Received: 3 June 2022 | Revised: 1 December 2022 | Accepted: 13 December 2022

DOI: 10.1111/sapm.12555

ORIGINAL ARTICLE

STUDIES IN APPLIED MATHEMATICS WILEY

Elastodynamical resonances and cloaking of negative material structures beyond quasistatic approximation

Hongjie Li[1] | **Hongyu Liu**[2] | **Jun Zou**[1]

[1]Department of Mathematics, The Chinese University of Hong Kong, Shatin, China

[2]Department of Mathematics, City University of Hong Kong, Kowloon, China

Correspondence
Hongyu Liu, Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong SAR, China.
Email: hongyliu@cityu.edu.hk

Funding information
France-Hong Kong ANR/RGC Joint Research Grant; NSFC/RGC Joint Research Fund; Hong Kong RGC General Research Funds; Direct Grant for Research, CUHK

Abstract
Given the flexibility of choosing negative elastic parameters, we construct material structures that can induce two resonance phenomena, referred to as the elastodynamical resonances. They mimic the emerging plasmon/polariton resonance and anomalous localized resonance in optics for subwavelength particles. However, we study the peculiar resonance phenomena for linear elasticity beyond the subwavelength regime. It is shown that the resonance behaviors possess distinct characters, with some similar to the subwavelength resonances, but some sharply different due to the frequency effect. It is particularly noted that we construct a core–shell material structure that can induce anomalous localized resonance as well as cloaking phenomena beyond the quasistatic limit. The study is boiled down to analyzing the so-called elastic Neumann–Poincaré (N-P) operator in the frequency regime. We provide an in-depth analysis of the spectral properties of the N-P operator on a circular domain beyond the quasistatic approximation, and these results are of independent interest to the spectral theory of layer potential operators.

KEYWORDS
anomalous localized resonance, beyond quasistatic limit, core–shell structure, invisibility cloaking, negative materials, Neumann–Poincaré operator, spectral

1 | INTRODUCTION

1.1 | Mathematical formulation and main findings

We initially focus on the mathematics, not on the physics, and present the Lamé system which governs the propagation of linear elastic deformation.

For $N = 2, 3$, we write $\mathbf{C}_{\lambda,\mu} := (\mathrm{C}_{ijkl})_{i,j,k,l=1}^N$ as a four-rank elastic material tensor defined by:

$$\mathrm{C}_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1}$$

where δ is the Kronecker delta, and (λ, μ) are referred to as the Lamé parameters. For a regular elastic material, the Lamé parameters satisfy the following strong convexity conditions:

$$\text{i)}\quad \mu > 0; \qquad \text{ii)}\quad 2\lambda + N\mu > 0. \tag{2}$$

Next, we introduce a core–shell–matrix material structure for our study. Let D and $\Omega \subset \mathbb{R}^N$ be two bounded $C^{1,\alpha}$-domains for $\alpha \in (0,1)$ such that $D \Subset \Omega$, and both $\Omega\backslash\overline{D}$ and $\mathbb{R}^N\backslash\overline{\Omega}$ are connected. Assume that the matrix $\mathbb{R}^N\backslash\overline{\Omega}$ is occupied by a regular elastic material parameterized by two Lamé constants (λ, μ) satisfying (2). The shell $\Omega\backslash\overline{D}$ is occupied by a metamaterial whose Lamé constants are given by $(\hat{\lambda}, \hat{\mu})$. It is assumed that $\hat{\lambda}$ and $\hat{\mu}$ can be flexibly chosen and do not necessarily fulfill the strong convexity conditions (2). In fact, they are complex-valued with $\Re\hat{\lambda}, \Re\hat{\mu}$ breaking the strong convexity conditions (2) and $\Im\hat{\lambda}, \Im\hat{\mu} \in \mathbb{R}_+$. This is critical in our study and will be further remarked in what follows. The inner core D is occupied by a regular elastic material whose Lamé constants $(\breve{\lambda}, \breve{\mu})$ satisfy the strong convex conditions (2). We introduce the notation defined by:

$$\mathbf{C}_{\mathbb{R}^N\backslash\overline{\Omega},\lambda,\mu} = \mathbf{C}_{\lambda,\mu}\chi(\mathbb{R}^N\backslash\overline{\Omega}), \tag{3}$$

where $\mathbf{C}_{\lambda,\mu}$ is given in (1) and $\chi(\mathbb{R}^N\backslash\overline{\Omega})$ denotes the indicator function of $\mathbb{R}^N\backslash\overline{\Omega}$. The same notation applies to the tensors $\mathbf{C}_{\Omega\backslash\overline{D},\hat{\lambda},\hat{\mu}}$ and $\mathbf{C}_{D,\breve{\lambda},\breve{\mu}}$. Now, we introduce the following elastic tensor:

$$\mathbf{C}_0 = \mathbf{C}_{\mathbb{R}^N\backslash\overline{\Omega},\lambda,\mu} + \mathbf{C}_{\Omega\backslash\overline{D},\hat{\lambda},\hat{\mu}} + \mathbf{C}_{D,\breve{\lambda},\breve{\mu}}. \tag{4}$$

The tensor $\mathbf{C}_0$ describes an elastic material configuration of a core–shell–matrix structure with the metamaterial located in the shell. We point out that it may happen that $D = \emptyset$ in our subsequent analysis. In such a case, $\mathbf{C}_0$ is said to be a metamaterial structure with no core. In what follows, material structures with a core or without a core can induce different resonance phenomena.

Let $\mathbf{f} \in L^{\infty}_{loc}(\mathbb{R}^N \backslash \overline{\Omega})^N$ signify an elastic source that is compactly supported in $\mathbb{R}^N \backslash \overline{\Omega}$. The elastic displacement field $\mathbf{u} = (u_i)_{i=1}^N \in H^1_{loc}(\mathbb{R}^N)^N$ induced by the interaction between the source $\mathbf{f}$ and the medium configuration $\mathbf{C}_0$ is governed by the following Lamé system:

$$\begin{cases} \nabla \cdot \mathbf{C}_0 \nabla^s \mathbf{u}(\mathbf{x}) + \omega^2 \mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \quad \text{in } \mathbb{R}^N, \\ \mathbf{u}(\mathbf{x}) \text{ satisfies the Kupradze radiation condition,} \end{cases} \tag{5}$$

where $\omega \in \mathbb{R}_+$ signifies an angular frequency. Here and also in what follows, the operator ∇^s is the symmetric gradient defined by:

$$\nabla^s \mathbf{u} := \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^t\right), \tag{6}$$

where $\nabla \mathbf{u}$ denotes the matrix $(\partial_j u_i)_{i,j=1}^N$ and the superscript t signifies the matrix transpose. It follows from Ref. 1 that the elastic displacement $\mathbf{u}(\mathbf{x})$ can be decomposed into $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s$ in $\mathbb{R}^N \backslash \overline{\Omega}$, where $\mathbf{u}_p$ and $\mathbf{u}_s$ are, respectively, referred to as the pressure and shear waves and satisfy the following equations:

$$\left(\triangle + k_p^2\right)\mathbf{u}_p = 0, \quad \nabla \times \mathbf{u}_p = 0; \ \left(\triangle + k_s^2\right)\mathbf{u}_s = 0, \quad \nabla \cdot \mathbf{u}_s = 0, \tag{7}$$

with

$$k_s := \omega/\sqrt{\mu} \quad \text{and} \quad k_p := \omega/\sqrt{\lambda + 2\mu}. \tag{8}$$

In (5), the Kupradze radiation condition is expressed as:

$$\nabla \mathbf{u}_p \hat{\mathbf{x}} - \mathrm{i} k_p \mathbf{u}_p = \mathcal{O}\left(|\mathbf{x}|^{-(N+1)/2}\right) \quad \text{and} \quad \nabla \mathbf{u}_s \hat{\mathbf{x}} - \mathrm{i} k_s \mathbf{u}_s = \mathcal{O}\left(|\mathbf{x}|^{-(N+1)/2}\right) \tag{9}$$

as $|\mathbf{x}| \to \infty$, which hold uniformly in $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^{N-1}$.

Next, we recall the quasistatic condition for the above elastic scattering problems:

$$\omega \cdot \mathrm{diam}(\Omega) \ll 1, \tag{10}$$

which signifies that the size of the material structure Ω, that is, the diameter of Ω, is much smaller than the operating wavelength $2\pi/\omega$. In the current article, we will instead mainly study the case beyond the quasistatic regime, namely,

$$\omega \cdot \mathrm{diam}(\Omega) \sim 1. \tag{11}$$

For simplicity, it is sufficient for us to require that

$$\omega \sim 1 \quad \text{and} \quad \mathrm{diam}(\Omega) \sim 1. \tag{12}$$

We proceed to introduce the following functional for $\mathbf{u}, \mathbf{v} \in (H^1(\Omega\backslash\overline{D}))^N$:

$$P_{\hat{\lambda},\hat{\mu}}(\mathbf{u},\mathbf{v}) = \int_{\Omega\backslash\overline{D}} \nabla^s\mathbf{u} : \mathbf{C}_0\overline{\nabla^s\mathbf{v}}d\mathbf{x} = \int_{\Omega\backslash\overline{D}} \left(\hat{\lambda}(\nabla\cdot\mathbf{u})\overline{(\nabla\cdot\mathbf{v})} + 2\hat{\mu}\nabla^s\mathbf{u} : \overline{\nabla^s\mathbf{v}}\right) d\mathbf{x}, \tag{13}$$

where $\mathbf{C}_0$ and ∇^s are defined in (4) and (6), respectively. In (13) and also in what follows, $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^N a_{ij}b_{ij}$ for two matrices $\mathbf{A} = (a_{ij})_{i,j=1}^N$ and $\mathbf{B} = (b_{ij})_{i,j=1}^N$. The energy dissipation of the elastic system (5)–(9) is given by:

$$E(\mathbf{u}) = \Im P_{\hat{\lambda},\hat{\mu}}(\mathbf{u},\mathbf{u}). \tag{14}$$

We are now in a position to give the precise meaning of the elastic resonances for our subsequent study.

Definition 1. Consider the Lamé system (5)–(9) associated with the material structure $\mathbf{C}_0$ in (4) under the assumption (12). We say that the *near-resonance* occurs if it holds that

$$E(\mathbf{u}) \geq M \tag{15}$$

for $M \gg 1$. If in addition to (15), the displacement field $\mathbf{u}$ further fulfills the following boundedness condition:

$$|\mathbf{u}| \leq C \quad \text{when} \quad |\mathbf{x}| > \tilde{R}, \tag{16}$$

for some $C \in \mathbb{R}_+$ and $\tilde{R} \in \mathbb{R}_+$ such that $\Omega \subset B_{\tilde{R}}$, then we say that *anomalous localized resonance* (ALR) occurs. Here and also in what follows, $B_{\tilde{R}}$ signifies a ball of radius $\tilde{R}$ and centered at the origin, and $C, \tilde{R}$ are constants independent of $\mathbf{C}_0$ and $\mathbf{f}$.

Remark 1. The terminology *near-resonance* is defined in (15). However, in order to be consistent with the relevant studies in the literature, we still call it *resonance* in what follows. In fact, the phenomenon of the near-resonance has many important applications. Here, we would like to mention one example, that is, the so-called anomalous localized resonance (ALR), which can induce the cloaking effect. The first mathematical work concerning the ALR in Ref. 2 defined the "resonance," which is also the limit case. From a practical point of view, it is sufficient to have the near-cloaking effect via the near-resonance. Nevertheless, in the setting of Ref. 2, it was shown that the dissipated energy can be made arbitrarily large by controlling some asymptotic parameter. In contrast, in the current article, the notion of near-resonance depends on the magnitude of M. Hence, it is a weaker resonance compared to that considered in Ref. 2.

Remark 2. It is noted that the resonant condition (15) indicates that the resonant field $\mathbf{u}$ exhibits highly oscillatory behavior. Moreover, in our subsequent study, it allows that $M \to +\infty$, which indicates that in the limiting case, the scattering system (5)–(9) loses its well-posedness. Indeed, it will be seen in what follows that in the limiting case, the solutions to the scattering system (5)–(9) are not unique. It is clear that the metamaterials located in $\Omega\backslash\overline{D}$ play a critical role for the occurrence of the resonance. In fact, if $\mathbf{C}_0$ is a regular elastic material configuration, then the Lamé system (5)–(9) is well-posed and the resonance does not occur.

Remark 3. If ALR occurs, one can show that invisibility cloaking phenomenon can be induced. In fact, by normalization, we set $\widetilde{\mathbf{f}} := \mathbf{f}/\sqrt{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} := \Im P_{\hat{\lambda},\hat{\mu}}(\mathbf{u},\mathbf{u}) \gg 1$. One can see that both $\widetilde{\mathbf{f}}$ and the material structure $\mathbf{C}_0$ are nearly invisible to observations made outside $B_{\tilde{R}}$. Indeed, it is easily seen that the induced elastic field $\widetilde{\mathbf{u}} = \mathbf{u}/\sqrt{\boldsymbol{\alpha}} \ll 1$ in $\mathbb{R}^N \backslash B_{\tilde{R}}$; see Refs. 2–5 for more relevant discussions. Hence, when ALR occurs according to Definition 1, we also say that cloaking due to anomalous localized resonance (CALR) occurs.

The major findings of this article can be briefly summarized as follows with the technical details supplied in the sequel; see Theorems 3–5:

Consider the Lamé system (5)–(9) associated with the material structure $\mathbf{C}_0$ in (4), under the assumption (12).

(1) Suppose that the material structure has no core, namely, $D = \emptyset$. There exist generic material structures of the form (4) such that resonance occurs.
(2) Under the same setup as the above (1), but with Ω being radially symmetric, we derive the explicit construction of all the material structures that can induce resonance. Moreover, we present a comprehensive analysis on the quantitative behaviors of the resonant field. It is shown that the resonance behaviors possess distinct characters, with some similar to the subwavelength resonances, but some sharply different due to the frequency effect.
(3) We construct a core–shell metamaterial structure that can induce CALR beyond the quasistatic limit.
(4) In establishing the resonance results, we derive comprehensive spectral properties of the nonstatic elastic Neumann–Poincaré (N-P) operator on a circular domain, which will be introduced in the sequel. These results are of independent interest to the spectral theory of layer potential operators.

Remark 4. According to our discussion in Remark 2, the main technical ingredient in our study is to derive some relations satisfied by the material parameters in $\mathbf{C}_0$, the geometric parameters of Ω/D as well as the frequency ω such that the resonance conditions (Definition 1) can be fulfilled. It is clear that these conditions are coupled nonlinearly and in fact they are essentially determined by the infinite-dimensional kernel of the partial differential equation (PDE) system (5), that is, the set of nontrivial solutions to (5) with $\mathbf{f} \equiv \mathbf{0}$.

Remark 5. We would like to make a remark on the metamaterial parameters in $\Omega\backslash\overline{D}$, namely, $\hat{\lambda}$ and $\hat{\mu}$. As pointed out in Remark 2, $\Re\hat{\lambda}$ and $\Re\hat{\mu}$ are allowed to break the strong convexity conditions in (2). This is critical for inducing the resonances. On the other hand, $\Im\hat{\lambda}$ and $\Im\hat{\mu}$ are required to be positive. In a certain sense, they play the role of regularization parameters that can retain the well-posedness of the Lamé system (5)–(9). Moreover, they are also critical physical parameters to induce the resonances. In fact, $\Im\hat{\lambda}$ and $\Im\hat{\mu}$ should be delicately chosen according to $\Re\hat{\lambda}$, $\Re\hat{\mu}$, and ω as well as the asymptotic parameter M in (15). This is in sharp contrast to the related studies in the static/quasistatic case, where $\Im\hat{\lambda}$ and $\Im\hat{\mu}$ play solely as the regularization parameters which are asymptotically small generic parameters and converge to zero in the limiting case. This will become clearer in our subsequent analysis.

1.2 | Connection to existing studies and discussions

Metamaterials are artificially engineered materials to have properties that are not found in naturally occurring materials. Negative materials are an important class of metamaterials which possess negative material parameters. Negative materials can be artificially engineered by assembling subwavelength resonators periodically or randomly; see, for example, Refs. 6–8 and the references cited therein. Negative materials are revolutionizing many industrial applications, including antennas,[9] absorber,[10] invisibility cloaking ,[2,11–15] superlens,[16,17] and super-resolution imaging,[18–20] to mention just a few.

We are mainly concerned with the quantitative theoretical understandings of negative metamaterials, which have received considerable interest recently in the literature. A variety of peculiar resonance phenomena form the fundamental basis for many striking applications of negative metamaterials. Intriguingly, those resonance phenomena are distinct and possess distinguishing characters. For a typical scenario, let us consider the Lamé system (5) in the static case, namely, $\omega \equiv 0$. If $\mathbf{C}_0$ is allowed to possess negative material parameters, it is no longer an elliptic tensor, that is, the strong convexity conditions (2) can be broken. In such a case, the PDE system (5) may possess (infinitely many) nontrivial solutions even with $\mathbf{f} \equiv \mathbf{0}$. Hence, the infinite kernel of the nonelliptic partial differential operator, namely, $\nabla \cdot \mathbf{C}_0 \nabla^s$, can induce certain resonances if the excitation term $\mathbf{f}$ is properly chosen. Similar resonance phenomena have been more extensively and intensively investigated for acoustic and electromagnetic metamaterials that are governed by the Helmholtz and Maxwell systems, respectively. They are referred to as the plasmon/polariton resonances in the literature; see Refs. 2, 21–25 for the Helmholtz equation, Refs. 19, 26, 27 for the Maxwell system, and Refs. 3, 5, 28–31 for the Lamé system. Most of the existing studies in the literature are concerned with the static or quasistatic cases (cf. (10)). A widely studied resonance phenomenon is induced by the interface of negative and positive materials, which is referred to as the plasmon/polariton resonance in the literature. It turns out that the plasmon/polariton resonant oscillations are localized around the metamaterial interface, and hence are usually called the surface plasmon/polariton resonances.

It is not surprising that the occurrence of plasmon/polariton resonances strongly depend on the medium configuration as well as the geometry of the metamaterial structure, which are delicately coupled together in certain nonlinear relations. In this article, we for the first time show the existence of generic metamaterial structures that can induce resonances in elasticity beyond quasistatic approximations in both 2D and 3D. It turns out that in addition to the medium and geometric parameters of the metamaterial structure, the operating frequency will also play a critical role and needs to be incorporated into the nonlinear coupling mentioned above. In addition to its theoretical significance, we would like to emphasize that our study also uncovers two interesting physical phenomena due to the frequency effect. First, the resonant oscillation outside the material structure is localized around the metamaterial interface, but inside the material structure, it is not localized around the interface, which is sharply different from the subwavelength resonances; see more detailed discussion at the end of Section 3. Second, as already commented in Remark 5, the loss parameters $\Im\hat{\lambda}$ and $\Im\hat{\mu}$ will also play an important role, and they generally are required to be nonzero constants in the limiting case; see Remark 11 in what follows for more details. Finally, as noted earlier, negative materials usually occur in the nanoscale, and hence, it is unobjectionable that many studies are concerned with subwavelength resonances. On the other hand, there are also conceptual and visionary studies which employ metamaterials for novel applications beyond the quasistatic limit, say, for example, the superlens.[16, 17] The proposed study in this article follows a similar spirit to the latter class mentioned above, though we are mainly concerned with the theoretical aspects.

If the metamaterial structure is constructed in the core–shell form, it may induce the cloaking phenomenon due to the ALR;[4, 29, 32] that is, the whole structure is invisible for an impinging wave. This is a much more delicate and subtle resonance phenomenon: the resonant oscillation is localized within a bounded region, that is, $B_{\tilde{R}}$ in Definition 1, and moreover, it not only depends on the material and geometric configurations of the core–shell structure, but also critically depends on the location of the excitation source. In addition to the invisibility phenomenon mentioned above, it is observed in that any small objects located near the material structure within the critical radius are also invisible to faraway observations; see Refs. 14, 15 for related discussions. All of the existing studies are confined within the radial geometries because on the one hand, one needs explicit expressions of the spectral system of certain integral operators[2,4,11,27,29] and on the other hand, it seems unnecessary for constructing material structures of general shapes for the cloaking purpose. The CALR was recently studied in Ref. 3 for 3D elasticity beyond the quasistatic approximation for the spherical structure. Hence, in the current article, we mainly consider the CALR in two dimensions. Nevertheless, we would like to remark that the derivation in 2D is more subtle and technically involved. The main reason is that in 3D elasticity there exists a certain class of shear waves that can be decoupled from the other shear waves and compressional waves.[3] The decoupling property significantly simplifies the analysis of the CALR. However, in 2D elasticity, all the shear waves and compressional waves are coupled together, which substantially increase the complexity of the relevant theoretical analysis. In fact, we develop several technically new ingredients in handling the 2D case in the present article.

Finally, we would like to discuss one more technical novelty of our study. In studying the metamaterial resonances, one powerful tool is to make use of the layer potential theory to reduce the underlying PDE system into a system of certain integral operators. In doing so, the resonance analysis is boiled down to analyzing the spectral properties of the integral operators. In this article, we provide an in-depth analysis of the so-called elastic Neumann–Poincaré (N-P) operator in the frequency regime. In particular, we derive the complete spectral system of the elastic N-P operator on a circular domain with several interesting observations. These results are new to the literature and are of independent interest to the spectral theory of elastic layer potential operators (cf. Refs. 29, 33–35).

The rest of this article is organized as follows. In Section 2, we present several technical auxiliary results. Section 3 is devoted to resonance analysis for material structures with no core. In Section 4, we construct a core–shell structure that can induce cloaking due to ALR.

2 | AUXILIARY RESULTS

In this section, we derive some key auxiliary results that will be needed for our subsequent analysis.

Set $\mathbf{x} = (x_j)_{j=1}^N \in \mathbb{R}^N$ to be the Euclidean coordinates and $r = |\mathbf{x}|$. Let $\theta_{\mathbf{x}}$ be the angle between $\mathbf{x}$ and x_1-axis. If there is no ambiguity, we write θ instead of $\theta_{\mathbf{x}}$ for simplicity. Let $\boldsymbol{\nu}$ signify the outward unit normal to a boundary $\partial\Omega$. If the domain Ω is a circle B_R, then $\boldsymbol{\nu} = (\cos(\theta), \sin(\theta))^t$ and the direction $\mathbf{t} = (-\sin(\theta), \cos(\theta))^t$ is the tangential direction on the boundary ∂B_R. Denote by $\nabla_{\mathbb{S}}$ the surface gradient.

The Lamé operator $\mathcal{L}_{\lambda,\mu}$ associated with the parameters (λ, μ) is defined by:

$$\mathcal{L}_{\lambda,\mu}\mathbf{w} := \mu \triangle \mathbf{w} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{w}. \tag{17}$$

The traction (the conormal derivative) of $\mathbf{w}$ on the boundary $\partial\Omega$ is defined by:

$$\partial_\nu \mathbf{w} = \lambda(\nabla \cdot \mathbf{w})\boldsymbol{\nu} + 2\mu(\nabla^s \mathbf{w})\boldsymbol{\nu}, \tag{18}$$

where the operator ∇^s is defined in (6). From Ref. 32, the fundamental solution $\boldsymbol{\Gamma}^\omega = (\Gamma^\omega_{i,j})^2_{i,j=1}$ to the operator $\mathcal{L}_{\lambda,\mu} + \omega^2$ in two dimensions is given by:

$$\left(\Gamma^\omega_{i,j}\right)^2_{i,j=1}(\mathbf{x}) = -\frac{\mathrm{i}\delta_{ij}}{4\mu} H_0(k_s|\mathbf{x}|) + \frac{\mathrm{i}}{4\pi\omega^2}\partial_i\partial_j\left(H_0\left(k_p|\mathbf{x}|\right) - H_0(k_s|\mathbf{x}|)\right), \tag{19}$$

where $H_0(\cdot)$ is the Hankel function of the first kind of order 0, and k_s and k_p are defined in (8). The corresponding fundamental solution $\boldsymbol{\Gamma}^\omega = (\Gamma^\omega_{i,j})^3_{i,j=1}$ in three dimensions is given by:

$$(\Gamma^\omega_{i,j})^3_{i,j=1}(\mathbf{x}) = -\frac{\delta_{ij}}{4\pi\mu|\mathbf{x}|} e^{\mathrm{i}k_s|\mathbf{x}|} + \frac{1}{4\pi\omega^2}\partial_i\partial_j \frac{e^{\mathrm{i}k_p|\mathbf{x}|} - e^{\mathrm{i}k_s|\mathbf{x}|}}{|\mathbf{x}|}. \tag{20}$$

Then, the single-layer potential associated with the fundamental solution $\boldsymbol{\Gamma}^\omega$ is defined as:

$$\mathbf{S}^\omega_{\partial\Omega}[\boldsymbol{\varphi}](\mathbf{x}) = \int_{\partial\Omega} \boldsymbol{\Gamma}^\omega(\mathbf{x} - \mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^N, \tag{21}$$

for $\boldsymbol{\varphi} \in L^2(\partial\Omega)^N$. On the boundary $\partial\Omega$, the conormal derivative of the single-layer potential satisfies the following jump formula:

$$\frac{\partial \mathbf{S}^\omega_{\partial\Omega}[\boldsymbol{\varphi}]}{\partial \boldsymbol{\nu}}\Big|_\pm(\mathbf{x}) = \left(\pm\frac{1}{2}\mathbf{I} + \mathbf{K}^{\omega,*}_{\partial\Omega}\right)[\boldsymbol{\varphi}](\mathbf{x}) \quad \mathbf{x} \in \partial\Omega, \tag{22}$$

where

$$\mathbf{K}^{\omega,*}_{\partial\Omega}[\boldsymbol{\varphi}](\mathbf{x}) = \text{p.v.} \int_{\partial\Omega} \frac{\partial \boldsymbol{\Gamma}^\omega}{\partial \boldsymbol{\nu}(\mathbf{x})}(\mathbf{x} - \mathbf{y})\boldsymbol{\varphi}(\mathbf{y})ds(\mathbf{y}), \tag{23}$$

with p.v. standing for the Cauchy principal value and the subscript $\pm$ indicating the limits from outside and inside Ω, respectively. The operator $\mathbf{K}^{\omega,*}_{\partial\Omega}$ is called the N-P operator associated with the Lamé system.

Next, we present some properties of the N-P operator $\mathbf{K}^{\omega,*}_{\partial\Omega}$. It is shown in Ref. 33 that the operator $\mathbf{K}^{\omega,*}_{\partial\Omega}$ is not compact and only polynomially compact in the following sense.

Lemma 1. *The N-P operator $\mathbf{K}^{\omega,*}_{\partial\Omega}$ is polynomially compact in the sense that in two dimensions, the operator $(\mathbf{K}^{\omega,*}_{\partial\Omega})^2 - k_0^2\mathbf{I}$ is compact; while in three dimensions, the operator $\mathbf{K}^{\omega,*}_{\partial\Omega}((\mathbf{K}^{\omega,*}_{\partial\Omega})^2 - k_0^2\mathbf{I})$ is* *compact, where*

$$k_0 := \frac{\mu}{2(\lambda + 2\mu)}. \tag{24}$$

Then, we can derive the following lemma for the spectrum of the N-P operator $\mathbf{K}_{\partial\Omega}^{\omega,*}$ (cf. Ref. 36).

Lemma 2. *In two dimensions, the spectrum $\sigma(\mathbf{K}_{\partial\Omega}^{\omega,*})$ consists of two nonempty sequences of eigenvalues that converge to k_0 and $-k_0$, respectively; while in three dimensions, the spectrum $\sigma(\mathbf{K}_{\partial\Omega}^{\omega,*})$ consists of three nonempty sequences of eigenvalues that converge to 0, k_0 and $-k_0$, respectively.*

Let $\Phi(\mathbf{x})$ be the fundamental solution to the operator $\triangle + k^2$ in two dimensions given by

$$\Phi(\mathbf{x}) = -\frac{\mathrm{i}}{4} H_0(k|\mathbf{x}|). \tag{25}$$

For $\varphi \in L^2(\partial\Omega)$, we define the single-layer potential associated with the fundamental solution $\Phi(\mathbf{x})$ by:

$$S_{\partial\Omega}^k[\varphi](\mathbf{x}) = \int_{\partial\Omega} \Phi(\mathbf{x} - \mathbf{y})\varphi(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{26}$$

Let $J_n(t)$ and $H_n(t)$, $n \in \mathbb{Z}$, denote the Bessel and Hankel functions of the first kind of order n, respectively. These functions satisfy the following Bessel differential equation:

$$t^2 f''(t) + t f'(t) + \left(t^2 - n^2\right) f(t) = 0, \tag{27}$$

for $f = J_n$ or H_n. When n is negative, there hold that $J_n(t) = (-1)^n J_{-n}(t)$ and $H_n(t) = (-1)^n H_{-n}(t)$. Moreover, the Bessel and Hankel functions $J_n(t)$ and $H_n(t)$ satisfy the recursion formulas (cf. Ref. 37):

$$\begin{aligned} f'_{n+1} &= f_n - (n+1)\frac{f_{n+1}}{t}, \quad f_{n+1} = n\frac{f_n}{t} - f'_n, \quad \text{for} \quad n \geq 0, \\ f'_{n-1} &= -f_n + (n-1)\frac{f_{n-1}}{t}, \quad f_{n-1} = n\frac{f_n}{t} + f'_n, \quad \text{for} \quad n \geq 1. \end{aligned} \tag{28}$$

The following asymptotic expansions hold for $t \ll 1$ (cf. Ref. 37),

$$\begin{aligned} J_n(t) &= \frac{t^n}{2^n n!}\left(1 - \frac{t^2}{4(n+1)} + o\left(t^2\right)\right), \quad \text{for } n \geq 0, \\ H_n(t) &= \frac{-\mathrm{i}2^n n!}{\pi t^n}\left(1 + \frac{t^2}{4(n-1)} + o\left(t^2\right)\right), \quad \text{for } n \geq 2, \end{aligned} \tag{29}$$

and

$$H_1(t) = -\frac{\mathrm{i}2}{\pi t}\left(1 - \frac{t^2}{2}\left(\ln t - \ln 2 + Eu - \frac{1 + \mathrm{i}\pi}{2}\right) + o\left(t^3\right)\right), \tag{30}$$

where Eu is the Euler constant. For larger n, the following asymptotic expansions hold:

$$J_n(t) = \frac{(t/2)^n}{n!}\left\{1 - \frac{t^2}{4n} + \frac{8t^2+t^4}{32n^2} + o\left\{\frac{1}{n^2}\right\}\right\},\ H_n(t) = \frac{-\mathrm{i}n!}{\pi(t/2)^n}\left\{\frac{1}{n} + \frac{t^2}{n^2} + o\left\{\frac{1}{n^2}\right\}\right\}. \quad (31)$$

The Hankel function $H_0(t)$ has the following expansion from Graf's formula(cf. Ref. 38):

$$H_0(k|\mathbf{x}-\mathbf{y}|) = \sum_{n\in\mathbb{Z}} H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta_{\mathbf{x}}}J_n(k|\mathbf{y}|)e^{-\mathrm{i}n\theta_{\mathbf{y}}}. \quad (32)$$

We will also need the following single-layer potential $S^k_{\partial B_R}$ acting on the density $e^{\mathrm{i}n\theta}$.

Lemma 3. *Let $S^k_{\partial B_R}$ be defined in (26), then it holds that*

$$S^k_{\partial B_R}[e^{\mathrm{i}n\theta}](\mathbf{x}) = -\frac{\mathrm{i}\pi R}{2}J_n(kR)H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta}, \quad \forall \mathbf{x}\in\mathbb{R}^2\backslash\overline{B}_R. \quad (33)$$

Proof. From the definition of the operator $S^k_{\partial\Omega}$ in (26) and the expansion for the function $H_0(k|\mathbf{x}-\mathbf{y}|)$ in (32), one has that

$$\begin{aligned} S^\omega_{\partial B_R}[e^{\mathrm{i}n\theta}](\mathbf{x}) &= -\frac{\mathrm{i}}{4}\int_{\partial B_R}\sum_{m\in\mathbb{Z}} H_m(k|\mathbf{x}|)e^{\mathrm{i}m\theta_{\mathrm{x}}}J_m(k|\mathbf{y}|)e^{-\mathrm{i}m\theta_{\mathrm{y}}}e^{\mathrm{i}n\theta_{\mathrm{y}}}ds_{\mathrm{y}} \\ &= -\frac{\mathrm{i}R}{4}\int_0^{2\pi}\sum_{m\in\mathbb{Z}} H_m(k|\mathbf{x}|)e^{\mathrm{i}m\theta_{\mathrm{x}}}J_m(kR)e^{\mathrm{i}(n-m)\theta_{\mathrm{y}}}ds_{\mathrm{y}} = -\frac{\mathrm{i}\pi R}{2}J_n(kR)H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta}. \end{aligned} \quad (34)$$

This completes the proof. ∎

For further calculations, we need the following identities.

Lemma 4. *There hold that: if $m = n-1$,*

$$\int_{\mathbb{S}} e^{-\mathrm{i}m\theta}e^{\mathrm{i}n\theta}\boldsymbol{\nu}ds = \pi\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}, \quad \int_{\mathbb{S}}\nabla_{\mathbb{S}}e^{-\mathrm{i}m\theta}e^{\mathrm{i}n\theta}ds = -(n-1)\pi\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}; \quad (35)$$

and if $m = n+1$,

$$\int_{\mathbb{S}} e^{-\mathrm{i}m\theta}e^{\mathrm{i}n\theta}\boldsymbol{\nu}ds = \pi\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}, \quad \int_{\mathbb{S}}\nabla_{\mathbb{S}}e^{-\mathrm{i}m\theta}e^{\mathrm{i}n\theta}ds = (n+1)\pi\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}. \quad (36)$$

Proof. Direct calculations yield that

$$\int_{\mathbb{S}} e^{-\mathrm{i}m\theta}e^{\mathrm{i}n\theta}\boldsymbol{\nu}ds = \int_{\mathbb{S}} e^{\mathrm{i}(n-m)\theta}\begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix}ds = \int_{\mathbb{S}} e^{\mathrm{i}(n-m)\theta}\begin{pmatrix}\frac{e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta}}{2}\\ \frac{e^{\mathrm{i}\theta}-e^{-\mathrm{i}\theta}}{2}\end{pmatrix}ds. \quad (37)$$

Thus, if $m = n - 1$, one has that

$$\int_{\mathbb{S}} e^{-\mathrm{i}m\theta} e^{\mathrm{i}n\theta} \boldsymbol{\nu} ds = \pi \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}; \tag{38}$$

and if $m = n + 1$, one has that

$$\int_{\mathbb{S}} e^{-\mathrm{i}m\theta} e^{\mathrm{i}n\theta} \boldsymbol{\nu} ds = \pi \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}. \tag{39}$$

Furthermore, one can have that

$$\int_{\mathbb{S}} \nabla_{\mathbb{S}} e^{-\mathrm{i}m\theta} e^{\mathrm{i}n\theta} \boldsymbol{\nu} ds = -\mathrm{i}m \int_{\mathbb{S}} e^{\mathrm{i}(n-m)\theta} \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} ds = -\mathrm{i}m \int_{\mathbb{S}} e^{\mathrm{i}(n-m)\theta} \begin{pmatrix} \frac{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}{-2\mathrm{i}} \\ \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2} \end{pmatrix} ds. \tag{40}$$

Thus, if $m = n - 1$, one has that

$$\int_{\mathbb{S}} \nabla_{\mathbb{S}} e^{-\mathrm{i}m\theta} e^{\mathrm{i}n\theta} ds = -(n-1)\pi \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}; \tag{41}$$

and if $m = n + 1$, one has that

$$\int_{\mathbb{S}} \nabla_{\mathbb{S}} e^{-\mathrm{i}m\theta} e^{\mathrm{i}n\theta} ds = (n+1)\pi \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}. \tag{42}$$

This completes the proof. ■

Lemma 5. *The following two identities hold for* $\mathbf{x} \in \mathbb{R}^2 \backslash \overline{B}_R$,

$$\int_{\partial B_R} H_0(k|\mathbf{x} - \mathbf{y}|) e^{\mathrm{i}n\theta_{\mathrm{y}}} \boldsymbol{\nu}_{\mathbf{y}} ds_{\mathbf{y}}$$
$$= \pi R H_{n-1}(k|\mathbf{x}|) e^{\mathrm{i}(n-1)\theta} J_{n-1}(kR) \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + \pi R H_{n+1}(k|\mathbf{x}|) e^{\mathrm{i}(n+1)\theta} J_{n+1}(kR) \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}, \tag{43}$$

and

$$\int_{\partial B_R} H_0(k|\mathbf{x} - \mathbf{y}|) e^{\mathrm{i}n\theta_{\mathrm{y}}} \mathbf{t}_{\mathbf{y}} ds_{\mathbf{y}}$$
$$= -\mathrm{i}\pi R H_{n-1}(k|\mathbf{x}|) e^{\mathrm{i}(n-1)\theta} J_{n-1}(kR) \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + \mathrm{i}\pi R H_{n+1}(k|\mathbf{x}|) e^{\mathrm{i}(n+1)\theta} J_{n+1}(kR) \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}. \tag{44}$$

Proof. From the expansion of the function $H_0(k|\mathbf{x}-\mathbf{y}|)$ in (32), and with the help of Lemma 4, one has that

$$\begin{aligned}\int_{\partial B_R} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}} &= \int_{\partial B_R}\sum_{m\in\mathbb{Z}} H_m(k|\mathbf{x}|)e^{\mathrm{i}m\theta_{\mathrm{x}}}J_m(k|\mathbf{y}|)e^{-\mathrm{i}m\theta_{\mathrm{y}}}e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}}\\ &= \sum_{m\in\mathbb{Z}} H_m(k|\mathbf{x}|)e^{\mathrm{i}m\theta_{\mathrm{x}}}R\int_{\mathbb{S}} J_m(k|\mathbf{y}|)e^{-\mathrm{i}m\theta_{\mathrm{y}}}e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}} \\ &= \pi R H_{n-1}(k|\mathbf{x}|)e^{\mathrm{i}(n-1)\theta}J_{n-1}(kR)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix} + \pi R H_{n+1}(k|\mathbf{x}|)e^{\mathrm{i}(n+1)\theta}J_{n+1}(kR)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}.\end{aligned} \tag{45}$$

Thus, the first identity is proved. Then, noticing that $\nabla_{\mathbb{S}}e^{\mathrm{i}n\theta} = \mathrm{i}ne^{\mathrm{i}n\theta}\mathbf{t}$, and following the same deduction, one can obtain the other identity, which completes the proof. ∎

Based on the previous two lemmas, we can further have following identities.

Lemma 6. *There holds the following identity for* $\mathbf{x}\in\mathbb{R}^2\backslash\overline{B}_R$,

$$\begin{aligned}&\int_{\partial B_R}\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}}\\ &= \frac{\pi H_{n-1}(k|\mathbf{x}|)e^{\mathrm{i}(n-1)\theta}}{R}\left((n(n-1)-k^2R^2)J_{n-1}(kR) - nkRJ'_{n-1}(kR)\right)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\\ &\quad + \frac{\pi H_{n+1}(k|\mathbf{x}|)e^{\mathrm{i}(n+1)\theta}}{R}\left((n(n+1)-k^2R^2)J_{n+1}(kR) + nkRJ'_{n+1}(kR)\right)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}.\end{aligned} \tag{46}$$

Proof. First, note that $\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}H_0(k|\mathbf{x}-\mathbf{y}|) = \nabla_{\mathbf{y}}\nabla_{\mathbf{y}}H_0(k|\mathbf{x}-\mathbf{y}|)$. Then, one has that

$$\begin{aligned}\int_{\partial B_R}\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}} &= \int_{\partial B_R}\nabla_{\mathbf{y}}\nabla_{\mathbf{y}}H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}}\\ &= \int_{\partial B_R}\left(\nabla_{\mathbb{S}}\nabla_{\mathbf{y}}H_0(k|\mathbf{x}-\mathbf{y}|)/|\mathbf{y}| + \partial_r\nabla_{\mathbf{y}}H_0(k|\mathbf{x}-\mathbf{y}|)\boldsymbol{\nu}_{\mathbf{y}}\right)e^{\mathrm{i}n\theta_{\mathrm{y}}}\nu_{\mathbf{y}}ds_{\mathbf{y}}\\ &= \int_{\partial B_R}\left(\partial_r\nabla_{\mathbf{y}}H_0(k|\mathbf{x}-\mathbf{y}|)\right)e^{\mathrm{i}n\theta_{\mathrm{y}}}ds_{\mathbf{y}}.\end{aligned} \tag{47}$$

Furthermore, from the expansion of the function $H_0(k|\mathbf{x}-\mathbf{y}|)$ in (32), one can have that

$$\begin{aligned}\partial_r\nabla_{\mathbf{y}}H_0(k|\mathbf{x}-\mathbf{y}|) &= \partial_r\nabla_{\mathbf{y}}\left(\sum_{n\in\mathbb{Z}}H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta_{\mathrm{x}}}J_n(k|\mathbf{y}|)e^{-\mathrm{i}n\theta_{\mathrm{y}}}\right)\\ &= \sum_{n\in\mathbb{Z}}H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta_{\mathrm{x}}}\partial_r\left(kJ'_n(k|\mathbf{y}|)e^{-\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}} + J_n(k|\mathbf{y}|)\nabla_{\mathbb{S}}e^{-\mathrm{i}n\theta_{\mathrm{y}}}/|\mathbf{y}|\right)\\ &= \sum_{n\in\mathbb{Z}}\frac{H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta_{\mathrm{x}}}}{|\mathbf{y}|^2}\left(k^2|\mathbf{y}|^2J''_n(k|\mathbf{y}|)e^{-\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}} + \left(k|\mathbf{y}|J'_n(k|\mathbf{y}|) - J_n(k|\mathbf{y}|)\right)\nabla_{\mathbb{S}}e^{-\mathrm{i}n\theta_{\mathrm{y}}}\right).\end{aligned} \tag{48}$$

From the identities (47) and (48), and together with the help of Lemma 4, one can obtain that

$$\begin{aligned}
&\int_{\partial B_R} \nabla_{\mathbf{x}}\nabla_{\mathbf{x}} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}} \\
&= \frac{\pi H_{n-1}(k|\mathbf{x}|)e^{\mathrm{i}(n-1)\theta}}{R}\left(k^2R^2J''_{n-1}(kR) - \left(kRJ'_{n-1}(kR) - J_{n-1}(kR)\right)(n-1)\right)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix} \\
&\quad + \frac{\pi H_{n+1}(k|\mathbf{x}|)e^{\mathrm{i}(n+1)\theta}}{R}\left(k^2R^2J''_{n+1}(kR) + \left(kRJ'_{n+1}(kR) - J_{n+1}(kR)\right)(n+1)\right)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}.
\end{aligned} \tag{49}$$

Now, using the identity (27), we can simplify the last equation as:

$$\begin{aligned}
&\int_{\partial B_R} \nabla_{\mathbf{x}}\nabla_{\mathbf{x}} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}}ds_{\mathbf{y}} \\
&= \frac{\pi H_{n-1}(k|\mathbf{x}|)e^{\mathrm{i}(n-1)\theta}}{R}\left(\left(n(n-1) - k^2R^2\right)J_{n-1}(kR) - nkRJ'_{n-1}(kR)\right)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix} \\
&\quad + \frac{\pi H_{n+1}(k|\mathbf{x}|)e^{\mathrm{i}(n+1)\theta}}{R}\left(\left(n(n+1) - k^2R^2\right)J_{n+1}(kR) + nkRJ'_{n+1}(kR)\right)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}.
\end{aligned} \tag{50}$$

The proof is completed. ■

Lemma 7. *There holds the following identity for* $\mathbf{x} \in \mathbb{R}^2\backslash\overline{B}_R$*:*

$$\begin{aligned}
&\int_{\partial B_R} \nabla_{\mathbf{x}}\nabla_{\mathbf{x}} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\mathbf{t}_{\mathbf{y}}ds_{\mathbf{y}} \\
&= \frac{-\mathrm{i}n\pi H_{n-1}(k|\mathbf{x}|)e^{\mathrm{i}(n-1)\theta}}{R}\left(kRJ'_{n-1}(kR) - J_{n-1}(kR)(n-1)\right)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix} \\
&\quad + \frac{-\mathrm{i}n\pi H_{n+1}(k|\mathbf{x}|)e^{\mathrm{i}(n+1)\theta}}{R}\left(kRJ'_{n+1}(kR) + J_{n+1}(kR)(n+1)\right)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}.
\end{aligned} \tag{51}$$

Proof. First, note that $\nabla_{\mathbf{x}}\nabla_{\mathbf{x}} H_0(k|\mathbf{x}-\mathbf{y}|) = \nabla_{\mathbf{y}}\nabla_{\mathbf{y}} H_0(k|\mathbf{x}-\mathbf{y}|)$. Then, one has that

$$\begin{aligned}
&\int_{\partial B_R} \nabla_{\mathbf{x}}\nabla_{\mathbf{x}} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\mathbf{t}_{\mathbf{y}}ds_{\mathbf{y}} = \int_{\partial B_R} \nabla_{\mathbf{y}}\nabla_{\mathbf{y}} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\mathbf{t}_{\mathbf{y}}ds_{\mathbf{y}} \\
&= \int_{\partial B_R} \left(\nabla_{\mathbb{S}}\nabla_{\mathbf{y}} H_0(k|\mathbf{x}-\mathbf{y}|)/|\mathbf{y}| + \partial_r\nabla_{\mathbf{y}} H_0(k|\mathbf{x}-\mathbf{y}|)\nu_{\mathbf{y}}\right)\frac{1}{\mathrm{i}n}\nabla_{\mathbb{S}}e^{\mathrm{i}n\theta_{\mathrm{y}}}ds_{\mathbf{y}} \\
&= -\frac{1}{\mathrm{i}n}\int_{\mathbb{S}} \left(\nabla_{\mathbf{y}} H_0(k|\mathbf{x}-\mathbf{y}|)\right)\Delta_S e^{\mathrm{i}n\theta_{\mathrm{y}}}ds_{\mathbf{y}} \\
&= -\mathrm{i}n\int_{\mathbb{S}} \nabla_{\mathbf{y}} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}ds_{\mathbf{y}}.
\end{aligned} \tag{52}$$

Then, from the expansion of the function $H_0(k|\mathbf{x}-\mathbf{y}|)$ in (32), it holds that

$$\begin{aligned}\nabla_{\mathbf{y}} H_0(k|\mathbf{x}-\mathbf{y}|) &= \nabla_{\mathbf{y}}\left(\sum_{n\in\mathbb{Z}} H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta_{\mathrm{x}}} J_n(k|\mathbf{y}|)e^{-\mathrm{i}n\theta_{\mathrm{y}}}\right)\\ &= \sum_{n\in\mathbb{Z}} \frac{H_n(k|\mathbf{x}|)e^{\mathrm{i}n\theta_{\mathrm{x}}}}{|\mathbf{y}|}\left(k|\mathbf{y}|J_n'(k|\mathbf{y}|)e^{-\mathrm{i}n\theta_{\mathrm{y}}}\boldsymbol{\nu}_{\mathbf{y}} + J_n(k|\mathbf{y}|)\nabla_{\mathbb{S}} e^{-\mathrm{i}n\theta_{\mathrm{y}}}\right).\end{aligned} \tag{53}$$

From Equations (52) and (53), and together with the help of Lemma 4, one can obtain that

$$\begin{aligned}&\int_{\partial B_R} \nabla_{\mathbf{x}}\nabla_{\mathbf{x}} H_0(k|\mathbf{x}-\mathbf{y}|)e^{\mathrm{i}n\theta_{\mathrm{y}}}\mathbf{t}_{\mathbf{y}}\,ds_{\mathbf{y}}\\ &= \frac{-\mathrm{i}n\pi H_{n-1}(k|\mathbf{x}|)e^{\mathrm{i}(n-1)\theta}}{R}\left(kRJ_{n-1}'(kR) - J_{n-1}(kR)(n-1)\right)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\\ &\quad + \frac{-\mathrm{i}n\pi H_{n+1}(k|\mathbf{x}|)e^{\mathrm{i}(n+1)\theta}}{R}\left(kRJ_{n+1}'(kR) + J_{n+1}(kR)(n+1)\right)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}.\end{aligned} \tag{54}$$

■

With these preparations, we can present the expressions of the single-layer potentials $\mathbf{S}_{\partial B_R}^{\omega}$ with two densities $e^{\mathrm{i}n\theta}\boldsymbol{\nu}$ and $e^{\mathrm{i}n\theta}\mathbf{t}$, and the proof follows directly from the definition of the single-layer potential operator $\mathbf{S}_{\partial B_R}^{\omega}$ in (21) and Lemmas 5–7.

Theorem 1. *The single-layer potentials* $\mathbf{S}_{\partial B_R}^{\omega}[e^{\mathrm{i}n\theta}\boldsymbol{\nu}]$ *and* $\mathbf{S}_{\partial B_R}^{\omega}[e^{\mathrm{i}n\theta}\mathbf{t}]$ *have the following expressions for* $\mathbf{x}\in\mathbb{R}^2\backslash\overline{B}_R$,

$$\begin{aligned}\mathbf{S}_{\partial B_R}^{\omega}[e^{\mathrm{i}n\theta}\boldsymbol{\nu}](\mathbf{x}) &= \frac{-\mathrm{i}\pi e^{\mathrm{i}(n-1)\theta}}{4\omega^2 R}\Big(nH_{n-1}(k_s|\mathbf{x}|)\big((n-1)J_{n-1}(k_sR) - k_sRJ_{n-1}'(k_sR)\big)\\ &\quad + H_{n-1}(k_p|\mathbf{x}|)\big((k_p^2R^2 - n^2 + n)J_{n-1}(k_pR) + nk_pRJ_{n-1}'(k_pR)\big)\Big)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix}\\ &\quad + \frac{-\mathrm{i}\pi e^{\mathrm{i}(n+1)\theta}}{4\omega^2 R}\Big(nH_{n+1}(k_s|\mathbf{x}|)\big((n+1)J_{n+1}(k_sR) + k_sRJ_{n+1}'(k_sR)\big)\\ &\quad + H_{n+1}(k_p|\mathbf{x}|)\big((k_p^2R^2 - n^2 - n)J_{n+1}(k_pR) - nk_pRJ_{n+1}'(k_pR)\big)\Big)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix},\end{aligned} \tag{55}$$

and

$$
\begin{aligned}
\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\mathbf{t}](\mathbf{x}) = {} & \frac{-\pi e^{\mathrm{i}(n-1)\theta}}{4\omega^2 R}\Big(H_{n-1}(k_s|\mathbf{x}|)\big(\big(k_s^2R^2 - n^2 + n\big)J_{n-1}(k_sR) + nk_sRJ'_{n-1}(k_sR)\big) \\
& + nH_{n-1}\big(k_p|\mathbf{x}|\big)\big((n-1)J_{n-1}\big(k_pR\big) - k_pRJ'_{n-1}\big(k_pR\big)\big)\Big)\begin{pmatrix}1\\ \mathrm{i}\end{pmatrix} \\
& + \frac{\pi e^{\mathrm{i}(n+1)\theta}}{4\omega^2 R}\Big(H_{n+1}(k_s|\mathbf{x}|)\big(\big(k_s^2R^2 - n^2 - n\big)J_{n+1}(k_sR) - nk_sRJ'_{n+1}(k_sR)\big) \\
& + nH_{n+1}\big(k_p|\mathbf{x}|\big)\big((n+1)J_{n+1}\big(k_pR\big) + k_pRJ'_{n+1}\big(k_pR\big)\big)\Big)\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}.
\end{aligned}
\tag{56}
$$

Remark 6. With the help of the recursion formulas in (28), the single-layer potentials $\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\boldsymbol{\nu}]$ and $\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\mathbf{t}]$ can be expressed as follows for $\mathbf{x}\in\mathbb{R}^2\backslash\overline{B}_R$:

$$
\begin{aligned}
\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\boldsymbol{\nu}](\mathbf{x}) = {} & \frac{-\mathrm{i}\pi}{4\omega^2 R}\big(nk_sRJ_n(k_sR)\mathbf{Q}^o_n(k_s|\mathbf{x}|) + k_p^2R^2J'_n\big(k_pR\big)\mathbf{P}^o_n(k_p|\mathbf{x}|)\big), \\
\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\mathbf{t}](\mathbf{x}) = {} & \frac{-\pi}{4\omega^2 R}\big(k_s^2R^2J'_n(k_sR)\mathbf{Q}^o_n(k_s|\mathbf{x}|) + nk_pRJ_n(k_pR)\mathbf{P}^o_n(k_p|\mathbf{x}|)\big),
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
\mathbf{Q}^o_n(k_s|\mathbf{x}|) = {} & \frac{2nH_n(k_s|\mathbf{x}|)}{k_s|\mathbf{x}|}e^{\mathrm{i}n\theta}\boldsymbol{\nu} + 2\mathrm{i}H'_n(k_s|\mathbf{x}|)e^{\mathrm{i}m\theta}\mathbf{t}, \\
\mathbf{P}^o_n(k_p|\mathbf{x}|) = {} & 2H'_n(k_p|\mathbf{x}|)e^{\mathrm{i}n\theta}\boldsymbol{\nu} + \frac{2\mathrm{i}nH_n(k_p|\mathbf{x}|)}{k_p|\mathbf{x}|}e^{\mathrm{i}m\theta}\mathbf{t}.
\end{aligned}
\tag{58}
$$

Moreover, these two functions $\mathbf{Q}^o_n(k_s|\mathbf{x}|)$ and $\mathbf{P}^o_n(k_p|\mathbf{x}|)$ are radiating solutions to the equation $(\mathcal{L}_{\lambda,\mu} + \omega^2)\mathbf{u} = 0$ in $\mathbf{x}\in\mathbb{R}^2\backslash\overline{B}_R$. The function $\mathbf{Q}^o_n(k_s|\mathbf{x}|)$ belongs to the s-wave and the function $\mathbf{P}^o_n(k_p|\mathbf{x}|)$ belongs to the p-wave.

Following similar deductions to the above, one can derive the following proposition.

Proposition 1. *For $\mathbf{x}\in B_R$, the single-layer potentials $\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\boldsymbol{\nu}]$ and $\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\mathbf{t}]$ have the following expressions:*

$$
\begin{aligned}
\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\boldsymbol{\nu}](\mathbf{x}) = {} & \frac{-\mathrm{i}\pi}{4\omega^2 R}\big(nk_sRH_n(k_sR)\mathbf{Q}^i_n(k_s|\mathbf{x}|) + k_p^2R^2H'_n\big(k_pR\big)\mathbf{P}^i_n(k_p|\mathbf{x}|)\big), \\
\mathbf{S}^{\omega}_{\partial B_R}[e^{\mathrm{i}n\theta}\mathbf{t}](\mathbf{x}) = {} & \frac{-\pi}{4\omega^2 R}\big(k_s^2R^2H'_n(k_sR)\mathbf{Q}^i_n(k_s|\mathbf{x}|) + nk_pRH_n(k_pR)\mathbf{P}^i_n(k_p|\mathbf{x}|)\big),
\end{aligned}
$$

where

$$
\mathbf{Q}^i_n(k_s|\mathbf{x}|) = \frac{2nJ_n(k_s|\mathbf{x}|)}{k_s|\mathbf{x}|}e^{\mathrm{i}n\theta}\boldsymbol{\nu} + 2\mathrm{i}J'_n(k_s|\mathbf{x}|)e^{\mathrm{i}n\theta}\mathbf{t},
\tag{59}
$$

$$\mathbf{P}_n^i(k_p|\mathbf{x}|) = 2J_n'(k_p|\mathbf{x}|)e^{in\theta}\boldsymbol{\nu} + \frac{2inJ_n(k_p|\mathbf{x}|)}{k_p|\mathbf{x}|}e^{in\theta}\mathbf{t}. \tag{60}$$

Moreover, these two functions $\mathbf{Q}_n^i(k_s|\mathbf{x}|)$ *and* $\mathbf{P}_n^i(k_p|\mathbf{x}|)$ *are entire solutions to the equation* $(\mathcal{L}_{\lambda,\mu} + \omega^2)\mathbf{u} = 0$ *in* $\mathbf{x} \in B_R$. *The function* $\mathbf{Q}_n^i(k_s|\mathbf{x}|)$ *belongs to the s-wave and the function* $\mathbf{P}_n^i(k_p|\mathbf{x}|)$ *belongs to the p-wave.*

Because the function $\mathbf{S}_{\partial B_R}^{\omega}[\boldsymbol{\psi}](\mathbf{x})$ with $\boldsymbol{\psi} \in L^2(\partial B_R)^2$ is continuous from $\mathbb{R}^2\backslash\overline{B}_R$ to $\mathbb{R}^2\backslash B_R$, by letting $|\mathbf{x}| = R$ in Theorem 1 and together with the help of recursion formulas given in (28), one has the following lemma.

Lemma 8. *The single-layer potentials* $\mathbf{S}_{\partial B_R}^{\omega}[e^{in\theta}\boldsymbol{\nu}]$ *and* $\mathbf{S}_{\partial B_R}^{\omega}[e^{in\theta}\mathbf{t}]$ *have the following expressions for* $|\mathbf{x}| = R$:

$$\mathbf{S}_{\partial B_R}^{\omega}[e^{in\theta}\boldsymbol{\nu}](\mathbf{x}) = \alpha_{1n}e^{in\theta}\boldsymbol{\nu} + \alpha_{2n}e^{im\theta}\mathbf{t} \quad \textit{and} \quad \mathbf{S}_{\partial B_R}^{\omega}[e^{in\theta}\mathbf{t}](\mathbf{x}) = \alpha_{3n}e^{in\theta}\boldsymbol{\nu} + \alpha_{4n}e^{im\theta}\mathbf{t}, \tag{61}$$

where

$$\begin{aligned}
\alpha_{1n} &= -\frac{i\pi}{2\omega^2 R}\left(n^2 J_n(k_sR)H_n(k_sR) + k_p^2R^2J_n'(k_pR)H_n'(k_pR)\right),\\
\alpha_{2n} &= \frac{n\pi}{2\omega^2}\left(k_sJ_n(k_sR)H_n'(k_sR) + k_pJ_n'(k_pR)H_n(k_pR)\right),\\
\alpha_{3n} &= -\frac{n\pi}{2\omega^2}\left(k_sJ_n'(k_sR)H_n(k_sR) + k_pJ_n(k_pR)H_n'(k_pR)\right),\\
\alpha_{4n} &= -\frac{i\pi}{2\omega^2 R}\left(k_s^2R^2J_n'(k_sR)H_n'(k_sR) + n^2J_n(k_pR)H_n(k_pR)\right).
\end{aligned} \tag{62}$$

Next, we calculate the tractions $\partial_{\boldsymbol{\nu}}S_{\partial B_R}^{\omega}[e^{in\theta}\boldsymbol{\nu}]|_{\pm}$ and $\partial_{\boldsymbol{\nu}}S_{\partial B_R}^{\omega}[e^{in\theta}\mathbf{t}]|_{\pm}$ on the boundary ∂B_R, where the traction operator $\partial_{\boldsymbol{\nu}}$ is defined in (18). First, we notice that

$$\frac{\partial}{\partial_{x_1}}\left(H_n(k|\mathbf{x}|)e^{in\theta}\right) = p_{1n} \quad \text{and} \quad \frac{\partial}{\partial_{x_2}}\left(H_n(k|\mathbf{x}|)e^{in\theta}\right) = p_{2n}, \tag{63}$$

where

$$\begin{aligned}
p_{1n} &= kH_n'(k|\mathbf{x}|)e^{in\theta}\cos(\theta) - inH_n(k|\mathbf{x}|)e^{in\theta}\sin(\theta)/|\mathbf{x}|,\\
p_{2n} &= kH_n'(k|\mathbf{x}|)e^{in\theta}\sin(\theta) + inH_n(k|\mathbf{x}|)e^{in\theta}\cos(\theta)/|\mathbf{x}|.
\end{aligned} \tag{64}$$

Hence, for

$$\mathbf{g} = e^{in\theta}H_n(k|\mathbf{x}|)\begin{pmatrix} a \\ b \end{pmatrix}, \tag{65}$$

where a and b are two constants, one has that

$$\nabla \cdot \mathbf{g} = ap_{1n} + bp_{2n}, \qquad 2\nabla^s\mathbf{g} = \begin{pmatrix} 2ap_{1n} & ap_{2n} + bp_{1n} \\ ap_{2n} + bp_{1n} & 2bp_{2n} \end{pmatrix}, \tag{66}$$

where p_{1n} and p_{2n} are defined in (63). During the simplification, we have used the recursion formulas given in (28). Then, we can obtain the following lemma.

Lemma 9. *There hold the following relations:*

$$\partial_{\boldsymbol{\nu}}\mathbf{S}^{\omega}_{\partial B_R}\left[e^{\mathrm{i}m\theta}\boldsymbol{\nu}\right]|_+ = g_{1,m}(|\mathbf{x}|)e^{\mathrm{i}m\theta}\boldsymbol{\nu} + g_{2,m}(|\mathbf{x}|)e^{\mathrm{i}m\theta}\mathbf{t},$$
$$\partial_{\boldsymbol{\nu}}\mathbf{S}^{\omega}_{\partial B_R}\left[e^{\mathrm{i}m\theta}\mathbf{t}\right]|_+ = g_{3,m}(|\mathbf{x}|)e^{\mathrm{i}m\theta}\boldsymbol{\nu} + g_{4,m}(|\mathbf{x}|)e^{\mathrm{i}m\theta}\mathbf{t}, \tag{67}$$

where

$$g_{1,m}(|\mathbf{x}|) = \frac{\mathrm{i}\pi}{2\omega^2R^2}\big(2\mu m^2 J_m(k_sR)\big(H_m(k_s|\mathbf{x}|) - k_sRH_m'(k_s|\mathbf{x}|)\big) + J_m'\big(k_pR\big)k_pR\big(H_m\big(k_p|\mathbf{x}|\big)\big(\omega^2R^2 - 2\mu m^2\big) + 2k_p\mu RH_m'\big(k_p|\mathbf{x}|\big)\big)\big),$$
$$g_{2,m}(|\mathbf{x}|) = -\frac{m\mu\pi}{2\omega^2R^2}\big(J_m(k_sR)H_m(k_s|\mathbf{x}|)\big(k_s^2R^2 - 2m^2\big) + 2R\big(k_sJ_m(k_sR)H_m'(k_s|\mathbf{x}|) + k_pJ_m'\big(k_pR\big)\big(H_m\big(k_p|\mathbf{x}|\big) - k_pRH_m'\big(k_p|\mathbf{x}|\big)\big)\big)\big), \tag{68}$$

$$g_{3,m}(|\mathbf{x}|) = \frac{m\pi}{2\omega^2R^2}\big(J_m\big(k_pR\big)H_m\big(k_p|\mathbf{x}|\big)\big((\lambda+2\mu)k_p^2R^2 - 2\mu m^2\big) + 2\mu R\big(k_pJ_m\big(k_pR\big)H_m'\big(k_p|\mathbf{x}|\big) + k_sJ_m'(k_sR)\big(H_m(k_s|\mathbf{x}|) - k_sRH_m'(k_s|\mathbf{x}|)\big)\big)\big),$$
$$g_{4,m}(|\mathbf{x}|) = \frac{\mathrm{i}\mu\pi}{2\omega^2R^2}\big(2m^2J_m\big(k_pR\big)\big(H_m\big(k_p|\mathbf{x}|\big) - k_pRH_m'\big(k_p|\mathbf{x}|\big)\big) + J_m'(k_sR)k_sR\big(k_s^2R^2H_m(k_s|\mathbf{x}|) + 2k_sRH_m'(k_s|\mathbf{x}|) - 2m^2H_m(k_s|\mathbf{x}|)\big)\big). \tag{69}$$

Remark 7. Taking the traction of the function $\mathbf{Q}_n^i$ and $\mathbf{P}_n^i$ defined in (59) and (60) on the boundary ∂B_R gives that

$$\partial_{\boldsymbol{\nu}}\mathbf{Q}_n^i = \gamma_{1n}e^{\mathrm{i}n\theta}\boldsymbol{\nu} + \gamma_{2n}e^{\mathrm{i}n\theta}\mathbf{t}, \quad \partial_{\boldsymbol{\nu}}\mathbf{P}_n^i = \gamma_{3n}e^{\mathrm{i}n\theta}\boldsymbol{\nu} + \gamma_{4n}e^{\mathrm{i}n\theta}\mathbf{t}, \tag{70}$$

where

$$\gamma_{1n} = \frac{4n\mu}{k_sR^2}\big(k_sRJ_n'(k_sR) - J_n(k_sR)\big), \quad \gamma_{2n} = \frac{2\mathrm{i}\mu}{k_sR^2}\big(\big(2n^2 - k_s^2R^2\big)J_n(k_sR) - 2k_sRJ_n'(k_sR)\big), \tag{71}$$

$$\gamma_{3n} = \frac{2\mu}{k_pR^2}\big(\big(2n^2 - k_s^2R^2\big)J_n\big(k_pR\big) - 2k_pRJ_n'(k_pR)\big), \quad \gamma_{4n} = \frac{4\mathrm{i}n\mu}{k_pR^2}\big(k_pRJ_n'(k_pR) - J_n(k_pR)\big). \tag{72}$$

With the help of Lemma 9 and the jump formula in (22), one can conclude the following lemma.

Lemma 10. *There hold that*

$$\mathbf{K}^{\omega,*}_{\partial B_R}\left[e^{\mathrm{i}n\theta}\boldsymbol{\nu}\right] = a_{1n}e^{\mathrm{i}n\theta}\boldsymbol{\nu} + a_{2n}e^{\mathrm{i}n\theta}\mathbf{t} \quad \text{and} \quad \mathbf{K}^{\omega,*}_{\partial B_R}\left[e^{\mathrm{i}n\theta}\mathbf{t}\right] = b_{1n}e^{\mathrm{i}n\theta}\boldsymbol{\nu} + b_{2n}e^{\mathrm{i}n\theta}\mathbf{t}, \tag{73}$$

where

$$a_{1n} = -\frac{1}{2} + g_{1,n}(R), \quad a_{2n} = g_{2,n}(R), \quad b_{1n} = g_{3,n}(R), \quad b_{2n} = -\frac{1}{2} + g_{4,n}(R); \tag{74}$$

with the functions $g_{i,n}(|\mathbf{x}|), 1 \le i \le 4$ *given in Lemma 9.*

Finally, we obtain the eigensystem for the N-P operator $\mathbf{K}^{\omega,*}_{\partial B_R}$.

Theorem 2. *Let* $a_{1n}, a_{2n}, b_{1n}, b_{2n}$ *be given in Lemma 10. The eigensystem for the N-P operator* $\mathbf{K}^{\omega,*}_{\partial B_R}$ *is given as follows:*

1) if $a_{2n} \neq 0$*, the eigenvalues are*

$$\begin{aligned} \xi_{1n} &= \frac{1}{2}\left(a_{1n} + b_{2n} - \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}\right), \\ \xi_{2n} &= \frac{1}{2}\left(a_{1n} + b_{2n} + \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}\right), \end{aligned} \tag{75}$$

and the corresponding eigenfunctions are

$$\begin{aligned} \mathbf{p}_{1n} &= \left(\frac{a_{1n} - b_{2n} - \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}}{2a_{2n}}\right) e^{in\theta}\boldsymbol{\nu} + e^{in\theta}\boldsymbol{t}, \\ \mathbf{p}_{2n} &= \left(\frac{a_{1n} - b_{2n} + \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}}{2a_{2n}}\right) e^{in\theta}\boldsymbol{\nu} + e^{in\theta}\boldsymbol{t}; \end{aligned} \tag{76}$$

2) if $a_{2n} = 0$*, and* $a_{1n} \neq b_{2n}$*, the eigenvalues are*

$$\xi_{1n} = a_{1n}, \quad \xi_{2n} = b_{2n}, \tag{77}$$

and the corresponding eigenfunctions are

$$\mathbf{p}_{1n} = e^{in\theta}\boldsymbol{\nu}, \quad \mathbf{p}_{2n} = \left(\frac{b_{1n}}{b_{2n} - a_{1n}}\right) e^{in\theta}\boldsymbol{\nu} + e^{in\theta}\mathbf{t}; \tag{78}$$

3) if $a_{2n} = 0, a_{1n} = b_{2n}$*, and* $b_{1n} = 0$*, the eigenvalues are*

$$\xi_{1n} = a_{1n}, \quad \xi_{2n} = a_{1n}, \tag{79}$$

and the corresponding eigenfunctions are

$$\mathbf{p}_{1n} = e^{in\theta}\boldsymbol{\nu}, \quad \mathbf{p}_{2n} = e^{in\theta}\mathbf{t}; \tag{80}$$

4) if $a_{2n} = 0, a_{1n} = b_{2n}$*, and* $b_{1n} \neq 0$*, the eigenvalues are*

$$\xi_{1n} = a_{1n}, \quad \xi_{2n} = a_{1n}, \tag{81}$$

there is only one eigenfunction

$$\mathbf{p}_{1n} = e^{in\theta}\boldsymbol{\nu}, \tag{82}$$

and another one is the generalized eigenfunction,

$$\mathbf{p}_{2n} = \frac{1}{b_{1n}} e^{in\theta}\mathbf{t}, \tag{83}$$

namely, $\mathbf{p}_{2n}$ *satisfies*

$$\left(\mathbf{K}^{\omega,*}_{\partial B_R} - \xi_{1n}\right)\mathbf{p}_{2n} = \mathbf{p}_{1n}. \tag{84}$$

Proof. We first know from Lemma 10 that

$$\mathbf{K}^{\omega,*}_{\partial B_R}[\boldsymbol{\nu}, \mathbf{t}] = (\boldsymbol{\nu}, \mathbf{t})T_n, \tag{85}$$

where T_n is a 2×2 matrix given by $T_n = (a_{1n}, b_{1n}; a_{2n}, b_{2n})$ Thus, we focus ourself on investigating the eigensystem of the matrix T_n, which could further lead to the eigensystem of the operator $\mathbf{K}^{\omega,*}_{\partial B_R}$. Specifically, we like to find the matrix $P_n = (\mathbf{p}_{1n}, \mathbf{p}_{2n})$ such that

$$T_n P_n = P_n \Lambda_n, \tag{86}$$

where the matrix Λ_n is a diagonal matrix, namely, $\Lambda_n = (\xi_{1n}, 0; 0, \xi_{2n})$. A direct calculation shows that if $a_{2n} \neq 0$, one has that

$$\begin{aligned}
\mathbf{p}_{1n} &= \left(\frac{a_{1n} - b_{2n} - \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}}{2a_{2n}}, 1\right)^t,\\
\mathbf{p}_{2n} &= \left(\frac{a_{1n} - b_{2n} + \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}}{2a_{2n}}, 1\right)^t,
\end{aligned} \tag{87}$$

$$\xi_{1n} = \frac{1}{2}\left(a_{1n} + b_{2n} - \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}\right), \tag{88}$$

$$\xi_{2n} = \frac{1}{2}\left(a_{1n} + b_{2n} + \sqrt{a_{1n}^2 - 2a_{1n}b_{2n} + 4a_{2n}b_{1n} + b_{2n}^2}\right). \tag{89}$$

For the case $a_{2n} = 0$ and $a_{1n} \neq b_{2n}$, one has that

$$\mathbf{p}_{1n} = (1, 0)^t, \quad \mathbf{p}_{2n} = \left(\frac{b_{1n}}{b_{2n} - a_{1n}}, 1\right)^t \quad \text{and} \quad \xi_{1n} = a_{1n}, \quad \xi_{2n} = b_{2n}. \tag{90}$$

Moreover, if $a_{2n} = 0, a_{1n} = b_{2n},$ and $b_{1n} = 0$, one has that

$$\mathbf{p}_{1n} = (0, 1)^t, \quad \mathbf{p}_{2n} = (1, 0)^t, \quad \xi_{1n} = a_{1n}, \quad \xi_{2n} = a_{1n}. \tag{91}$$

For the last case $a_{2n} = 0, a_{1n} = b_{2n},$ and $b_{1n} \neq 0$, the situation is different. The matrix Λ_n given in (86) is not a diagonal matrix anymore, but a Jordan matrix, given as follows:

$$\Lambda_n = \begin{pmatrix} a_{1n} & 1 \\ 0 & a_{2n} \end{pmatrix}. \tag{92}$$

Then, the generalized eigenvectors are given as $\mathbf{p}_{1n} = (1,0)^t$ and $\mathbf{p}_{2n} = (0, 1/b_{1n})^t$. Finally, with the help of the relationship (85), one can prove the statement of the theorem and the proof is completed. ■

Remark 8. We present the asymptotic expansion for the eigenvalues when the frequency $\omega \ll 1$. From the asymptotic expansions of the Bessel function and Hankel function in (29) and (30) for $\omega \ll 1$, one has that when $|n| \geq 2$,

$$|a_{2n}| = \frac{\mu}{2(\lambda + 2\mu)} + \mathcal{O}(\omega) \neq 0, \tag{93}$$

which is the first case in Theorem 2, thus the eigenvalues are

$$\xi_{1n} = -\frac{\mu}{2(\lambda + 2\mu)} + \mathcal{O}\big(\omega^2\big), \quad \xi_{2n} = \frac{\mu}{2(\lambda + 2\mu)} + \mathcal{O}\big(\omega^2\big). \tag{94}$$

When $|n| = 1$, one has that

$$|a_{2n}| = \frac{\mu}{2(\lambda + 2\mu)} + \mathcal{O}(\omega) \neq 0, \tag{95}$$

which is the first case in Theorem 2, thus the eigenvalues are

$$\xi_{1n} = \frac{\mu}{2(\lambda + 2\mu)} + o(\omega), \quad \xi_{2n} = \frac{1}{2} + \mathcal{O}\big(\omega^2\big). \tag{96}$$

When $n = 0$, one has that

$$a_{2n} = b_{1n} = 0, \quad \text{and} \quad a_{1n} \neq b_{2n}, \tag{97}$$

which is the second case in Theorem 2, thus the eigenvalues are

$$\xi_{1n} = -\frac{\lambda}{2(\lambda + 2\mu)} + o(\omega), \quad \xi_{2n} = \frac{1}{2} + \mathcal{O}\big(\omega^2\big). \tag{98}$$

These conclusions recover the results concerning the spectrum of the N-P operator in the static regime (cf. Refs. 4, 39).

3 | ELASTIC RESONANCES FOR MATERIAL STRUCTURES WITH NO CORE

In this section, we construct a broad class of elastic structures of the form $\mathbf{C}_0$ in (4) with no core, namely, $D = \emptyset$ that can induce resonances. All the notations below are carried over from Sections 1 and 2. Suppose that a source term $\mathbf{f}$ is supported outside Ω. In such a case, the elastic system (5) can be reduced into the following transmission problem:

$$\begin{cases} \mathcal{L}_{\hat{\lambda},\hat{\mu}}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ \mathcal{L}_{\lambda,\mu}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = \mathbf{f}, & \mathbf{x} \in \mathbb{R}^N \backslash \overline{\Omega}, \\ \mathbf{u}(\mathbf{x})|_- = \mathbf{u}(\mathbf{x})|_+, & \mathbf{x} \in \partial\Omega, \\ \partial_{\hat{\nu}}\mathbf{u}(\mathbf{x})|_- = \partial_{\nu}\mathbf{u}(\mathbf{x})|_+, & \mathbf{x} \in \partial\Omega\,. \end{cases} \tag{99}$$

3.1 Existence of resonances in generic scenarios

Using the single-layer potential in (21), the solution to this system can be written as:

$$\mathbf{u} = \begin{cases} \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\psi}_1](\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\psi}_2](\mathbf{x}) + \mathbf{F}, & \mathbf{x} \in \mathbb{R}^N \backslash \overline{\Omega}, \end{cases} \tag{100}$$

where $\mathbf{F}$ is called the Newtonian potential of the source $\mathbf{f}$ and $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in L^2(\partial\Omega)^N$:

$$\mathbf{F}(\mathbf{x}) := \int_{\mathbb{R}^N} \boldsymbol{\Gamma}^{\omega}(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^N. \tag{101}$$

One can readily verify that the solution defined in (100) satisfies the first two conditions in (99). For the third and fourth conditions in (99) across $\partial\Omega$, namely, the transmission conditions, one can obtain that

$$\begin{cases} \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\psi}_1] - \mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\psi}_2] = \mathbf{F}, \\ \partial_{\hat{\nu}}\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\boldsymbol{\psi}_1]|_- - \partial_{\nu}\mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\psi}_2]|_+ = \partial_{\nu}\mathbf{F}, \end{cases} \quad \mathbf{x} \in \partial\Omega. \tag{102}$$

With the help of the jump formula (22), Equation (102) can be rewritten as:

$$\mathbf{A}^{\omega}\begin{bmatrix}\boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2\end{bmatrix} = \begin{bmatrix}\mathbf{F} \\ \partial_{\nu}\mathbf{F}\end{bmatrix}, \tag{103}$$

where

$$\mathbf{A}^{\omega} = \begin{bmatrix} \hat{\mathbf{S}}^{\omega}_{\partial\Omega} & -\mathbf{S}^{\omega}_{\partial\Omega} \\ -1/2I + \hat{\mathbf{K}}^{\omega,*}_{\partial\Omega} & -1/2I - \mathbf{K}^{\omega,*}_{\partial\Omega} \end{bmatrix}. \tag{104}$$

Next, we show that the resonance could occur even for the domain Ω to be of a generic geometry. It is noted that $\hat{\mathbf{S}}^{\omega}_{\partial\Omega}$ and $\mathbf{S}^{\omega}_{\partial\Omega}$ are compact operators on $L^2(\partial\Omega)^N$ (cf. Ref. 32). Following the similar argument as that in the proof of Lemma 2, one can readily show that the spectrum of the operator $\mathbf{A}^{\omega}$ consists of the point spectrum only. Denoting by $\mathcal{H}_j$ the generalized eigenspace of $\mathbf{A}^{\omega}$ for the eigenvalue ξ_j, we can obtain the following result, by applying the Jordan theory directly to the operator $\mathbf{A}^{\omega}_{\delta}|_{\mathcal{H}_j} : \mathcal{H}_j \to \mathcal{H}_j$.

Lemma 11. *The generalized eigenspace $\mathcal{H}_j = \{\boldsymbol{\Psi}_{j,l,k}\}$, $1 \le l \le m_j$, $1 \le k \le n_{j,l}$ satisfies*

$$\mathbf{A}^{\omega}\left(\boldsymbol{\Psi}_{j,1,1}, \dots, \boldsymbol{\Psi}_{j,m_j,n_{j,m_j}}\right) = \left(\boldsymbol{\Psi}_{j,1,1}, \dots, \boldsymbol{\Psi}_{j,m_j,n_{j,m_j}}\right)\begin{pmatrix} J_{j,1} & & \\ & \ddots & \\ & & J_{j,m_j}\end{pmatrix}, \tag{105}$$

where $J_{j,l}$ is the canonical Jordan matrix of size $n_{j,l}$ in the form

$$J_{j,l} = \begin{pmatrix} \xi_j & 1 & & \\ & \ddots & \ddots & \\ & & \xi_j & 1 \\ & & & \xi_j \end{pmatrix}. \tag{106}$$

The following theorem presents the existence of resonances in generic scenarios.

Theorem 3. *Let $(\xi_j, \mathbf{\Psi})$ be the eigenpair defined in Lemma 11. Assume that the source term is chosen as follows:*

$$\begin{bmatrix} \mathbf{F} \\ \partial_\nu \mathbf{F} \end{bmatrix} = \sum_{k=1}^{p_j} f_k \mathbf{\Psi}_{j,1,k}, \tag{107}$$

where f_k are the coefficients and $p_j = \max\{n_{j,l}\}_{l=1}^{m_j}$ with $n_{j,l}$ defined in Lemma 11. If the parameters are properly chosen such that for $M \gg 1$,

$$\frac{f_{p_j}^2}{\xi_j^{2p_j}} \Im\left(\int_{\partial\Omega} \hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\mathbf{\Psi}_{j,1,1}] \cdot \overline{\partial_{\hat{\nu}} \hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\mathbf{\Psi}_{j,1,1}]|_-} \right) > M, \tag{108}$$

then the resonance occurs in the sense of Definition 1.

Proof. From Lemma 11 and the choice of the source in (107), the density function can be written as:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \sum_{k=1}^{p_j} g_k \mathbf{\Psi}_{j,1,k}, \tag{109}$$

where g_k are the coefficients to be determined. With the help of (103) and Lemma 11, one has that

$$g_k = \frac{1}{\xi_j^{p_j-k+1}} \sum_{i=k}^{p_j} (f_i \xi_j^{p_j-i} (-1)^{i-k+1}). \tag{110}$$

Thus, g_1 has the following expression:

$$g_1 = \frac{f_{p_j}}{\xi_j^{p_j}} (-1)^{p_j} + \mathcal{O}(\xi_j^{1-p_j}). \tag{111}$$

Then, we have the following estimate for the dissipation energy:

$$\begin{aligned} E(\mathbf{u}) &= \Im P_{\hat{\lambda},\hat{\mu}}(\mathbf{u},\mathbf{u}) = \Im\left(\omega^2 \int_\Omega |\mathbf{u}|^2 d\mathbf{x} + \int_{\partial\Omega} \mathbf{u} \cdot \partial_\nu \overline{\mathbf{u}} \right) \\ &\geq |g_1|^2 \Im\left(\int_{\partial\Omega} \hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\mathbf{\Psi}_{j,1,1}] \cdot \overline{\partial_{\hat{\nu}} \hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\mathbf{\Psi}_{j,1,1}]|_-} \right) \\ &\geq \frac{f_{p_j}^2}{\xi_j^{2p_j}} \Im\left(\int_{\partial\Omega} \hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\mathbf{\Psi}_{j,1,1}] \cdot \overline{\partial_{\hat{\nu}} \hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\mathbf{\Psi}_{j,1,1}]|_-} \right). \end{aligned} \tag{112}$$

This completes the proof following from the condition (108). ■

Remark 9. The condition (108) generally can be satisfied. In fact, because the Lamé parameters $(\hat{\lambda}, \hat{\mu})$ in the domain Ω can break the strong convexity conditions in (2), hence the system (99) is allowed to lose the ellipticity. Thus, there exists a certain eigenvalue satisfying the condition $\xi_j \ll 1$. Next, we choose the domain Ω to be a circle to strictly verify the statement in Theorem 3 in two dimensions. For the three dimensions, readers may refer to the article.[3]

3.2 | Resonance and its quantitative behavior for circular domain

In this subsection, we consider the specific case that the domain Ω is a circle B_R. In such a case, we can have a deep understanding of the occurrence of the resonance as well as its quantitative behaviors. Because the source term $\mathbf{f}$ is supported outside B_R, there exists $\epsilon > 0$ such that when $\mathbf{x} \in B_{R+\epsilon}$, the Newtonian potential $\mathbf{F}$ defined in (101) satisfies

$$\mathcal{L}_{\lambda,\mu}\mathbf{F} + \omega^2\mathbf{F} = 0. \tag{113}$$

Thus, $\mathbf{F}$ can be written as:

$$\mathbf{F} = \sum_{n=-\infty}^{\infty}\left(\frac{\kappa_{1,n}k_sR}{nJ_n(k_sR)}\mathbf{Q}_n^i + \frac{\kappa_{2,n}k_pR}{nJ_n(k_pR)}\mathbf{P}_n^i\right), \tag{114}$$

where $\kappa_{1,n}, \kappa_{2,n}$ are the coefficients, and the functions $\mathbf{Q}_n^i$ and $\mathbf{P}_n^i$ are defined in (59) and (60). Here, $\frac{k_sR}{nJ_n(k_sR)}$ and $\frac{k_pR}{nJ_n(k_pR)}$ are the normalization constants. From the expressions for the functions $\mathbf{Q}_n^i$ and $\mathbf{P}_n^i$ in (59) and (60), one has that on the boundary ∂B_R:

$$\mathbf{F} = \sum_{n=-\infty}^{\infty}\mathbf{b}_n^t\mathbf{f}_n, \tag{115}$$

where

$$\mathbf{b}_n = \begin{pmatrix} e^{in\theta}\boldsymbol{\nu} \\ e^{in\theta}\mathbf{t}\end{pmatrix}, \quad \mathbf{f}_n = \begin{pmatrix} f_{1,n} \\ f_{2,n}\end{pmatrix} = \begin{pmatrix} \kappa_{1,n}\eta_{1,n} + \kappa_{2,n}\eta_{3,n} \\ \kappa_{1,n}\eta_{2,n} + \kappa_{2,n}\eta_{4,n}\end{pmatrix}, \tag{116}$$

with

$$\eta_{1,n} = 2, \quad \eta_{2,n} = \frac{2\mathrm{i}k_sRJ_n'(k_sR)}{nJ_n(k_sR)}, \quad \eta_{3,n} = \frac{2k_pRJ_n'(k_pR)}{nJ_n(k_pR)}, \quad \eta_{4,n} = 2\mathrm{i}. \tag{117}$$

Moreover, from (70), one has that on the boundary ∂B_R:

$$\partial_{\boldsymbol{\nu}}\mathbf{F} = \sum_{n=-\infty}^{\infty}\mathbf{b}_n^t\tilde{\mathbf{f}}_n, \tag{118}$$

where

$$\tilde{\mathbf{f}}_n = \begin{pmatrix} \tilde{f}_{1,n} \\ \tilde{f}_{2,n}\end{pmatrix} = \begin{pmatrix} \frac{\kappa_{1,n}\gamma_{1,n}k_sR}{nJ_n(k_sR)} + \frac{\kappa_{2,n}\gamma_{3,n}k_pR}{nJ_n(k_pR)} \\ \frac{\kappa_{1,n}\gamma_{2,n}k_sR}{nJ_n(k_sR)} + \frac{\kappa_{2,n}\gamma_{4,n}k_pR}{nJ_n(k_pR)}\end{pmatrix}, \tag{119}$$

with $\gamma_{i,n}$, $1 \le i \le 4$ given in (70).

From Lemmas 8 and 9, one has that under the basis $(e^{in\theta}\boldsymbol{\nu}, e^{in\theta}\mathbf{t})$, the operators $\mathbf{S}_{\partial\Omega}^{\omega}, \hat{\mathbf{S}}_{\partial\Omega}^{\omega}, \partial_{\boldsymbol{\nu}}\mathbf{S}_{\partial\Omega}^{\omega}[\boldsymbol{\psi}_2]|_+, \partial_{\hat{\boldsymbol{\nu}}}\hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\boldsymbol{\psi}_1]|_-$ have the following expressions:

$$\mathbf{S}_{\partial\Omega}^{\omega} = \mathcal{T}_{1n}, \quad \hat{\mathbf{S}}_{\partial\Omega}^{\omega} = \hat{\mathcal{T}}_{1n}, \quad \partial_{\boldsymbol{\nu}}\mathbf{S}_{\partial\Omega}^{\omega}[\boldsymbol{\psi}_2]|_+ = \mathcal{T}_{2n}, \quad \partial_{\hat{\boldsymbol{\nu}}}\hat{\mathbf{S}}_{\partial\Omega}^{\omega}[\boldsymbol{\psi}_1]|_- = \hat{\mathcal{T}}_{2n}, \tag{120}$$

where

$$\mathcal{T}_{1n} = \begin{pmatrix} \alpha_{1,n} & \alpha_{3,n} \\ \alpha_{2,n} & \alpha_{4,n} \end{pmatrix}, \quad \widehat{\mathcal{T}}_{1n} = \begin{pmatrix} \hat{\alpha}_{1,n} & \hat{\alpha}_{3,n} \\ \hat{\alpha}_{2,n} & \hat{\alpha}_{4,n} \end{pmatrix},$$

$$\mathcal{T}_{2n} = \begin{pmatrix} g_{1,n} & g_{3,n} \\ g_{2,n} & g_{4,n} \end{pmatrix}, \quad \widehat{\mathcal{T}}_{2n} = \begin{pmatrix} \hat{g}_{1,n} - 1 & \hat{g}_{3,n} \\ \hat{g}_{2,n} & \hat{g}_{4,n} - 1 \end{pmatrix}. \tag{121}$$

In the last equation, α_{in} and $g_{i,n}$ with $i = 1, 2, 3, 4$ are given in Lemmas 8 and 9, and $\hat{\alpha}_{in}$ and $\hat{g}_{i,n}$ with $i = 1, 2, 3, 4$ are also given in Lemmas 8 and 9 with (μ, λ) replaced by $(\hat{\mu}, \hat{\lambda})$.

Hence, the density functions $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ can be expressed by:

$$\boldsymbol{\psi}_1 = \sum_{n=-\infty}^{\infty} \mathbf{b}_n^t \boldsymbol{\psi}_{1,n}, \qquad \boldsymbol{\psi}_2 = \sum_{n=-\infty}^{\infty} \mathbf{b}_n^t \boldsymbol{\psi}_{2,n}, \tag{122}$$

where

$$\mathbf{b}_n = \begin{pmatrix} e^{in\theta}\boldsymbol{\nu} \\ e^{in\theta}\mathbf{t} \end{pmatrix}, \quad \boldsymbol{\psi}_{1,n} = \begin{pmatrix} \psi_{1,1,n} \\ \psi_{1,2,n} \end{pmatrix}, \quad \boldsymbol{\psi}_{2,n} = \begin{pmatrix} \psi_{2,1,n} \\ \psi_{2,2,n} \end{pmatrix}, \tag{123}$$

and the coefficients $\psi_{i,j,n}$, $1 \le i, j \le 2$ are needed to be determined. Thus, the system (102) can be written as for $-\infty < n < \infty$:

$$\begin{cases} \widehat{\mathcal{T}}_{1n}\boldsymbol{\psi}_{1,n} = \mathcal{T}_{1n}\boldsymbol{\psi}_{2,n} + \mathbf{f}_n, \\ \widehat{\mathcal{T}}_{2n}\boldsymbol{\psi}_{1,n} = \mathcal{T}_{2n}\boldsymbol{\psi}_{2,n} + \tilde{\mathbf{f}}_n. \end{cases} \tag{124}$$

Directly solving Equation (124) gives that

$$\psi_{1,1,n} = \frac{c_{1,n}}{d_n}, \quad \psi_{1,2,n} = \frac{c_{2,n}}{d_n}, \tag{125}$$

where

$$\begin{aligned} c_{1,n} = {} & (f_{2,n}\hat{\alpha}_{3,n} - f_{1,n}\hat{\alpha}_{4,n})(g_{3,n}g_{2,n} - g_{1,n}g_{4,n}) + (f_{2,n}(\hat{g}_{4,n} - 1) - \tilde{f}_{2,n}\hat{\alpha}_{4,n})(g_{1,n}\alpha_{3,n} - g_{3,n}\alpha_{1,n}) \\ & + (f_{2,n}\hat{g}_{3,n} - \tilde{f}_{1,n}\hat{\alpha}_{4,n})(g_{4,n}\alpha_{1,n} - g_{2,n}\alpha_{3,n}) + (f_{1,n}(\hat{g}_{4,n} - 1) - \tilde{f}_{2,n}\hat{\alpha}_{3,n})(g_{3,n}\alpha_{2,n} - g_{1,n}\alpha_{4,n}) \\ & + (f_{1,n}\hat{g}_{3,n} - \tilde{f}_{1,n}\hat{\alpha}_{3,n})(g_{2,n}\alpha_{4,n} - g_{4,n}\alpha_{2,n}) + (\tilde{f}_{1,n}(\hat{g}_{4,n} - 1) - \tilde{f}_{2,n}\hat{g}_{3,n})(\alpha_{1,n}\alpha_{4,n} - \alpha_{3,n}\alpha_{2,n}), \end{aligned} \tag{126}$$

$$\begin{aligned} c_{2,n} = {} & (f_{2,n}\hat{\alpha}_{1,n} - f_{1,n}\hat{\alpha}_{2,n})(g_{1,n}g_{4,n} - g_{3,n}g_{2,n}) + (\tilde{f}_{2,n}(\hat{g}_{1,n} - 1) - \tilde{f}_{1,n}\hat{g}_{2,n})(\alpha_{1,n}\alpha_{4,n} - \alpha_{3,n}\alpha_{2,n}) \\ & + (f_{1,n}\hat{g}_{2,n} - \tilde{f}_{2,n}\hat{\alpha}_{1,n})(g_{1,n}\alpha_{4,n} - g_{3,n}\alpha_{2,n}) + (f_{2,n}(\hat{g}_{1,n} - 1) - \tilde{f}_{1,n}\hat{\alpha}_{2,n})(g_{2,n}\alpha_{3,n} - g_{4,n}\alpha_{1,n}) \\ & + (f_{2,n}\hat{g}_{2,n} - \tilde{f}_{2,n}\hat{\alpha}_{2,n})(g_{3,n}\alpha_{1,n} - g_{1,n}\alpha_{3,n}) + (f_{1,n}(\hat{g}_{1,n} - 1) - \tilde{f}_{1,n}\hat{\alpha}_{1,n})(g_{4,n}\alpha_{2,n} - g_{2,n}\alpha_{4,n}), \end{aligned} \tag{127}$$

and

$$\begin{aligned} d_n = & (\hat{\alpha}_{1,n}\hat{\alpha}_{4,n} - \hat{\alpha}_{3,n}\hat{\alpha}_{2,n})(g_{1,n}g_{4,n} - g_{3,n}g_{2,n}) \\ & + (\hat{g}_{3,n}\hat{\alpha}_{2,n} - \hat{\alpha}_{4,n}(\hat{g}_{1,n} - 1))(g_{4,n}\alpha_{1,n} - g_{2,n}\alpha_{3,n}) \\ & + (\hat{g}_{3,n}\hat{\alpha}_{2,n} - \hat{\alpha}_{4,n}(\hat{g}_{1,n} - 1))(\alpha_{1,n}g_{4,n} - g_{2,n}\alpha_{3,n}) \\ & + (\hat{\alpha}_{3,n}\hat{g}_{2,n} - \hat{\alpha}_{1,n}(\hat{g}_{4,n} - 1))(g_{1,n}\alpha_{4,n} - g_{3,n}\alpha_{2,n}) \\ & + (\hat{\alpha}_{1,n}\hat{g}_{3,n} - \hat{\alpha}_{3,n}(\hat{g}_{1,n} - 1))(g_{2,n}\alpha_{4,n} - g_{4,n}\alpha_{2,n}) \\ & + (\hat{g}_{2,n}\hat{g}_{3,n}(\hat{g}_{4,n} - 1)(\hat{g}_{1,n} - 1))(\alpha_{3,n}\alpha_{2,n} - \alpha_{1,n}\alpha_{4,n}). \end{aligned} \tag{128}$$

Theorem 4. *Consider the configuration $\mathbf{C}_0$ with $D = \emptyset$ defined in (4) and a source term $\mathbf{f}$ supported outside the domain Ω. If the Lamé parameters $(\hat{\lambda}, \hat{\mu})$ inside the domain Ω are chosen such that for any $M \in \mathbb{R}_+$:*

$$|\psi_{1,1,n}|^2 \Im\left(\int_{\partial\Omega} \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{in\theta}\boldsymbol{\nu}] \cdot \overline{\partial_{\hat{\nu}}\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{in\theta}\boldsymbol{\nu}]}|_-\right) > M, \tag{129}$$

for some $n_0 \in \mathbb{N}$, where $\psi_{1,1,n_0}$ is defined in (125), then the elastic resonance occurs.

Proof. With the help of the Green's formula, the dissipation energy defined in (14) can be written as:

$$\begin{aligned} E(\mathbf{u}) = \Im P_{\hat{\lambda},\hat{\mu}}(\mathbf{u}, \mathbf{u}) = \Im\left(\omega^2 \int_{\Omega} |\mathbf{u}|^2 d\mathbf{x} + \int_{\partial\Omega} \mathbf{u} \cdot \partial_{\boldsymbol{\nu}}\overline{\mathbf{u}}\right) = \Im\left(\int_{\partial\Omega} \mathbf{u} \cdot \partial_{\boldsymbol{\nu}}\overline{\mathbf{u}}\right) \\ \geq |\psi_{1,1,n}|^2 \Im\left(\int_{\partial\Omega} \hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{in\theta}\boldsymbol{\nu}] \cdot \overline{\partial_{\hat{\nu}}\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{in\theta}\boldsymbol{\nu}]}|_-\right), \end{aligned} \tag{130}$$

which shows that the resonance occurs thanks to (129) and completes the proof. ■

Remark 10. If the Lamé parameters $(\hat{\lambda}, \hat{\mu})$ inside the domain Ω are chosen as follows:

$$(\hat{\lambda}, \hat{\mu}) = c(\lambda, \mu), \tag{131}$$

where (λ, μ) are the Lamé parameters in $\mathbb{R}^2 \backslash \overline{\Omega}$. For a large order n such that the asymptotic expansions (31) hold, the parameter c should have the following asymptotic expansion such that the condition (129) holds:

$$c = -\frac{\lambda + 3\mu}{\lambda + \mu} + \vartheta_n, \tag{132}$$

where $\vartheta_n = \mathcal{O}(1/n)$. In fact, for a large order n, the solutions of the equation (124) have the following asymptotic expansions:

$$\psi_{1,1,n} = \frac{\varepsilon_{1,n}}{((1+c)\lambda + (3+c)\mu)\varrho + \mathcal{O}(1/n)}, \quad \psi_{1,2,n} = \frac{\varepsilon_{2,n}}{((1+c)\lambda + (3+c)\mu)\varrho + \mathcal{O}(1/n)}, \tag{133}$$

where ϱ is a constant not depending on n and

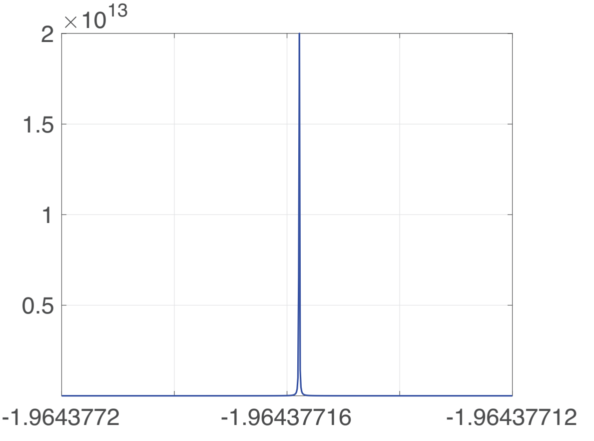

FIGURE 1 The value of the LHS in (129) with respect to $\Re c$. Horizontal axis: value of $\Re c$; Vertical axis: value of the LHS in (129)

$$\begin{aligned} \varepsilon_{1,n} =& \big((c-1)\omega^2R^2(\lambda+\mu)(\lambda+3\mu) - 8c\mu(\lambda+2\mu)((c+3)\lambda+(c+7)\mu)\big) \\ & \times 256c\mu^2 n(c_1+c_2)(\lambda+2\mu)^3\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \end{aligned} \tag{134}$$

$$\begin{aligned} \varepsilon_{2,n} =& -\big((c-1)\omega^2R^2(\lambda+\mu)(\lambda+3\mu) + 8c\mu(\lambda+2\mu)(3c\lambda+(3c+5)\mu+\lambda)\big) \\ & \times 256ic\mu^2 n(c_1+c_2)(\lambda+2\mu)^3\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \end{aligned} \tag{135}$$

Thus, one can readily conclude that the parameter c should have the following asymptotic expansion:

$$c = -\frac{\lambda+3\mu}{\lambda+\mu} + \mathcal{O}\left(\frac{1}{n}\right), \tag{136}$$

such that the condition (129) holds.

Next, we show that the condition (129) can be achieved. The Lamé parameters inside the domain Ω are chosen as those in (131). The other parameters are chosen as follows:

$$n = 5, \quad \lambda = \mu = \omega = R = 1, \quad \Im c = 2.08 \times 10^{-9}. \tag{137}$$

This is the case beyond the quasistatic approximation from the values of ω and R. The value of the LHS in (129) with respect to the real part of c, that is, $\Re c$, is plotted in Figure 1. This clearly shows that the condition (129) is fulfilled and thus the resonance occurs.

Remark 11. To ensure the occurrence of the resonance, that is, the condition (15) is fulfilled, in the quasistatic case, the condition $\Im c \to 0$ is required (cf. Refs. 13, 29). However, in our current case beyond the quasistatic regime, one usually requires $\Im c \to c^*$ with $c^* \neq 0$. This is a sharp difference from the quasistatic case. Next, we conduct a numerical simulation to verify this statement. The parameters are chosen as follows:

$$n = 5, \quad \lambda = \mu = \omega = R = 1, \quad \Re c = -1.9643, \tag{138}$$

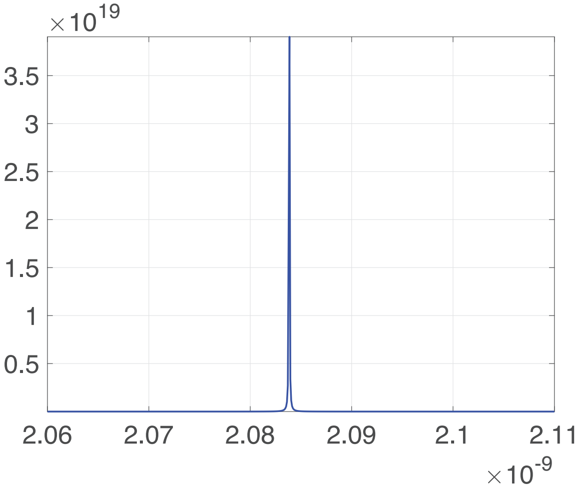

FIGURE 2 The value of the LHS in (129) with respect to $\Im c$. Horizontal axis: value of $\Im c$; Vertical axis: value of the LHS in (129)

which is the case beyond the quasistatic approximation from the values of ω and R. The value of the LHS in (129) with respect to the imaginary part of c, that is, $\Im c$, is plotted in Figure 2. This clearly shows that the resonance occurs and the critical value $\Im c \neq 0$.

Finally, we consider the quantitative behaviors of the resonant fields when resonance occurs. It is recalled that in the static/quasistatic regime, the plasmon/polariton resonances are localized around the metamaterial interface. However, we will show that in the frequency regime beyond the quasistatic approximation, the resonant oscillation outside the material structure is localized around the metamaterial interface, but inside the material structure, it is not localized around the interface, which is in sharp contrast to the subwavelength resonance. In fact, from the expression of the solution in (100) and the density functions in (122), it is sufficient to analyze the properties of single-layer potentials $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\boldsymbol{\nu}](\mathbf{x})$ and $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\mathbf{t}](\mathbf{x})$ expressed in Theorem 1 and Proposition 1 for $\mathbf{x}$ lying in different regions. Here, we only take the term $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\boldsymbol{\nu}](\mathbf{x})$ to illustrate the phenomenon as the discussion is the same for the term $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\mathbf{t}](\mathbf{x})$. The parameters are chosen as follows:

$$n = 5, \quad \lambda = \mu = R = 1, \quad \omega = 20, \tag{139}$$

which is the case beyond the quasistatic approximation from the values of ω and R. The amplitude of the single-layer potential $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\boldsymbol{\nu}](\mathbf{x})$ for $|\mathbf{x}| \leq 1$ and $1 < |\mathbf{x}| < 3$ is plotted in Figure 3A and B, respectively. From the plot, one can conclude that the field outside B_1 is localized around the surface ∂B_1, while the field inside B_1 is not localized around the boundary.

If we choose the parameters as follows:

$$n = 5, \quad \lambda = \mu = R = 1, \quad \omega = 0.1, \tag{140}$$

which is the case of the quasistatic approximation. The amplitude of the single-layer potential $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\boldsymbol{\nu}](\mathbf{x})$ for $|\mathbf{x}| \leq 1$ and $1 < |\mathbf{x}| < 2$ is plotted in Figure 4A and B, respectively. From the plot, one can conclude that the fields both inside and outside B_1 are localized around the surface ∂B_1. Finally, we would like to remark that by using the relevant results in Ref. 29, one can show that the elastodynamical resonances in 3D reveal similar behaviors as the 2D case discussed above.

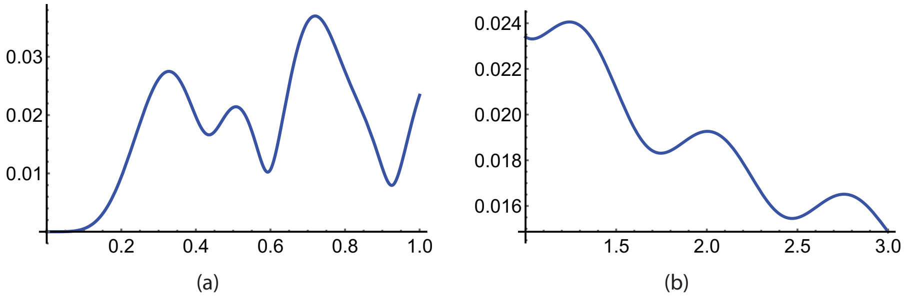

FIGURE 3 The amplitude of the single-layer potential $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\boldsymbol{\nu}](\mathbf{x})$ with parameters chosen in (139) for (A) $|\mathbf{x}| \le 1$; (B) $1 < |\mathbf{x}| \le 3$

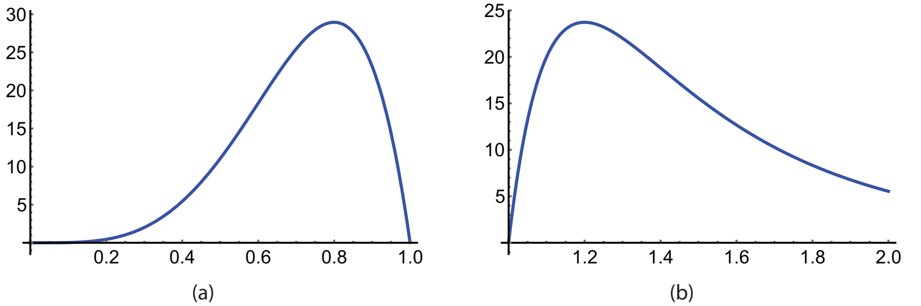

FIGURE 4 The amplitude of the single-layer potential $\mathbf{S}^{\omega}_{\partial B_R}[e^{in\theta}\boldsymbol{\nu}](\mathbf{x})$ with parameters chosen in (140) for (A) $|\mathbf{x}| \le 1$; (B) $1 < |\mathbf{x}| \le 2$

4 | CALR FOR A CORE–SHELL STRUCTURE BEYOND THE QUASISTATIC APPROXIMATION

In this section, we construct a core–shell elastic structure that can induce ALR; see Definition 1. We confine our study in two dimensions and as mentioned earlier, we refer to Ref. 3 for related studies in the three-dimensional case. In what follows, we let $D = B_{r_i}$ and $\Omega = B_{r_e}$, $r_e > r_i$. Moreover, we let $\mathcal{L}_{\breve{\lambda},\breve{\mu}}$, $\partial_{\breve{\boldsymbol{\nu}}}$, $\breve{\mathbf{S}}_{\partial D,}$ and $(\breve{\mathbf{K}}^{\omega}_{\partial D})^*$, respectively, denote the Lamé operator, the associated conormal derivative, the single-layer potential operator, and the N-P operator associated with the Lamé parameters $(\breve{\lambda}, \breve{\mu})$.

Assume that the source $\mathbf{f}$ is supported outside Ω. Associated with the material structure $\mathbf{C}_0$ in (4) with D and Ω given above, the elastic system (5) becomes

$$
\begin{cases}
\mathcal{L}_{\breve{\lambda},\breve{\mu}}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = 0, & \text{in } D,\\
\mathcal{L}_{\hat{\lambda},\hat{\mu}}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = 0, & \text{in } \Omega\backslash\overline{D},\\
\mathcal{L}_{\lambda,\mu}\mathbf{u}(\mathbf{x}) + \omega^2\mathbf{u}(\mathbf{x}) = \mathbf{f}, & \text{in } \mathbb{R}^2\backslash\overline{\Omega},\\
\mathbf{u}|_- = \mathbf{u}|_+, \quad \partial_{\breve{\boldsymbol{\nu}}}\mathbf{u}|_- = \partial_{\hat{\boldsymbol{\nu}}}\mathbf{u}|_+ & \text{on } \partial D,\\
\mathbf{u}|_- = \mathbf{u}|_+, \quad \partial_{\hat{\boldsymbol{\nu}}}\mathbf{u}|_- = \partial_{\boldsymbol{\nu}}\mathbf{u}|_+ & \text{on } \partial\Omega.
\end{cases}
\tag{141}
$$

With the help of the potential theory, the solution to the equation system (141) can be represented by:

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \breve{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_1](\mathbf{x}), & \mathbf{x} \in D, \\ \hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2](\mathbf{x}) + \hat{\mathbf{S}}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_3](\mathbf{x}), & \mathbf{x} \in \Omega \backslash \overline{D}, \\ \mathbf{S}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_4](\mathbf{x}) + \mathbf{F}(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^2 \backslash \overline{\Omega}, \end{cases} \tag{142}$$

where $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \boldsymbol{\varphi}_3, \boldsymbol{\varphi}_4 \in L^2(\partial D)^2$ and $\mathbf{F}$ is the Newtonian potential of the source $\mathbf{f}$ defined in (101). One can easily see that the solution given (142) satisfies the first three conditions in (141) and the last two conditions on the boundary yield that

$$\begin{cases} \breve{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_1] = \hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_3], & \text{on} \quad \partial D, \\ \partial_{\breve{\nu}} \breve{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_1]|_{-} = \partial_{\hat{\nu}} (\hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_3])|_{+}, & \text{on} \quad \partial D, \\ \hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_3] = \mathbf{S}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_4] + \mathbf{F}, & \text{on} \quad \partial \Omega, \\ \partial_{\hat{\nu}} (\hat{\mathbf{S}}^{\omega}_{\partial D}[\boldsymbol{\varphi}_2] + \hat{\mathbf{S}}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_3])|_{-} = \partial_{\nu} (\mathbf{S}^{\omega}_{\partial \Omega}[\boldsymbol{\varphi}_4] + \mathbf{F})|_{+}, & \text{on} \quad \partial \Omega. \end{cases} \tag{143}$$

With the help of the jump formula in (22), the equation system (143) further yields the following integral system:

$$\begin{bmatrix} \breve{\mathbf{S}}^{\omega}_{\partial D} & -\hat{\mathbf{S}}^{\omega}_{\partial D,i} & -\hat{\mathbf{S}}^{\omega}_{\partial \Omega,i} & 0 \\ -\frac{1}{2} + (\breve{\mathbf{K}}^{\omega}_{\partial D})^* & -\frac{1}{2} - (\hat{\mathbf{K}}^{\omega}_{\partial D})^* & -\partial_{\hat{\nu}_i} \hat{\mathbf{S}}^{\omega}_{\partial \Omega} & 0 \\ 0 & \hat{\mathbf{S}}^{\omega}_{\partial D,e} & \hat{\mathbf{S}}^{\omega}_{\partial \Omega,e} & -\mathbf{S}^{\omega}_{\partial \Omega} \\ 0 & \partial_{\hat{\nu}_e} \hat{\mathbf{S}}^{\omega}_{\partial D} & -\frac{1}{2} + (\hat{\mathbf{K}}^{\omega}_{\partial \Omega})^* & -\frac{1}{2} - (\mathbf{K}^{\omega}_{\partial \Omega})^* \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\varphi}_3 \\ \boldsymbol{\varphi}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{F} \\ \partial_{\nu} \mathbf{F} \end{bmatrix}, \tag{144}$$

where $\partial_{\hat{\nu}_i}$ and $\partial_{\hat{\nu}_e}$ signify the conormal derivatives on the boundaries of D and Ω, respectively.

Following similar arguments as those in the previous section, there exists $\epsilon > 0$ such that when $\mathbf{x} \in B_{r_e+\epsilon}$ the Newtonian potential $\mathbf{F}$ can be written as:

$$\mathbf{F} = \sum_{n \geq N} \left(\frac{\varkappa_{1,n} k_s r_e}{n J_n(k_s r_e)} \mathbf{Q}^i_n \right), \tag{145}$$

where $\varkappa_{1,n}$ are the coefficients, the functions $\mathbf{Q}^i_n$ are defined in (59), and N is large enough such that the spherical Bessel and Hankel functions, $J_n(t)$ and $H_n(t)$, fulfill the asymptotic expansions shown in (29). We would like to remark that the Newtonian potential $\mathbf{F}$ only contains the term $\mathbf{Q}^i_n$. Indeed, one can also include the term $\mathbf{P}^i_n$ and the analysis will be similar. To ease the exposition, we only consider the case that the Newtonian potential $\mathbf{F}$ contains the term $\mathbf{Q}^i_n$ only. From the expressions for the functions $\mathbf{Q}^i_n$ in (59), one has that on the boundary ∂B_R:

$$\mathbf{F} = \sum_{n \geq N} \mathbf{b}^t_n \mathbf{f}_n, \tag{146}$$

where

$$\mathbf{b}_n = \begin{pmatrix} e^{in\theta} \boldsymbol{\nu} \\ e^{in\theta} \mathbf{t} \end{pmatrix}, \quad \mathbf{f}_n = \begin{pmatrix} f_{1,n} \\ f_{2,n} \end{pmatrix} = \begin{pmatrix} \varkappa_{1,n} \eta_{1,n} \\ \varkappa_{1,n} \eta_{2,n} \end{pmatrix}, \tag{147}$$

with

$$\eta_{1,n} = 2, \quad \eta_{2,n} = \frac{2ik_s r_e J_n'(k_s r_e)}{nJ_n(k_s r_e)}. \tag{148}$$

Moreover, from the identities in (70), one has that on the boundary ∂B_{r_e}:

$$\partial_\nu \mathbf{F} = \sum_{n\geq N} \mathbf{b}_n^t \tilde{\mathbf{f}}_n, \tag{149}$$

where

$$\tilde{\mathbf{f}}_n = \begin{pmatrix} \tilde{f}_{1,n} \\ \tilde{f}_{2,n} \end{pmatrix} = \begin{pmatrix} \frac{\varkappa_{1,n}\gamma_{1,n}k_s r_e}{nJ_n(k_s r_e)} \\ \frac{\varkappa_{1,n}\gamma_{2,n}k_s r_e}{nJ_n(k_s r_e)} \end{pmatrix}, \tag{150}$$

with $\gamma_{i,n}$, $1 \leq i \leq 2$ given in (70) with R replaced by r_e.

Lemmas 8 and 9 show that based on the basis $(e^{in\theta}\boldsymbol{\nu}, e^{in\theta}\mathbf{t})$, the operators in the system (144) have the following expressions:

$$\begin{aligned}
&\breve{\mathbf{S}}_{\partial D}^{\omega} = \mathcal{R}_{11n}, \quad \hat{\mathbf{S}}_{\partial D,i}^{\omega} = \mathcal{R}_{12n}, \quad \hat{\mathbf{S}}_{\partial \Omega,i}^{\omega} = \mathcal{R}_{13n}, \quad (\breve{\mathbf{K}}_{\partial D}^{\omega})^* = \mathcal{R}_{21n}, \\
&(\hat{\mathbf{K}}_{\partial D}^{\omega})^* = \mathcal{R}_{22n}, \quad \partial_{\hat{\nu}_i}\hat{\mathbf{S}}_{\partial \Omega}^{\omega} = \mathcal{R}_{23n}, \quad \hat{\mathbf{S}}_{\partial D,e}^{\omega} = \mathcal{R}_{32n}, \quad \hat{\mathbf{S}}_{\partial \Omega,e}^{\omega} = \mathcal{R}_{33n}, \\
&\mathbf{S}_{\partial \Omega}^{\omega} = \mathcal{R}_{34n}, \quad \partial_{\hat{\nu}_e}\hat{\mathbf{S}}_{\partial D}^{\omega} = \mathcal{R}_{42n}, \quad (\hat{\mathbf{K}}_{\partial \Omega}^{\omega})^* = \mathcal{R}_{43n}, \quad (\mathbf{K}_{\partial \Omega}^{\omega})^* = \mathcal{R}_{44n},
\end{aligned} \tag{151}$$

where

$$\mathcal{R}_{11n} = \begin{pmatrix} \breve{\alpha}_{1ni} & \breve{\alpha}_{3ni} \\ \breve{\alpha}_{2ni} & \breve{\alpha}_{4ni} \end{pmatrix}, \quad \mathcal{R}_{12n} = \begin{pmatrix} \hat{\alpha}_{1ni} & \hat{\alpha}_{3ni} \\ \hat{\alpha}_{2ni} & \hat{\alpha}_{4ni} \end{pmatrix}, \quad \mathcal{R}_{13n} = \begin{pmatrix} \hat{\eta}_{1ni} & \hat{\eta}_{3ni} \\ \hat{\eta}_{2ni} & \hat{\eta}_{4ni} \end{pmatrix}, \tag{152}$$

$$\mathcal{R}_{21n} = \begin{pmatrix} \breve{a}_{1ni} & \breve{b}_{1ni} \\ \breve{a}_{2ni} & \breve{b}_{2ni} \end{pmatrix}, \quad \mathcal{R}_{22n} = \begin{pmatrix} \hat{a}_{1ni} & \hat{b}_{1ni} \\ \hat{a}_{2ni} & \hat{b}_{2ni} \end{pmatrix}, \quad \mathcal{R}_{23n} = \begin{pmatrix} \hat{\zeta}_{1ni} & \hat{\zeta}_{3ni} \\ \hat{\zeta}_{2ni} & \hat{\zeta}_{4ni} \end{pmatrix}, \tag{153}$$

$$\mathcal{R}_{32n} = \begin{pmatrix} \hat{\eta}_{1ne} & \hat{\eta}_{3ne} \\ \hat{\eta}_{2ne} & \hat{\eta}_{4ne} \end{pmatrix}, \quad \mathcal{R}_{33n} = \begin{pmatrix} \hat{\alpha}_{1ne} & \hat{\alpha}_{3ne} \\ \hat{\alpha}_{2ne} & \hat{\alpha}_{4ne} \end{pmatrix}, \quad \mathcal{R}_{34n} = \begin{pmatrix} \alpha_{1ne} & \alpha_{3ne} \\ \alpha_{2ne} & \alpha_{4ne} \end{pmatrix}, \tag{154}$$

$$\mathcal{R}_{42n} = \begin{pmatrix} \hat{\zeta}_{1ne} & \hat{\zeta}_{3ne} \\ \hat{\zeta}_{2ne} & \hat{\zeta}_{4ne} \end{pmatrix}, \quad \mathcal{R}_{43n} = \begin{pmatrix} \hat{a}_{1ne} & \hat{b}_{1ne} \\ \hat{a}_{2ne} & \hat{b}_{2ne} \end{pmatrix}, \quad \mathcal{R}_{44n} = \begin{pmatrix} a_{1ne} & b_{1ne} \\ a_{2ne} & b_{2ne} \end{pmatrix}. \tag{155}$$

In the above expressions, $\breve{\alpha}_{jni}$ with $j = 1,2,3,4$ are given in Lemma 8 with R replaced by r_i and with (μ, λ) replaced by $(\breve{\mu}, \breve{\lambda})$, and $\breve{a}_{jni}$ $\breve{b}_{jni}$ with $j = 1,2$ are given in Lemma 10 with R replaced by r_i and with (μ, λ) replaced by $(\breve{\mu}, \breve{\lambda})$. The same principle holds for parameters $\hat{\alpha}_{jni}, \hat{a}_{jni}, \hat{b}_{jni}, \hat{\alpha}_{jne}, \hat{a}_{jne}, \hat{b}_{jne}, \alpha_{jne}, a_{jne}, b_{jne}$ and the other parameters are given as follows:

$$\hat{\eta}_{1ni} = -\frac{i\pi}{2\omega^2 r_i}\left(n^2 J_n(\hat{k}_s r_i)H_n(\hat{k}_s r_e) + \hat{k}_p^2 r_i r_e J_n'(\hat{k}_p r_i)H_n'(\hat{k}_p r_e)\right), \tag{156}$$

$$\hat{\eta}_{2ni} = \frac{n\pi}{2\omega^2 r_i}\left(\hat{k}_s r_i J_n'(\hat{k}_s r_i) H_n(\hat{k}_s r_e) + \hat{k}_p r_e J_n(\hat{k}_p r_i) H_n'(\hat{k}_p r_e)\right), \tag{157}$$

$$\hat{\eta}_{3ni} = -\frac{n\pi}{2\omega^2 r_i}\left(\hat{k}_s r_e J_n(\hat{k}_s r_i) H_n'(\hat{k}_s r_e) + \hat{k}_p r_i J_n'(\hat{k}_p r_i) H_n(\hat{k}_p r_e)\right), \tag{158}$$

$$\hat{\eta}_{4ni} = -\frac{\mathrm{i}\pi}{2\omega^2 r_i}\left(\hat{k}_s^2 r_e r_i J_n'(\hat{k}_s r_i) H_n'(\hat{k}_s r_e) + n^2 J_n(\hat{k}_p r_i) H_n(\hat{k}_p r_e)\right); \tag{159}$$

$$\hat{\eta}_{1ne} = -\frac{\mathrm{i}\pi}{2\omega^2 r_e}\left(n^2 J_n(\hat{k}_s r_i) H_n(\hat{k}_s r_e) + \hat{k}_p^2 r_i r_e J_n'(\hat{k}_p r_i) H_n'(\hat{k}_p r_e)\right), \tag{160}$$

$$\hat{\eta}_{2ne} = \frac{n\pi}{2\omega^2 r_e}\left(\hat{k}_s r_e J_n(\hat{k}_s r_i) H_n'(\hat{k}_s r_e) + \hat{k}_p r_i J_n'(\hat{k}_p r_i) H_n(\hat{k}_p r_e)\right), \tag{161}$$

$$\hat{\eta}_{3ne} = -\frac{n\pi}{2\omega^2 r_e}\left(\hat{k}_s r_i J_n'(\hat{k}_s r_i) H_n(\hat{k}_s r_e) + \hat{k}_p r_e J_n(\hat{k}_p r_i) H_n'(\hat{k}_p r_e)\right), \tag{162}$$

$$\hat{\eta}_{4ne} = -\frac{\mathrm{i}\pi}{2\omega^2 r_e}\left(\hat{k}_s^2 r_e r_i J_n'(\hat{k}_s r_i) H_n'(\hat{k}_s r_e) + n^2 J_n(\hat{k}_p r_i) H_n(\hat{k}_p r_e)\right); \tag{163}$$

$$\begin{aligned}\hat{\zeta}_{1ni} =& \frac{\mathrm{i}\pi}{2\omega^2 r_2^2}\Big(2\hat{\mu} n^2 J_n(\hat{k}_s r_1)\big(H_n(\hat{k}_s r_2) - \hat{k}_s r_2 H_n'(\hat{k}_s r_2)\big) + \\ & J_n'(\hat{k}_p r_1)\hat{k}_p r_1\big(H_n(\hat{k}_p r_2)\big(\omega^2 r_2^2 - 2\hat{\mu} n^2\big) + 2\hat{k}_p \hat{\mu} r_2 H_n'(\hat{k}_p r_2)\big)\Big),\end{aligned} \tag{164}$$

$$\begin{aligned}\hat{\zeta}_{2ni} =& \frac{n\hat{\mu}\pi}{2\omega^2 r_2^2}\Big(H_n(\hat{k}_s r_2) J_n(\hat{k}_s r_1)\big(2n^2 - \hat{k}_s^2 {r_2}^2\big) - 2J_n'(\hat{k}_p r_1)\hat{k}_p r_1 \times \\ & \big(H_n(\hat{k}_p r_2) - H_n'(\hat{k}_p r_2)\hat{k}_p r_2\big) - 2H_n'(\hat{k}_s r_2) J_n(\hat{k}_s r_1)\hat{k}_s r_2\Big),\end{aligned} \tag{165}$$

$$\begin{aligned}\hat{\zeta}_{3ni} =& \frac{-n\pi}{2\omega^2 r_2^2}\Big(H_n(\hat{k}_p r_2) J_n(\hat{k}_p r_1)\big(2\hat{\mu} n^2 - \omega^2 {r_2}^2\big) - 2\hat{\mu} J_n'(\hat{k}_s r_1)\hat{k}_s r_1 \times \\ & \big(H_n(\hat{k}_s r_2) - H_n'(\hat{k}_s r_2)\hat{k}_s r_2\big) - 2\hat{\mu} H_n'(\hat{k}_p r_2) J_n(\hat{k}_p r_1)\hat{k}_p r_2\Big),\end{aligned} \tag{166}$$

$$\begin{aligned}\hat{\zeta}_{4ni} =& \frac{\mathrm{i}\hat{\mu}\pi}{2\omega^2 r_2^2}\Big(2n^2 J_n(\hat{k}_p r_1)\big(H_n(\hat{k}_p r_2) - \hat{k}_p r_2 H_n'(\hat{k}_p r_2)\big) + \\ & J_n'(\hat{k}_s r_1)\hat{k}_s r_1\big(H_n(\hat{k}_s r_2)\big(k_s^2 r_2^2 - 2n^2\big) + 2\hat{k}_s r_2 H_n'(\hat{k}_s r_2)\big)\Big);\end{aligned} \tag{167}$$

$$\begin{aligned}\hat{\zeta}_{1ne} =& \frac{\mathrm{i}\pi}{2\omega^2 r_1^2}\Big(2\hat{\mu} n^2 H_n(\hat{k}_s r_2)\big(J_n(\hat{k}_s r_1) - \hat{k}_s r_1 J_n'(\hat{k}_s r_1)\big) + \\ & H_n'(\hat{k}_p r_2)\hat{k}_p r_2\big(J_n(\hat{k}_p r_1)\big(\omega^2 r_1^2 - 2\hat{\mu} n^2\big) + 2\hat{k}_p \hat{\mu} r_1 H_n'(\hat{k}_p r_1)\big)\Big),\end{aligned} \tag{168}$$

$$\begin{aligned}\hat{\zeta}_{2ne} =& \frac{n\hat{\mu}\pi}{2\omega^2 r_2^2}\Big(J_n(\hat{k}_s r_1) H_n(\hat{k}_s r_2)\big(2n^2 - \hat{k}_s^2 {r_1}^2\big) - 2H_n'(\hat{k}_p r_2)\hat{k}_p r_2 \times \\ & \big(J_n(\hat{k}_p r_1) - J_n'(\hat{k}_p r_1)\hat{k}_p r_1\big) - 2J_n'(\hat{k}_s r_1) H_n(\hat{k}_s r_2)\hat{k}_s r_1\Big),\end{aligned} \tag{169}$$

$$\begin{aligned}\hat{\zeta}_{3ne} =& \frac{-n\pi}{2\omega^2 r_1^2}\Big(J_n(\hat{k}_p r_1) H_n(\hat{k}_p r_2)\big(2\hat{\mu} n^2 - \omega^2 {r_1}^2\big) - 2\hat{\mu} H_n'(\hat{k}_s r_2)\hat{k}_s r_2 \times \\ & \big(J_n(\hat{k}_s r_1) - J_n'(\hat{k}_s r_1)\hat{k}_s r_1\big) - 2\hat{\mu} J_n'(\hat{k}_p r_1) H_n(\hat{k}_p r_2)\hat{k}_p r_1\Big),\end{aligned} \tag{170}$$

$$\hat{\zeta}_{4ne} = \frac{\mathrm{i}\hat{\mu}\pi}{2\omega^2 r_1^2}\big(2n^2 J_n(\hat{k}_p r_2)\big(H_n(\hat{k}_p r_1) - \hat{k}_p r_1 H_n'(\hat{k}_p r_1)\big) + \\ J_n'(\hat{k}_s r_2)\hat{k}_s r_2\big(H_n(\hat{k}_s r_1)\big(k_s^2 r_1^2 - 2n^2\big) + 2\hat{k}_s r_1 H_n'(\hat{k}_s r_1)\big)\big). \tag{171}$$

The density functions $\boldsymbol{\varphi}_i$ with $i = 1, 2, 3, 4$ can be written as:

$$\boldsymbol{\varphi}_i = \sum_{n=-\infty}^{\infty} \mathbf{b}_n^t \boldsymbol{\varphi}_{i,n}, \tag{172}$$

where

$$\mathbf{b}_n = \begin{pmatrix} e^{\mathrm{i}n\theta}\boldsymbol{\nu} \\ e^{\mathrm{i}n\theta}\mathbf{t} \end{pmatrix}, \quad \boldsymbol{\varphi}_{i,n} = \begin{pmatrix} \psi_{i,1,n} \\ \psi_{i,2,n} \end{pmatrix}, \tag{173}$$

and the coefficients $\psi_{i,j,n}$, $1 \le i \le 4$, $j = 1, 2$ are needed to be determined from the system (144). Based on the discussion above, the system (144) is equivalent to solving the system

$$\mathbf{M}\begin{bmatrix} \boldsymbol{\varphi}_{1,n} \\ \boldsymbol{\varphi}_{2,n} \\ \boldsymbol{\varphi}_{3,n} \\ \boldsymbol{\varphi}_{4,n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{f}_n \\ \tilde{\mathbf{f}}_n \end{bmatrix} \tag{174}$$

where $\mathbf{f}$ and $\tilde{\mathbf{f}}$ are given in (146) and (149), respectively, and the matrix $\mathbf{M}$ is given by:

$$\mathbf{M} = \begin{bmatrix} \mathcal{R}_{11n} & \mathcal{R}_{12n} & \mathcal{R}_{13n} & 0 \\ \mathcal{R}_{21n} & \mathcal{R}_{22n} & \mathcal{R}_{23n} & 0 \\ 0 & \mathcal{R}_{32n} & \mathcal{R}_{33n} & \mathcal{R}_{34n} \\ 0 & \mathcal{R}_{42n} & \mathcal{R}_{43n} & \mathcal{R}_{44n} \end{bmatrix}. \tag{175}$$

Theorem 5. *Consider the configuration* $(\mathbf{C}_0, \mathbf{f})$ *where* $\mathbf{C}_0$ *is given in (4) and the Newtonian potential* $\mathbf{F}$ *of the source term* $\mathbf{f}$ *has the expression shown in (145). If the parameters in* $\mathbf{C}_0$ *are chosen as follows:*

$$\breve{\lambda} = \lambda,\ \ \breve{\mu} = \mu,\ \ \hat{\mu} = \left(-\frac{\lambda+\mu}{\lambda+3\mu} + \mathrm{i}\delta + p_{n_0}\right)\mu,\ \hat{\lambda} = \left(-\frac{\lambda+\mu}{\lambda+3\mu} + \mathrm{i}\delta + p_{n_0}\right)\lambda,\ \ \delta = \rho^{n_0}, \tag{176}$$

for some n_0, *where* $\rho = r_i/r_e$ *and* $p_{n_0} = \mathcal{O}(1/n_0)$ *are chosen such that*

$$\det \mathbf{M} = \mathcal{O}\big(\rho^{4n_0}/n_0\big), \tag{177}$$

then ALR occurs if the source $\mathbf{f}$ *is supported inside the critical radius* $r_* = \sqrt{r_e^3/r_i}$. *Moreover, if the source is supported outside* B_{r_*}, *then no resonance occurs.*

Proof. The proof of this theorem is divided into three parts. In the first part, we solve the system (174) to obtain the coefficients $\varphi_{i,j,n}$, $2 \le i \le 4$, $j = 1, 2$ of the density functions $\boldsymbol{\varphi}_{i,n}$. In the second part, we show that the ALR occurs if the source $\mathbf{f}$ is supported inside the critical radius. In the third part, we prove the nonresonance result.

Part 1:

To ease the exposition, we first introduce some notations. We write $a \lesssim b$ and $a \simeq b$ to denote $a \le c_1 b$ and $a = c_2 b$, respectively, where c_1 and c_2 are constants depending on parameters $\lambda, \mu, r_i, r_e, \omega$ and independent of the order n.

To solve the system (174), the inverse of the matrix $\mathbf{M}$, that is, $\mathbf{M}^{-1} = \{\widetilde{m}_{ij} / \det \mathbf{M}\}_{1\le i,j\le 8,}$ needs be calculated. If the parameters in $\mathbf{C}_0$ are chosen as in (176), by tedious calculation and together with the help of the asymptotic expansion in (31), the determinant of the matrix $\mathbf{M}$ has the following asymptotic expansion:

$$\det \mathbf{M} \simeq \rho^{2n}\left(\delta^2 + \rho^{2n} + p_{n_0} + q_n\right)/n, \tag{178}$$

where $q_n = \mathcal{O}(1/n)$. Clearly, if p_{n_0} is chosen such that the term $p_{n_0} + q_{n_0}$ vanishes for some n_0, that is, the condition (177) is fulfilled, we have that

$$\det \mathbf{M} \simeq \begin{cases} \rho^{2n_0}\left(\delta^2 + \rho^{2n_0}\right)/n_0, & n = n_0, \\ \rho^{2n}\left(p_{n_0} + q_n\right)/n(1 + o(1)), & n \ne n_0. \end{cases} \tag{179}$$

Further tedious calculation shows that the terms $\widetilde{m}_{ij}$ with $3 \le i \le 8$, $5 \le j \le 8$ have the following asymptotic expansion:

$$\begin{aligned} \widetilde{m}_{35} &= \frac{\rho^{3n} n(\lambda+\mu)\varpi_1}{r_2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{36} = \frac{-\mathrm{i}\rho^{3n} n(\lambda+\mu)\varpi_1}{r_2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \\ \widetilde{m}_{37} &= \frac{\rho^{3n}(\lambda+3\mu)\varpi_1}{2\mu}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{38} = \frac{-\mathrm{i}\rho^{3n}(\lambda+3\mu)\varpi_1}{2\mu}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \end{aligned} \tag{180}$$

$$\begin{aligned} \widetilde{m}_{45} &= \frac{-\mathrm{i}\rho^{3n} n(\lambda+\mu)\varpi_1}{r_2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{46} = \frac{-\rho^{3n} n(\lambda+\mu)\varpi_1}{r_2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \\ \widetilde{m}_{47} &= \frac{-\mathrm{i}\rho^{3n}(\lambda+3\mu)\varpi_1}{2\mu}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{48} = \frac{-\rho^{3n}(\lambda+3\mu)\varpi_1}{2\mu}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \end{aligned} \tag{181}$$

$$\begin{aligned} \widetilde{m}_{55} &= \frac{\mathrm{i}\delta\rho^{2n}(\lambda+\mu)\varpi_2}{n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{56} = \frac{\delta\rho^{2n}(\lambda+\mu)\varpi_2}{n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \\ \widetilde{m}_{57} &= \frac{\mathrm{i}\delta\rho^{2n}(\lambda+3\mu)\varpi_2}{2\mu}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{58} = \frac{\delta\rho^{2n}(\lambda+3\mu)\varpi_2}{2\mu}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \end{aligned} \tag{182}$$

$$\begin{aligned} \widetilde{m}_{65} &= \frac{\delta\rho^{2n}(\lambda+\mu)\varpi_3}{r_2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{66} = \frac{\mathrm{i}\delta\rho^{2n}(\lambda+\mu)\varpi_3}{n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \\ \widetilde{m}_{67} &= \frac{\delta\rho^{2n}(\lambda+3\mu)\varpi_3}{2\mu n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{68} = \frac{\mathrm{i}\delta\rho^{2n}(\lambda+3\mu)\varpi_3}{2\mu n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \end{aligned} \tag{183}$$

$$\widetilde{m}_{75} = \frac{\mathrm{i}\delta\rho^{2n}(\lambda+\mu)\varpi_4}{r_2}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{76} = \frac{\delta\rho^{2n}(\lambda+\mu)\varpi_4}{r_2}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$
$$\widetilde{m}_{77} = \frac{\mathrm{i}\delta\rho^{2n}(\lambda+3\mu)\varpi_4}{2\mu n}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{78} = \frac{\delta\rho^{2n}(\lambda+3\mu)\varpi_4}{2\mu n}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \tag{184}$$

$$\widetilde{m}_{85} = \frac{\delta\rho^{2n}(\lambda+\mu)\varpi_5}{r_2}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{86} = \frac{\mathrm{i}\delta\rho^{2n}(\lambda+\mu)\varpi_5}{r_2}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$
$$\widetilde{m}_{87} = \frac{\delta\rho^{2n}(\lambda+3\mu)\varpi_5}{2\mu n}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad \widetilde{m}_{88} = \frac{\mathrm{i}\delta\rho^{2n}(\lambda+3\mu)\varpi_5}{2\mu n}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \tag{185}$$

where

$$\varpi_1 = \frac{(\lambda+\mu)^2}{\mu r_1^2(\lambda+3\mu)^3}\varpi_6, \quad \varpi_2 = -\frac{(\lambda+3\mu)^5\left(r_1^2-r_2^2\right)^2}{512\mu^3 r_2(\lambda+2\mu)^5(\lambda+\mu)}, \tag{186}$$

$$\varpi_3 = \frac{-1}{\mu(\lambda+2\mu)}\varpi_6, \quad \varpi_4 = \frac{1}{(\lambda+\mu)(\lambda+2\mu)}\varpi_6, \quad \varpi_5 = \frac{1}{\mu(\lambda+\mu)}\varpi_6, \tag{187}$$

with

$$\varpi_6 = \frac{(\lambda+3\mu)^5\left(\sqrt{\lambda+2\mu}-\sqrt{-\frac{\lambda^2+3\lambda\mu+2\mu^2}{\lambda+3\mu}}\right)\left(r_2^2-r_1^2\right)}{512\mu\omega^2(\lambda+\mu)(\lambda+2\mu)^{9/2}}. \tag{188}$$

Moreover, from (146) and (149), the coefficients of the source term in (174) have the following asymptotic expansions:

$$f_{1,n} = 2\kappa_{1,n}, \quad f_{2,n} = 2\mathrm{i}\kappa_{1,n}(1+\mathcal{O}(1/n)),$$
$$\tilde{f}_{1,n} = 4\mu n\kappa_{1,n}/r_2(1+\mathcal{O}(1/n)), \quad \tilde{f}_{1,n} = 4\mu n\kappa_{1,n}/r_2(1+\mathcal{O}(1/n)). \tag{189}$$

Thus, from Equations (179) to (189), we have that the coefficients $\varphi_{i,j,n}$, $2 \le i \le 4$, $j = 1,2$ in (174) enjoy the following asymptotic expressions for $n = n_0$,

$$\varphi_{2,1,n} \simeq \frac{-\kappa_{1,n}\rho^n n^2}{(\delta^2+\rho^{2n})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad \varphi_{2,2,n} \simeq \frac{\mathrm{i}\kappa_{1,n}\rho^n n^2}{(\delta^2+\rho^{2n})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$
$$\varphi_{3,1,n} \simeq \frac{\mathrm{i}\kappa_{1,n}\delta}{(\delta^2+\rho^{2n})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad \varphi_{3,2,n} \simeq \frac{-\kappa_{1,n}\delta n}{(\delta^2+\rho^{2n})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \tag{190}$$
$$\varphi_{4,1,n} \simeq \frac{-\mathrm{i}\kappa_{1,n}\delta n}{(\delta^2+\rho^{2n})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad \varphi_{4,2,n} \simeq \frac{\kappa_{1,n}\delta n}{(\delta^2+\rho^{2n})}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),$$

and for $n \neq n_0$

$$
\begin{aligned}
|\varphi_{2,1,n}| &\lesssim \left|\frac{\kappa_{1,n}\rho^n n^2}{(\delta^2+\rho^{2n})}\right|\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad |\varphi_{2,2,n}| \lesssim \left|\frac{\mathrm{i}\kappa_{1,n}\rho^n n^2}{(\delta^2+\rho^{2n})}\right|\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),\\
|\varphi_{3,1,n}| &\lesssim \left|\frac{\mathrm{i}\kappa_{1,n}\delta}{(\delta^2+\rho^{2n})}\right|\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad |\varphi_{3,2,n}| \lesssim \left|\frac{\kappa_{1,n}\delta n}{(\delta^2+\rho^{2n})}\right|\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),\\
|\varphi_{4,1,n}| &\lesssim \left|\frac{\mathrm{i}\kappa_{1,n}\delta n}{(\delta^2+\rho^{2n})}\right|\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad |\varphi_{4,2,n}| \lesssim \left|\frac{\kappa_{1,n}\delta n}{(\delta^2+\rho^{2n})}\right|\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).
\end{aligned} \tag{191}
$$

We would like to point out that in (190), the terms on the LHS share the same sign with the expression on the RHS.

Part 2: In this part, we show that the polarition resonance could occur when the source is located inside the critical radius r^*. Denote by

$$
\mathbf{u}_n = \sum_{n\geq N}\left(\hat{\mathbf{S}}^{\omega}_{\partial D}[\varphi_{2,1,n}e^{\mathrm{i}n\theta}\boldsymbol{\nu}+\varphi_{2,2,n}e^{\mathrm{i}n\theta}\mathbf{t}](\mathbf{x})+\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[\varphi_{3,1,n}e^{\mathrm{i}n\theta}\boldsymbol{\nu}+\varphi_{3,2,n}e^{\mathrm{i}n\theta}\mathbf{t}](\mathbf{x})\right), \tag{192}
$$

where the coefficients $\varphi_{i,j,n}$, $2\leq i\leq 4$, $j=1,2$ satisfy the asymptotic expansions in (191). Thus, from (142), the displacement field $\mathbf{u}$ to the system (141) in the shell $\Omega\backslash\overline{D}$ can be represented as:

$$
\mathbf{u} = \sum_{n\geq N}\mathbf{u}_n = \widetilde{\mathbf{u}}_{n_0} + \mathbf{u}_{n_0}, \tag{193}
$$

where

$$
\widetilde{\mathbf{u}}_{n_0} = \sum_{n\geq N, n\neq n_0}\mathbf{u}_n. \tag{194}
$$

With the help of Green's formula, the orthogonality of $(e^{\mathrm{i}n\theta}\boldsymbol{\nu}, e^{\mathrm{i}n\theta}\mathbf{t})$ on $L^2(\partial B_1)^2$, and Lemmas 8 as well as 10, the dissipation energy $E(\mathbf{u})$ defined in (14) can be written as:

$$
\begin{aligned}
E(\mathbf{u}) &= \Im P_{\hat{\lambda},\hat{\mu}}(\mathbf{u},\mathbf{u}) = \Im\left(\int_{\partial\Omega}\partial_{\hat{\nu}}\mathbf{u}\overline{\mathbf{u}}ds - \int_{\partial D}\partial_{\hat{\nu}}\mathbf{u}\overline{\mathbf{u}}ds\right)\\
&= \Im\left(\int_{\partial\Omega}\partial_{\hat{\nu}}\mathbf{u}_{n_0}\overline{\mathbf{u}}_{n_0}ds - \int_{\partial D}\partial_{\hat{\nu}}\mathbf{u}_{n_0}\overline{\mathbf{u}}_{n_0}ds\right) + \Im\left(\int_{\partial\Omega}\partial_{\hat{\nu}}\widetilde{\mathbf{u}}_{n_0}\overline{\widetilde{\mathbf{u}}}_{n_0}ds - \int_{\partial D}\partial_{\hat{\nu}}\widetilde{\mathbf{u}}_{n_0}\overline{\widetilde{\mathbf{u}}}_{n_0}ds\right)\\
&\geq \Im\left(\int_{\partial\Omega}\partial_{\hat{\nu}}\mathbf{u}_{n_0}\overline{\mathbf{u}}_{n_0}ds - \int_{\partial D}\partial_{\hat{\nu}}\mathbf{u}_{n_0}\overline{\mathbf{u}}_{n_0}ds\right).
\end{aligned} \tag{195}
$$

In the derivation of the last equation, we have used the following fact

$$
\Im\left(\int_{\partial\Omega}\partial_{\hat{\nu}}\widetilde{\mathbf{u}}_{n_0}\overline{\widetilde{\mathbf{u}}}_{n_0}ds - \int_{\partial D}\partial_{\hat{\nu}}\widetilde{\mathbf{u}}_{n_0}\overline{\widetilde{\mathbf{u}}}_{n_0}ds\right) = \Im P_{\hat{\lambda},\hat{\mu}}(\widetilde{\mathbf{u}}_{n_0},\widetilde{\mathbf{u}}_{n_0}) > 0, \tag{196}
$$

which follows from the definition of $P_{\hat{\lambda},\hat{\mu}}(\widetilde{\mathbf{u}}_{n_0},\widetilde{\mathbf{u}}_{n_0})$ in (13) and $\Im\hat{\lambda}, \Im\hat{\mu} \in \mathbb{R}_+$. By Lemmas 8 as well as 10, the asymptotic expansion in (31), and tedious calculation, we have that

$$\begin{aligned}\Im\left(\int_{\partial\Omega} \partial_{\hat{\nu}}\mathbf{u}_{n_0}\overline{\mathbf{u}}_{n_0}ds\right) &\simeq \Im\left(\left|\varphi_{3,1,n_0}\right|^2 \int_{\partial\Omega} \partial_{\hat{\nu}}\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\boldsymbol{\nu}]\cdot\overline{\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\boldsymbol{\nu}]}ds\right)\\ &+\Im\left(\left|\varphi_{3,2,n_0}\right|^2 \int_{\partial\Omega} \partial_{\hat{\nu}}\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\mathbf{t}]\cdot\overline{\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\mathbf{t}]}ds\right)\\ &+\Im\left(\varphi_{3,2,n_0}\overline{\varphi_{3,1,n_0}} \int_{\partial\Omega} \partial_{\hat{\nu}}\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\mathbf{t}]\cdot\overline{\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\boldsymbol{\nu}]}ds\right)\\ &+\Im\left(\varphi_{3,1,n_0}\overline{\varphi_{3,2,n_0}} \int_{\partial\Omega} \partial_{\hat{\nu}}\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\boldsymbol{\nu}]\cdot\overline{\hat{\mathbf{S}}^{\omega}_{\partial\Omega}[e^{\mathrm{i}n_0\theta}\mathbf{t}]}ds\right)\\ &\simeq \left|\kappa_{1,n_0}\right|^2\left(\frac{\delta^2}{(\delta^2+\rho^{2n_0})^2}\frac{\delta}{n_0}+\frac{\delta^2 n_0^2}{(\delta^2+\rho^{2n_0})^2}\frac{\delta}{n_0}+\frac{\delta^2 n_0}{(\delta^2+\rho^{2n_0})^2}\frac{\delta}{n_0}+\frac{\delta^2 n_0}{(\delta^2+\rho^{2n_0})^2}\frac{\delta}{n_0}\right)\\ &\simeq \left|\kappa_{1,n_0}\right|^2\frac{\delta^3 n_0}{(\delta^2+\rho^{2n_0})^2}.\end{aligned} \tag{197}$$

Following similar discussion, we have that

$$\Im\left(-\int_{\partial D} \partial_{\hat{\nu}}\mathbf{u}_{n_0}\overline{\mathbf{u}}_{n_0}ds\right) \simeq \left|\kappa_{1,n_0}\right|^2\frac{\rho^{2n_0}\delta n_0^3}{(\delta^2+\rho^{2n_0})^2}. \tag{198}$$

Combining (195), (197), and (198) yields that

$$E(\mathbf{u}) \geq \left|\kappa_{1,n_0}\right|^2\frac{\rho^{2n_0}\delta n_0^3}{(\delta^2+\rho^{2n_0})^2} \geq \left|\kappa_{1,n_0}\right|^2\left(\frac{r_e}{r_i}\right)^{n_0}. \tag{199}$$

If the source $\mathbf{f}$ is supported inside the critical radius $r_* = \sqrt{r_e^3/r_i}$, by (145) and the asymptotic properties of $J_n(t)$ and $H_n(t)$ in (31), one can verify that there exists $\tau_1 \in \mathbb{R}_+$ such that

$$\limsup_{n\to\infty}\left(\frac{\kappa_{1,n}}{r_e^n}\right)^{1/n} = \sqrt{\frac{r_i}{r_e^3}+\tau_1}. \tag{200}$$

Combining (195) and (200), one can obtain that

$$E(\mathbf{u}) \geq \left(\frac{r_i}{r_e}+\tau_1 r_e^2\right)^{n_0}\left(\frac{r_e}{r_i}\right)^{n_0} = \left(1+\frac{\tau_1 r_e^3}{r_i}\right)^{n_0}, \tag{201}$$

which exactly shows that the polariton resonance occurs, namely, the condition (15) is fulfilled.

Then, we prove the boundedness of the solution $\mathbf{u}$ when $|x| > r_e^2/r_i$; that is, the bounded condition (16) is satisfied. From (142) and (172), the displacement field $\mathbf{u}$ in $\mathbb{R}^2\backslash\overline{\Omega}$ can be represented as:

$$\mathbf{u} = \sum_{|n|\geq N}\left(\mathbf{S}^{\omega}_{\partial\Omega}[\boldsymbol{\varphi}_{4,1,n}e^{\mathrm{i}n\theta}\boldsymbol{\nu}+\boldsymbol{\varphi}_{4,2,n}e^{\mathrm{i}n\theta}\mathbf{t}](\mathbf{x})\right)+\mathbf{F}(\mathbf{x}). \tag{202}$$

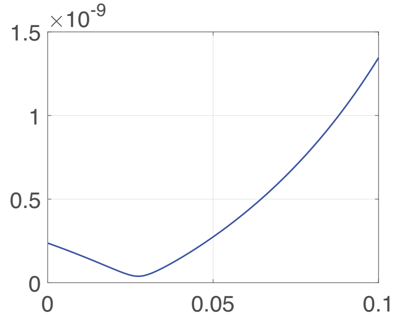

FIGURE 5 The absolute value of (det **M**) given in (177) with respect to p_{n_0}. Horizontal axis: value of $|\det \mathbf{M}|$; Vertical axis: absolute value of p_{n_0}

Moreover, from (191) and Theorem 1, one can obtain that

$$|\mathbf{u}| \leq \sum_{|n|\geq N} |\varkappa_{1,n}| \frac{r_e^{2n}}{r_i^n} \frac{1}{r^n} + |\mathbf{F}| \leq C, \tag{203}$$

when $|x| > r_e^2/r_i$. Thus, from (201) and (203), one can directly conclude that the CALR could occur when the source is located inside the radius $r_* = \sqrt{r_e^3/r_i}$.

Part 3: In this part, we show that there is no resonance when the source is supported outside the critical radius r_*. From (145) and the asymptotic properties of $J_n(t)$ and $H_n(t)$ in (31), one can show that there exists $\tau_2 > 0$ such that

$$\limsup_{n\to\infty} \left(\frac{\varkappa_{1,n}}{r_e^n}\right)^{1/n} \leq \frac{1}{r_* + \tau_2}, \tag{204}$$

and the dissipation energy $E(\mathbf{u})$ can be estimated as follows:

$$E(\mathbf{u}) \leq \sum_{n\geq N} \varkappa_{1,n}^2 \left(\frac{r_e}{r_i}\right)^n \leq \sum_{n\geq N} \left(\frac{1}{(r_* + \tau_2)^2} \frac{r_e}{r_i}\right)^n \leq \sum_{n\geq N} \left(\frac{1}{\left(\sqrt{r_e^3/r_i} + \tau_2\right)^2} \frac{r_e^3}{r_i}\right)^n \leq C, \tag{205}$$

which means that the polariton resonance does not occur. This completes the proof. ■

Remark 12. The choice of n_0 in Theorem 5 is such that the dissipation energy $E(\mathbf{u})$ expressed in (201) satisfies $E(\mathbf{u}) \geq M$, for some $M \gg 1$; that is, the resonance occurs from the definition (15). Because the base term $1 + \frac{\tau_1 r_e^3}{r_i} > 1$, the value of n_0 may not be large.

Remark 13. We can verify the condition (177) numerically. For this, we choose the following parameters:

$$n_0 = 25,\ \ \omega = 5,\ \ r_i = 0.8,\ \ r_e = 1,\ \ \breve{\mu} = \breve{\lambda} = \lambda = \mu = 1,\ \ \delta = (r_i/r_e)^{n_0} = 0.0038. \tag{206}$$

From the values of the parameters ω and r_e, one can readily verify that this is the case beyond quasistatic approximation. The value of $|\det \mathbf{M}|$ given in (177) in terms of the parameter p_{n_0} is plotted in Figure 5, which apparently demonstrates that the condition (177) is satisfied.

ACKNOWLEDGMENTS

The work of H. Li was supported by a Direct Grant for Research, CUHK (project 4053518). The work of H. Liu was supported by the Hong Kong RGC General Research Funds (projects 12302919, 12301420, and 11300821), the NSFC/RGC Joint Research Fund (project N_CityU 101/21), and the France-Hong Kong ANR/RGC Joint Research Grant, A-CityU203/19. The work of J. Zou was substantially supported by the Hong Kong RGC General Research Fund (projects 14306921 and 14306719).

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

ORCID

Hongyu Liu https://orcid.org/0000-0002-2930-3510

REFERENCES

1. Kupradze VD. *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*. Amsterdam: North-Holland; 1979.
2. Ammari H, Ciraolo G, Kang H, Lee H, Milton GW. Spectral theory of a Neumann–Poincaré-type operator and analysis of cloaking due to anomalous localized resonance. *Arch Ration Mech Anal*. 2013;208:667-692.
3. Deng Y, Li H, Liu H. Spectral properties of Neumann–Poincaré operator and anomalous localized resonance in elasticity beyond quasi-static limit. *J Elasticity*. 2020;140:213-242.
4. Li H, Liu H. On anomalous localized resonance for the elastostatic system. *SIAM J Math Anal*. 2016;48:3322-3344.
5. Li H, Li J, Liu H. On novel elastic structures inducing polariton resonances with finite frequencies and cloaking due to anomalous localized resonance. *J Math Pures Appl*. 2018;120:195-219.
6. Ammari H, Zhang H. Effective medium theory for acoustic waves in bubbly fluids near Minnaert resonant frequency. *SIAM J Math Anal*. 2017;49:3252-3276.
7. Li H, Liu H, Zou J. Minnaert resonances for bubbles in soft elastic materials. *SIAM J Appl Math*. 2022;82(1):119-141.
8. Smith DR, Padilla W, Vier D, Nemat-Nasser S, Schultz S. Composite medium with simultaneously negative permeability and permittivity. *Phys Rev Lett*. 2000;84:4184.
9. Enoch S, Tayeb G, Sabouroux P, Guérin N, Vincent P. A metamaterial for directive emission. *Phys Rev Lett*. 2002;89:213902.
10. Li W, Valentine J. Metamaterial perfect absorber based hot electron photodetection. *Nano Lett*. 2014;14:351-3514.
11. Li H, Liu H. On anomalous localized resonance and plasmonic cloaking beyond the quasistatic limit. *Proc Roy Soc A*. 474:20180165.
12. Li H. Recent progress on the mathematical study of anomalous localized resonance in elasticity. *Electron Res Archive*. 2020;28:1257-1272.
13. Li H, Li J, Liu H. On quasi-static cloaking due to anomalous localized resonance in $\mathbb{R}^3$. *SIAM J Appl Math*. 2015;75:1245-1260.
14. Milton GW, Nicorovici NAP. On the cloaking effects associated with anomalous localized resonance. *Proc R Soc A*. 2006;462:3027-3059.
15. Nicorovici NAP, McPhedran RC, Milton GW. Optical and dielectric properties of partially resonant composites. *Phys Rev B*. 1994;49:8479-8482.
16. Fang N, Lee H, Sun C, Zhang X. Sub-diffraction-limited optical imaging with a silver superlens. *Science*. 2005;308:53-537.

17. Pendry JB. Negative refraction makes a perfect lens. *Phys Rev Lett*. 2000;85:3966-3969.
18. Ammari H, Chow YT, Liu H, Sunkula M. Quantum integrable systems and concentration of plasmon resonance. *J Eur Math Soc*. 2022.
19. Ammari H, Deng Y, Millien P. Surface plasmon resonance of nanoparticles and applications in imaging. *Arch Ration Mech Anal*. 2016;220(1):109-153.
20. Ding M, Liu H, Zheng G. Shape reconstructions by using plasmon resonances. *ESAIM: Math Model Numer Anal*. 2022;56(2):705-726.
21. Ando K, Kang H, Liu H. Plasmon resonance with finite frequencies: a validation of the quasi-static approximation for diametrically small inclusions. *SIAM J Appl Math*. 2016;76:731-749.
22. Blåsten E, Li H, Liu H, Wang Y. Localization and geometrization in plasmon resonances and geometric structures of Neumann–Poincaré eigenfunctions. *ESAIM: Math Model Numer Anal*. 2020;54:957-976.
23. Deng Y, Liu H, Zheng G. Mathematical analysis of plasmon resonances for curved nanorods. *J Math Pures Appl*. 2021;153:248-280.
24. Fang X, Deng Y, Liu H. Sharp estimate of electric field from a conductive rod and application. *Stud Appl Math*. 2021;146:279-297.
25. Zheng G. Mathematical analysis of plasmonic resonance for 2-D photonic crystal. *J Differ Equ*. 2019;266(8):5095-5117.
26. Deng Y, Liu H, Zheng G. Plasmon resonances of nanorods in transverse electromagnetic scattering. *J Differ Equ*. 2022;318:502-536.
27. Li H, Li S, Liu H, Wang X. Analysis of electromagnetic scattering from plasmonic inclusions at optical frequencies and applications. *ESAIM: Math Model Numer Anal*. 2018;53(4):1351-1371.
28. Chen B, Gao Y, Liu H. Modal approximation for time-domain elastic scattering from metamaterial quasiparticles. *J Math Pures Appl*. 2022;165:148-189.
29. Deng Y, Li H, Liu H. On spectral properties of Neumann–Poincaré operator and plasmonic resonances in 3D elastostatics. *J Spectr Theory*. 2019;9(3):767-789.
30. Deng Y, Li H, Liu H. Analysis of surface polariton resonance for nanoparticles in elastic system. *SIAM J Math Anal*. 2020;52:1786-1805.
31. Li H, Liu H. On three-dimensional plasmon resonances in elastostatics. *Ann Mat Pura Appl*. 2017;196(3):1113-1135.
32. Ammari H, Bretin E, Garnier J, Kang H, Lee H, Wahab A. *Mathematical Methods in Elasticity Imaging*, Princeton University Press; 2015.
33. Ando K, Kang H, Miyanishi Y, Putinar M. *Spectral Analysis of Neumann–Poincaré Operator*, arXiv:2003.14387.
34. Miyanishi Y, Rozenblum G. Spectral properties of the Neumann–Poincaré operator in 3D elasticity. *Int Math Res Not IMRN*. 2021;11:8715-8740.
35. Rozenblum G. Eigenvalue asymptotics for polynomially compact pseudodifferential operators and applications. *Algebra i Analiz*. 2021;33(2):215-232.
36. Konvalinka M. Triangularizability of polynomially compact operators. *Integr Equ Oper Theory*. 2005;52:271-284.
37. Colton D, Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory*. 2nd ed. Berlin: Springer-Verlag; 1998.
38. Fang X, Deng Y, Chen X. Asymptotic behavior of spectral of Neumann–Poincaré operator in Helmholtz system. *Math Meth Appl Sci*. 2019;42:942-953.
39. Ando K, Ji Y, Kang H, Kim K, Yu S. Spectral properties of the Neumann–Poincaré operator and cloaking by anomalous localized resonance for the elasto-static system. *European J Appl Math*. 2018;29(2):189-225.

How to cite this article: Li H, Liu H, Zou J. Elastodynamical resonances and cloaking of negative material structures beyond quasistatic approximation. *Stud Appl Math*. 2023;150:716–754. https://doi.org/10.1111/sapm.12555