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HSMMC Pre-workshop Exercise (Linear Algebra) Solutions

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1. Compute $-2u + 4v$ where $u = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$.

   Answer: $-2u + 4v = -2\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 16 \\ 20 \end{pmatrix} = \begin{pmatrix} 4 \\ 20 \\ 16 \end{pmatrix}$

2. (a) Calculate $\begin{pmatrix} 3 & 7 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix}$.

   (b) Calculate the inverse of $\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$.

   Answer:

   (a) $\begin{pmatrix} 3 & 7 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} (3)(-2) + (7)(4) & (3)(1) + 7(2) \\ (-1)(-2) + (4)(4) & (-1)(1) + (4)(2) \end{pmatrix} = \begin{pmatrix} 22 & 17 \\ 18 & 7 \end{pmatrix}$

   (b) $\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}^{-1} = \frac{1}{(5)(4) - (3)(6)}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix}$

3. Define $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

   (a) Calculate A^2, A^3, A^4.

   (b) Hence, write down A^n (where n is a positive integer).

   (c) Similarly, find $(A^{-1})^n$.

   Answer:

   (a)

$$A^2 = AA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ (note: it can be proved by mathematical induction)

(c) $\begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$ (note: it can be proved by mathematical induction)

4. Please try the computational exercises in the Google Colab notebooks.

   Answer: See the solution notebook.

5. (Bonus) Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $C\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for some non-zero vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and distinct scalars λ_1 and λ_2.

   (a) Prove that $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$ and $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$.

   (b) Hence, prove that λ_1 and λ_2 are the roots of the equation $\lambda^2 - tr(C)\cdot\lambda + \det(C) = 0$. (Here, $tr(C)$ = the sum of all diagonal entries of $C = p + s$, which is also known as the *trace* of a matrix.)

   Answer:

   (a) Since $C\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ for some non-zero vector $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, we have $(C - \lambda_1 I)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$, i.e.,

$$\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0.$$

   If $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} \neq 0$, then the matrix inverse $\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix}^{-1}$ exists. Multiplying both sides of the above equation by $\begin{pmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{pmatrix}^{-1}$ gives $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$, which contradicts the condition that the vector is a non-zero vector. Therefore, we have $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = 0$.

   Similarly, $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix} = 0$.

(b) From (a), we have

$$\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix} = (p-\lambda_1)(s-\lambda_1) - rq = 0,$$

from which we have

$$\lambda_1^2 - (p+s)\lambda_1 + ps - rq = 0.$$

Similarly,

$$\lambda_2^2 - (p+s)\lambda_2 + ps - rq = 0.$$

Also, note that $p + s = tr(C)$ and $ps - rq = \det(C)$. Therefore, λ_1, λ_2 are the two roots of the solution

$$\lambda^2 - tr(C) \cdot \lambda + \det(C) = 0.$$