   ii. Use (a)(ii) with $(h,k)=(7,4)$ and $r=1$.
   We have $(4-7m)^2=m^2+1$, which gives $m=\frac{3}{4}$ or $\frac{5}{12}$.
   Hence, $m_{PQ}=\frac{5}{12}$.

   iii. Use (a)(ii) with $(h,k)=R=(-5,5)$ and $r=1$.
   We have $(-5+5m)^2=m^2+1$, which gives $m=\frac{3}{4}$ or $\frac{4}{3}$.
   Hence, $m_{QR}=\frac{4}{3}$.
   Let $Q=(a,b)$.
   We have $\frac{b-4}{a-7}=\frac{5}{12}$ and $\frac{b+5}{a+5}=\frac{4}{3}$.
   Solve a and b, we have $Q=(\frac{-7}{11},\frac{9}{11})$.

10. We first compute the coordinates of P, which is $(\frac{5}{3},\frac{7}{3})$.

    Since L has equal positive intercepts, its slope is -1.

    Hence, the equation of L is $x+y-4=0$.

11. (a) The coordinates of P is $(\frac{7k+1}{k+1},\frac{4k+2}{k+1})$.

    (b) When P lies on $7x-3y-28=0$,

$$7(\frac{7k+1}{k+1})-3(\frac{4k+2}{k+1})-28=0$$

    We get $k=3$.

    Hence, the ratio is $3:1$.

12. (a) The equation of Γ is:

    $(x-5)^2+(y-7)^2=(x-13)^2+(y-1)^2$

    $4x-3y-24=0$.

    (b) $H(6,0)$ and $K(0,-8)$

    Since $\angle HOK=90^\circ$, HK is a diameter of C.

    Diameter$=\sqrt{6^2+8^2}=10$

    Hence, the circumference of C is $10\pi=31.4>30$. The claim is correct.