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TWO SINGLE-MEASUREMENT UNIQUENESS RESULTS FOR INVERSE SCATTERING PROBLEMS WITHIN POLYHEDRAL GEOMETRIES

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Dedicated to the memory of Professor Victor Isakov (1947–2021)

Abstract. We consider the unique determinations of impenetrable obstacles or diffraction grating profiles in $\mathbb{R}^3$ by a single far-field measurement within polyhedral geometries. We are particularly interested in the case that the scattering objects are of impedance type. We derive two new unique identifiability results by a single measurement for the inverse scattering problem in the aforementioned two challenging setups. The main technical idea is to exploit certain quantitative geometric properties of the Laplacian eigenfunctions which were initiated in our recent works [12, 13]. In this paper, we derive novel geometric properties that generalize and extend the related results in [13], which further enable us to establish the new unique identifiability results. It is pointed out that in addition to the shape of the obstacle or the grating profile, we can simultaneously recover the boundary impedance parameters.

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1. **Introduction.**

1.1. **Mathematical setup and main results for the inverse obstacle problem.**

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain such that $\mathbb{R}^3 \backslash \overline{\Omega}$ is connected. Let u^i be an incident plane field of the form

$$u^i := u^i(\mathbf{x}; k, \mathbf{d}) = e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}, \quad \mathbf{x} \in \mathbb{R}^3, \tag{1.1}$$

where $k \in \mathbb{R}_+$ denotes the wavenumber and $\mathbf{d} \in \mathbb{S}^2$ signifies the incident direction. Denote u^s as the scattered wave field generated from the interruption of the propagation of u^i by an impedance obstacle. Define $u := u^i + u^s$ to be the total wave field. The forward scattering problem is described as follows:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ u = u^i + u^s & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \\ \partial_\nu u + \eta u = 0 & \text{on} \quad \partial\Omega, \\ \lim\limits_{r\to\infty} r\left(\dfrac{\partial u^s}{\partial r} - \mathrm{i}k u^s\right) = 0, & r := |x|, \end{cases} \tag{1.2}$$

where the last limit is known as the Sommerfeld radiation condition that holds uniformly in $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}| \in \mathbb{S}^2$. The Robin type boundary condition is referred to as the impedance boundary condition for an impedance obstacle, where ν denotes the exterior unit normal vector to $\partial\Omega$ and $\eta \in L^\infty(\partial\Omega)$ represents the corresponding boundary impedance parameter. If $\eta \neq 0$, Ω is said to be an impedance obstacle.

The wellposedness of the forward scattering problem (1.2) is known by [16, 45] and there exists a unique solution $u \in H^1_{loc}(\mathbb{R}^3 \backslash \overline{\Omega})$ fulfilling the following expansion, which holds uniformly for $\hat{\mathbf{x}}$:

$$u^s(\mathbf{x}; k, \mathbf{d}) = \frac{e^{\mathrm{i}kr}}{4\pi r} u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}) + \mathcal{O}\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty, \tag{1.3}$$

where u_∞ is known as the associated far-field pattern or the scattering amplitude.

The corresponding inverse obstacle scattering problem to (1.2) is to recover the shape of the obstacle Ω as well as the associated impedance parameter η by the knowledge of the far-field pattern $u_\infty(\hat{\mathbf{x}}; k, \mathbf{d})$. By introducing an abstract operator $\mathcal{F}$ which sends the obstacle to the corresponding far-field pattern, the aforementioned inverse problem can be formulated as

$$\mathcal{F}(\Omega, \eta) = u_\infty(\hat{\mathbf{x}}; k, \mathbf{d}), \tag{1.4}$$

which is nonlinear and ill-posed (cf. [36, 49]).

We first introduce the concept of "admissible" obstacles by presenting certain a-priori conditions on the underlying obstacles in deriving our main unique determination results for the inverse obstacle problem.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^3$ be an open polyhedron associated with the boundary impedance condition (i.e. the third equation) in (1.2). Denote $\partial\Omega = \cup_{m=1}^p \Sigma_m$ ($p \in \mathbb{N}$ and $p \geq 4$), where Σ_m is a face of Ω, and $\mathcal{E}(\Omega) = \{\boldsymbol{l}_1, \ldots, \boldsymbol{l}_q\}$ is the set of edges of Ω. Ω is said to be an* admissible polyhedral obstacle, *if the following conditions are fulfilled:*

(a) $\eta \in C(\partial\Omega)$;
(b) for any face $\Sigma_m \subset \partial\Omega$, the surface impedance parameter η on $\overline{\Sigma_m}$ is a real-analytic function;
(c) for any edge $\boldsymbol{l}_j \in \mathcal{E}(\Omega)$, $\eta(\mathbf{x}_0) \neq 0$ for any $\mathbf{x}_0 \in \boldsymbol{l}_j$.

Throughout this paper, for notational unification, we write an admissible polyhedral obstacle as (Ω, η).

Remark 1.2. It is noted that the conditions on the impedance parameter η in Definition 1.1 can be fulfilled when η is a non-zero constant on $\partial\Omega$. That is, $\eta(\mathbf{x}) \equiv \eta_0 \in \mathbb{C}\backslash\{0\}$ for $\mathbf{x} \in \partial\Omega$. Compared with the admissible polyhedral obstacles introduced in [13, Definition 6.1], we completely remove the root assumption of the associated Legendre polynomials on the dihedral angle between any two adjacent faces of the underlying polyhedron, which may not be easily verified in practical applications. Therefore, our new unique identifiability result in what follows for determining the admissible obstacle as introduced above can be applied to more general scenarios in the inverse obstacle scattering problem.

Following the notations in [13], we let Π_1, Π_2 be any two adjacent faces of a polyhedron Ω. Denote $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ to be an edge corner associated with Π_1 and Π_2. $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ signifies the vertex corner formulated by $\Pi_1, \Pi_2, \cdots, \Pi_n$ at the vertex $\mathbf{x}_0 \in \partial\Omega$. It is clear that a vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is composed of finite many edge corners, which intersect at $\mathbf{x}_0$, ; see [13, Definitions 1.2 and 1.3] for more details. Now, recall the definitions for rational and irrational obstacles based on the concept of the rational and irrational corners introduced in [13] as follows.

Definition 1.3. *[13, Definition 4.1] Let $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ be an edge corner associated with Π_1 and Π_2. Denote the dihedral angle of Π_1 and Π_2 by $\phi = \alpha\cdot\pi$, $\alpha \in (0,1)$. If ϕ is an irrational dihedral angle, namely, α is an irrational number, then $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ is said to be an irrational edge corner. Otherwise it is said to be a rational edge corner. For a rational edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$, it is called a rational angle of degree p of $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ if $\alpha = q/p$ with $p, q \in \mathbb{N}$ being irreducible.*

Definition 1.4. *[13, Definition 4.2] Let $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ be a vertex corner, where $n \in \mathbb{N}$ and $n \geq 3$. It is clear that $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is composed of the following n edge corners:*

$$\mathcal{E}_\ell := \mathcal{E}(\Pi_\ell, \Pi_{\ell+1}, \boldsymbol{l}_\ell), \quad \mathcal{E}_n := \mathcal{E}(\Pi_n, \Pi_1, \boldsymbol{l}_n), \quad \Pi_{n+1} := \Pi_1, \quad \ell = 1, 2, \ldots, n-1,$$

where $\boldsymbol{l}_\ell$ is the line segment of $\Pi_\ell \cap \Pi_{\ell+1}$ and $\boldsymbol{l}_n$ is the line segment of $\Pi_n \cap \Pi_1$, respectively. Denote

$$\begin{aligned} I_{\mathsf{IR}} &= \{\ell \in \mathbb{N} \mid 1 \leq \ell \leq n, \quad \mathcal{E}_\ell \text{ is an irrational edge corner}\}, \\ I_{\mathsf{R}} &= \{\ell \in \mathbb{N} \mid 1 \leq \ell \leq n, \quad \mathcal{E}_\ell \text{ is a rational edge corner}\}. \end{aligned} \tag{1.5}$$

If $\#I_{\mathsf{IR}} \geq 1$, then $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is said to be an irrational vertex corner. If $\#I_{\mathsf{IR}} \equiv 0$, then $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ is said to be a rational vertex corner. For a rational vertex corner $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$ composed of edge corners $\mathcal{E}_\ell := \mathcal{E}(\Pi_\ell, \Pi_{\ell+1}, \boldsymbol{l}_\ell)$, the largest degree of $\mathcal{E}_\ell$ $(\ell = 1, \ldots, n)$ is referred to as the rational degree *of $\mathcal{V}(\{\Pi_\ell\}_{\ell=1}^n, \mathbf{x}_0)$.*

Next, we provide the definition for rational and irrational admissible obstacles.

Definition 1.5. *Let (Ω, η) be an admissible polyhedral obstacle. If there exists a rational vertex corner, then it is said to be a* rational obstacle*. If all the vertex corners of Ω are irrational, then it is called an* irrational obstacle*. The smallest degree of the rational corner of Ω is referred to as the* rational degree *of Ω.*

We also introduce the admissible complex polyhedral obstacles as follows.

Definition 1.6. *Ω is said to be an admissible complex polyhedral obstacle if it consists of finitely many pairwise disjoint admissible polyhedral obstacles Ω_j ($j = 1, \ldots, l$) such that Ω_j is simply connected and $\partial\Omega_i \cap \partial\Omega_j = \emptyset$ ($\forall i, j \in \{1, \ldots, l\}$ with $i \neq j$). That is,*

$$(\Omega, \eta) = \bigcup_{j=1}^{l} (\Omega_j, \eta_j), \quad \eta = \sum_{j=1}^{l} \eta_j \chi_{\partial\Omega_j \cap \partial\Omega}$$

where $l \in \mathbb{N}$ and each (Ω_j, η_j) is an admissible polyhedral obstacle. Moreover, Ω is said to be irrational if all of its component polyhedral obstacles are irrational, otherwise it is said to be rational. For the latter case, the smallest degree among all the degrees of its rational components is defined to be the degree of the complex obstacle Ω, which is denoted by $\mathsf{deg}(\Omega)$. Furthermore, Ω is said to be convex if all of its component polyhedral obstacles are convex.

With all the necessary notations and definitions introduced above, we are now in a position to give the local unique identifiability results for an admissible complex polyhedral obstacle by a single far-field measurement with respect to rational and irrational cases, separately.

Theorem 1.7. *Considering the scattering problem (1.2) associated with two admissible irrational complex polyhedral obstacles (Ω_j, η_j) in $\mathbb{R}^3$, $j = 1, 2$. Let $u_\infty^j(\hat{\mathbf{x}}; k, \mathbf{d})$ be the corresponding far-field patterns associated with (Ω_j, η_j) and the incident wave u^i is defined in (1.1). Let $\mathbf{G}$ be the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega_1 \cup \Omega_2)}$. Suppose that*

$$u_\infty^1(\hat{\mathbf{x}}; k, \mathbf{d}) = u_\infty^2(\hat{\mathbf{x}}; k, \mathbf{d}), \quad \textit{for} \quad \textit{all } \hat{\mathbf{x}} \in \mathbb{S}^2, \tag{1.6}$$

then $(\partial\Omega_1 \backslash \partial\Omega_2) \bigcup (\partial\Omega_2 \backslash \partial\Omega_1)$ cannot possess an edge corner on $\partial\mathbf{G}$.

Moreover,

$$\eta_1 = \eta_2 \quad \textit{on} \quad \partial\Omega_1 \cap \partial\Omega_2. \tag{1.7}$$

For any vertex $\mathbf{x}_c$ of a polyhedron $\Omega \in \mathbb{R}^3$, we denote for $r \in \mathbb{R}_+$ that $\Omega_r(\mathbf{x}_c) = B_r(\mathbf{x}_c) \cap \mathbb{R}^3 \backslash \overline{\Omega}$. Define

$$\mathcal{L}(f)(\mathbf{x}_c) := \lim_{r \to +0} \frac{1}{|\Omega_r(\mathbf{x}_c)|} \int_{\Omega_r(\mathbf{x}_c)} f(\mathbf{x}) \, \mathrm{d}\mathbf{x} \tag{1.8}$$

if the limit exists for any $f \in L^2_{loc}(\mathbb{R}^3 \backslash \overline{\Omega})$. It is easy to see that if $f(\mathbf{x})$ is continuous in $\overline{\Omega_{\epsilon_0}(\mathbf{x}_c)}$ for a sufficiently small $\epsilon_0 \in \mathbb{R}_+$, then $\mathcal{L}(f)(\mathbf{x}_c) = f(\mathbf{x}_c)$. According to [13, Remark 6.11], (1.8) can be fulfilled in certain practical scenarios.

Using the technical assumption (1.8), the unique determination of rational obstacles can be stated as follows.

Theorem 1.8. *For a fixed $k \in \mathbb{R}_+$, we let (Ω_j, η_j), $j = 1, 2$, be two admissible rational complex obstacles, with $u_\infty^j(\hat{\mathbf{x}}; k, \mathbf{d})$ being their corresponding far-field patterns associated with the incident field u^i defined in (1.1), where $\mathbf{d}$ is the incident direction. Assume that*

$$\mathsf{deg}(\Omega_j) \geq 3, \quad j = 1, 2, \tag{1.9}$$

where $\mathsf{deg}(\Omega_j)$ is the rational degree of Ω_j defined in Definition 1.6. We further write $\mathbf{G}$ for the unbounded connected component of $\mathbb{R}^3 \backslash \overline{(\Omega_1 \cup \Omega_2)}$. If

$$u_\infty^1(\hat{\mathbf{x}}; k, \mathbf{d}) = u_\infty^2(\hat{\mathbf{x}}; k, \mathbf{d}), \quad \hat{\mathbf{x}} \in \mathbb{S}^2, \quad \textit{and} \quad \mathcal{L}(\nabla u^j)(\mathbf{x}_c) \neq 0, \ j = 1, 2, \tag{1.10}$$

for all vertices $\mathbf{x}_c$ *of* Ω, *then the set* $(\partial\Omega_1\backslash\partial\Omega_2)\cup(\partial\Omega_2\backslash\partial\Omega_1)$ *can not possess an edge corner on* $\partial\mathbf{G}$. *Moreover,*

$$\eta_1=\eta_2 \quad on \quad \partial\Omega_1\cap\partial\Omega_2.$$

Remark 1.9. We would like to point out that the technical condition $\mathcal{L}(\nabla u^j)(\mathbf{x}_c)\neq 0$ in (1.10) can be satisfied under some generic conditions on Ω. For example, if the diameter of the obstacle Ω is relatively small compared with the wavelength in the certain regime that $k\cdot\operatorname{diam}(\Omega)\ll 1$, then (1.10) can hold.

Theorems 1.7 and 1.8 actually reveal the unique determination for a certain kind of impedance obstacles locally in the neighborhood of the corner. It is easy to verify that if the underlying admissible complex obstacles are convex, then there holds the global uniqueness results by a single far-field measurement accordingly. Indeed, one has

Corollary 1.10. *For a fixed* $k\in\mathbb{R}_+$, *let* (Ω,η) *and* $(\widetilde{\Omega},\widetilde{\eta})$ *be two convex admissible irrational polyhedral obstacles, with* $u^j_\infty(\hat{\mathbf{x}};k,\mathbf{d})$, $j=1,2$, *being their corresponding far-field patterns associated with the incident field* u^i *defined in* (1.1), *where* $\mathbf{d}$ *is the incident direction. If* (1.6) *is fulfilled, then*

$$\Omega=\widetilde{\Omega},\quad \eta=\widetilde{\eta}.$$

Compared with the unique identifiability study in [13, Section 6], the results presented in this paper significantly relax the number of the measurements from "at most two far-field patterns" to "only one far-field pattern". This is made possible by relaxing the technical condition $u(\mathbf{0})=0$ in the vanishing properties of Laplacian eigenfunctions in $\mathbb{R}^3$, which will be systematically investigated in the subsequent sections. Moreover, the a-priori conditions imposed on "admissible obstacles" are also relaxed by removing the conditions on the associated Legendre polynomials. This is derived from the fact that the vanishing orders of the Laplacian eigenfunction u at an edge corner can be determined by corresponding two intersecting adjacent planes without any a-prior information of the other planes; see [13, Theorem 3.1] for more relevant discussions.

1.2. **Mathematical setup and main results for the inverse diffraction grating problem.** First, we give a brief review on the mathematical setup of the diffraction grating profile. Assume that the diffraction grating involves an impenetrable surface Λ_f which is 2π-periodic with respect to $\mathbf{x}':=(x_1,x_2)$. Precisely speaking, denote

$$\Lambda_f=\{\mathbf{x}:=(\mathbf{x}',x_3)\in\mathbb{R}^3;x_3=f(\mathbf{x}')\},\tag{1.11}$$

where f is a bi-periodic Lipschitz function with period 2π with respect to x_1 and x_2. Let

$$\Omega_f:=\{\mathbf{x}\in\mathbb{R}^3;\mathbf{x}'\in\mathbb{R}^2,x_3>f(\mathbf{x}')\}\tag{1.12}$$

be the unbounded domain filled with an isotropic homogeneous medium. Suppose that the incident plane wave $u^i(\mathbf{x};k,\mathbf{d})$, where $k\in\mathbb{R}_+$ and

$$\begin{aligned}&\mathbf{d}:=\mathbf{d}(\theta_d,\phi_d)=(\sin\phi_d\cos\theta_d,\sin\phi_d\sin\theta_d,-\cos\phi_d),\\&\theta_d\in[0,2\pi),\ \phi_d\in(-\pi/2,\pi/2),\end{aligned}\tag{1.13}$$

propagates to Λ_f from the top. The total wave field fulfills the following Helmholtz system

$$\Delta u+k^2u=0\ \text{ in }\ \Omega_f;\quad \partial_\nu u+\eta u=0\ \text{ on }\ \Lambda_f,\tag{1.14}$$

where η denotes the surface impedance parameter and $\eta \in L^\infty(\Lambda_f)$. Throughout the rest of the paper, we let

$$\boldsymbol{\alpha} = k(\sin\phi_d \cos\theta_d, \sin\phi_d \sin\theta_d).$$

To ensure the well-posedness of (1.14), the total wave field u is supposed to be $\boldsymbol{\alpha}$-quasiperiodic with respect to x_1 and x_2, which can be defined more rigorously as

Definition 1.11. *u is said to be $\boldsymbol{\alpha}$-quasiperiodic with respect to $\mathbf{x}' = (x_1, x_2)$, if there holds*

$$u(\mathbf{x}' + 2\pi\mathbf{n}, x_3) = e^{\mathrm{i}2\pi\boldsymbol{\alpha}\cdot\mathbf{n}} u(\mathbf{x}', x_3)$$

for any $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$.

The corresponding scattered wave u^s satisfies the following Rayleigh series expansion

$$u^s(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} u_{\mathbf{n}} e^{\mathrm{i}\boldsymbol{\alpha}_{\mathbf{n}}\cdot\mathbf{x}' + \mathrm{i}\boldsymbol{\beta}_{\mathbf{n}} x_3} := \sum_{\mathbf{n}\in\mathbb{Z}^2} u_{\mathbf{n}} e^{\mathrm{i}\boldsymbol{\xi}_{\mathbf{n}}\cdot\mathbf{x}}, \quad x_3 > \max_{\mathbf{x}'\in[0,2\pi)^2} f(\mathbf{x}'), \tag{1.15}$$

where $u_{\mathbf{n}} := u_{\mathbf{n}}(k) \in \mathbb{C}$ are the Rayleigh coefficients of u^s and

$$\boldsymbol{\xi}_{\mathbf{n}} = (\boldsymbol{\alpha}_{\mathbf{n}}, \boldsymbol{\beta}_{\mathbf{n}}), \text{ with } \boldsymbol{\alpha}_{\mathbf{n}} := \mathbf{n} + \boldsymbol{\alpha}, \boldsymbol{\beta}_{\mathbf{n}}^2 = k^2 - |\boldsymbol{\alpha}_{\mathbf{n}}|^2, \tag{1.16}$$

where $\Im\boldsymbol{\beta}_{\mathbf{n}} \geq 0$ if $|\boldsymbol{\alpha}_{\mathbf{n}}|^2 > k^2$. The existence and uniqueness of the $\boldsymbol{\alpha}$-quasiperiodic solution to (1.14) for the sound-soft ($\eta \equiv \infty$) or impedance boundary condition with a constant η fulfilling $\Im\eta > 0$ can be seen in [1, 9, 26, 28].

Define

$$\Gamma_b := \{(\mathbf{x}', x_3); \mathbf{x}' \in [0, 2\pi)^2, x_3 = b\}, \quad b > \max_{\mathbf{x}'\in[0,2\pi)^2} |f(\mathbf{x}')| \tag{1.17}$$

The inverse problem associated with (1.14) is to determine Λ_f from the knowledge of $u(\mathbf{x}|_{\Gamma_b}; k, \mathbf{d})$. By introducing an abstract operator $\mathcal{F}$ which sends the information of Λ_f to the measurement $u(\mathbf{x}; k, \mathbf{d})$ on Γ_b, the inverse problem can be formulated as

$$\mathcal{F}(\Lambda_f, \eta) = u(\mathbf{x}|_{\Gamma_b}; k, \mathbf{d}). \tag{1.18}$$

The unique identifiability for the diffraction grating with impedance boundary condition by finite measurements has been an open problem for a long time. In order to investigate this problem, similar to the study on the inverse obstacle problems, we first propose the necessary definition of so-called "admissible" polyhedral gratings considered in our work as follows.

Definition 1.12. *Let (Λ_f, η) be a bi-periodic grating as described in (1.11). It is said to be an admissible polyhedral diffraction grating if Λ_f is a polyhedral Lipschitz surface in $\mathbb{R}^3$, consisting of a finite number of planar faces in one periodic cell $[0, 2\pi) \times [0, 2\pi)$, and the surface impedance parameter η satisfies the following assumptions:*

(a) $\eta \in C(\Lambda_f)$;
(b) for any face $\Sigma_m \subset \Lambda_f$, the surface impedance parameter η on $\overline{\Sigma_m}$ is a real-analytic function;
(c) for any edge $\boldsymbol{l}_j \in \mathcal{E}(\Lambda_f)$, $\eta(\mathbf{x}_0) \neq 0$ for any $\mathbf{x}_0 \in \boldsymbol{l}_j$.

Henceforth, for notational unification, we write an admissible polyhedral diffraction grating as (Λ_f, η).

Following the definitions for irrational and rational vertex corners in Definition 1.4, we have the following concept for admissible irrational and rational polyhedral diffraction gratings.

Definition 1.13. *Let (Λ_f, η) be an admissible polyhedral diffraction grating. If there exists a rational vertex corner in one period, then it is said to be a* rational polyhedral diffraction grating. *If all the vertex corners of Λ_f in one period are irrational, then it is called an* irrational polyhedral diffraction grating. *The smallest degree of the rational corners of Λ_f is referred to as the* rational degree *of Λ_f, which is denoted by $\mathsf{deg}(\Lambda_f)$.*

We would like to point out that the vertex corner considered in Definition 1.13 includes the edge corner which is intersected by two adjacent planes as a special case.

Now, we can provide our main unique determination results for the inverse diffraction grating problem (1.18) with respect to irrational and rational structures, respectively.

Theorem 1.14. *Let (Λ_f, η_f) and (Λ_g, η_g) be two admissible irrational polyhedral diffraction gratings and $\mathbf{G}$ be the unbounded connected component of $\Omega_f \cap \Omega_g$. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d} \in \mathbb{S}^2$ be the corresponding incident direction defined in (1.13). Let Γ_b be a measurement boundary given by (1.17) with*

$$b > \max\left\{ \max_{\mathbf{n}' \in [0,2\pi)^2} |f(\mathbf{n}')|, \max_{\mathbf{n}' \in [0,2\pi)^2} |g(\mathbf{n}')| \right\}.$$

Suppose that $u_f(\mathbf{x}; k, \mathbf{d})$ and $u_g(\mathbf{x}; k, \mathbf{d})$ are the total wave fields measured on Γ_b associate with (Λ_f, η_f) and (Λ_g, η_g), respectively. If there holds that

$$u_f(\mathbf{x}; k, \mathbf{d}) = u_g(\mathbf{x}; k, \mathbf{d}) \quad \text{for} \quad \mathbf{x} \in \Gamma_b, \tag{1.19}$$

then $\partial\mathbf{G} \backslash \partial\Lambda_f$ can not possess an edge corner of Λ_g and $\partial\mathbf{G} \backslash \partial\Lambda_g$ can not possess an edge corner of Λ_f. Moreover,

$$\eta_f = \eta_g \quad \text{on} \quad \Lambda_f \cap \Lambda_g.$$

Similar to Theorem 1.8, if we assume the total wave field to (1.14) satisfies the condition (1.8) on all the vertices of an admissible polyhedral diffraction grating, the uniqueness results for admissible rational polyhedral diffraction gratings can be stated as

Theorem 1.15. *Let (Λ_f, η_f) and (Λ_g, η_g) be two admissible rational polyhedral diffraction gratings and $\mathbf{G}$ be the unbounded connected component of $\Omega_f \cap \Omega_g$. Let $k \in \mathbb{R}_+$ be fixed and $\mathbf{d} \in \mathbb{S}^2$ be the corresponding incident direction defined in (1.13). Let Γ_b be a measurement boundary given by (1.17) with*

$$b > \max\left\{ \max_{\mathbf{n}' \in [0,2\pi)^2} |f(\mathbf{n}')|, \max_{\mathbf{n}' \in [0,2\pi)^2} |g(\mathbf{n}')| \right\}.$$

Assume that

$$\mathsf{deg}(\Lambda_f) \geq 3 \text{ and } \mathsf{deg}(\Lambda_g) \geq 3, \tag{1.20}$$

where $\mathsf{deg}(\Lambda_f)$ and $\mathsf{deg}(\Lambda_g)$ are the rational degrees of Λ_f and Λ_g defined in Definition 1.13, respectively. Suppose that $u_f(\mathbf{x}; k, \mathbf{d})$ and $u_g(\mathbf{x}; k, \mathbf{d})$ are the total wave *fields measured on Γ_b associate with (Λ_f, η_f) and (Λ_g, η_g), respectively. If there holds that*

$$u_f(\mathbf{x}; k, \mathbf{d}) = u_g(\mathbf{x}; k, \mathbf{d}) \quad \textit{for} \quad \mathbf{x} \in \Gamma_b, \quad \mathcal{L}(\nabla u_f)(\mathbf{x}_c^f) \neq 0, \quad \mathcal{L}(\nabla u_g)(\mathbf{x}_c^g) \neq 0, \tag{1.21}$$

where $\mathbf{x}_c^f$ and $\mathbf{x}_c^g$ are arbitrary vertices of the admissible rational polyhedral diffraction gratings (Λ_f, η_f) and (Λ_g, η_g), respectively, then $\partial\mathbf{G}\backslash\partial\Lambda_f$ can not possess an edge corner of Λ_g and $\partial\mathbf{G}\backslash\partial\Lambda_g$ can not possess an edge corner of Λ_f. Moreover

$$\eta_f = \eta_g \quad \textit{on} \quad \Lambda_f \cap \Lambda_g.$$

Similar to Corollary 1.10, we have the following global uniqueness result for convex irrational polyhedral diffraction gratings by a single measurement. Before that, we first introduce the definition of a convex polyhedral diffraction grating.

Definition 1.16. *Let Λ_f be an admissible polyhedral diffraction grating. If f is convex in the periodic cell $[0, 2\pi] \times [0, 2\pi]$, then we say that Λ_f is a convex polyhedral diffraction grating. Otherwise Λ_f is said to be a non-convex polyhedral diffraction grating.*

By virtue of Theorem 1.14, using Definition 1.16, one has the following corollary.

Corollary 1.17. *For a fixed $k \in \mathbb{R}_+$, let (Λ_f, η_f) and (Λ_g, η_g) be two convex irrational polyhedral diffraction gratings, with $u_f(\mathbf{x}; k, \mathbf{d})$ and $u_g(\mathbf{x}; k, \mathbf{d})$ being the corresponding measurements on Γ_b given by (1.17) associated with the incident plane wave field $u^i(\mathbf{x}; k, \mathbf{d})$, where $\mathbf{d}$ is the incident direction (1.13). If (1.19) is fulfilled, then*

$$\Lambda_f = \Lambda_g, \quad \eta_f = \eta_g.$$

1.3. **Background and discussions.** Determining an impenetrable obstacle by a minimal/optimal number of scattering measurements is a long-standing problem in the inverse scattering theory. We refer to [17, 27, 43] for historical accounts as well as surveys on some existing developments in the literature. This problem has been resolved if a-priori geometric conditions are imposed on the underlying obstacle, say e.g. smallness in size (compared to the wavelength), radial symmetry or polyhedral shapes. We refer to [2,12–15,18,21–23,25,29,30,34–44,48–50] for existing theoretical results and [31–33, 47] for related numerical studies in the literature. In two recent papers [12, 13], a different perspective was proposed and the uniqueness study for the inverse scattering problem is delicately connected to the geometric properties of Laplacian eigenfunctions in certain specific setups. Within such a framework, one can establish the unique determination of polyhedral obstacles of impedance type by at most a few far-field measurements that were unable to be tackled by other means developed in the previous studies. The corresponding studies have been extended to solving a large class of inverse scattering problems associated with electromagnetic and elastic waves as well [7, 19–23]. The piecewise constant refractive indices by a single far field pattern can be uniquely identified; see [5, 11] for more details. The unique determination for a scatterer with a high curvature point by a single measurement is established in [6]. Recent progresses on uniqueness results on the shape determination by finite far field measurements for the inverse scattering problems within the polyhedral geometry can be found in [10]. As also discussed in Section 1.1, we derive in this paper some novel uniqueness results which extend the related results in [13] by relaxing certain technical conditions. This is made possible by exploiting the geometric structures of Laplacian eigenfunctions in a less restrictive setup compared to that in [13].

There are also rich results on the unique determination of periodic structures. It is known that in general a grating profile can be uniquely identified by infinitely many quasi-periodic incident plane waves with a fixed phase-shift [3, 28]. In [26], it is shown that uniqueness by a finite number of incident plane waves could be attained under the Dirichlet boundary condition if some a-priori information about the height of the grating curve is known in $\mathbb{R}^2$. The global uniqueness can also be established for the inverse scattering with a finite number of incident plane waves if the grating profiles are piecewise linear; see [24]. Recently, in [12], the unique recovery for the inverse diffraction grating with generalized impedance boundary condition (including the Dirichlet and Neumann boundary conditions) has been proved in a unified way by at most two far-field measurements under some mild assumptions. The corresponding development for uniqueness results of inverse elastic polygonal diffractive grating problems by finite many far field measurements can be found in [22]. In this paper, we establish the unique identifiability of a periodical diffraction grating with impedance boundary condition in $\mathbb{R}^3$ by a single far-field measurement, which makes a significant progress compared with the existing results.

The rest of this paper is organized as follows. In Section 2, we present the extension and generalization of our results in [13] on the geometric structures of Laplacian eigenfunctions. Section 3 is devoted to the proofs of the unique identifiability results for the inverse scattering problems.

2. **Geometric properties of Laplacian eigenfunctions.**

In [13], certain geometric properties of Laplacian eigenfunctions in $\mathbb{R}^3$ were investigated. Specifically, we studied the cases for edge corners and vertex corners respectively and derived a rigorous characterization of the relationship between the analytic behaviour of Laplacian eigenfunctions at the underlying corner point and the geometric quantities of that corner. In fact, in the edge corner case, the vanishing order of the eigenfunction is related to the rationality of the intersecting dihedral angle, whereas in the vertex corner case, the vanishing order of the eigenfunction follows a more complicated manner through the roots of the Legendre polynomials. In this section, as an extension of the study in [13], we establish the vanishing properties of the Laplacian eigenfunction by getting rid of the technical condition that $u(\mathbf{x}_c) = 0$ at the intersecting point as well as relaxing the technical restrictions associated with the Legendre polynomials.

Let Ω be an open set in $\mathbb{R}^3$. Consider $u \in L^2(\Omega)$ and $\lambda \in \mathbb{R}_+$ such that

$$-\Delta u = \lambda u. \tag{2.1}$$

The solution u to (2.1) is referred to as a (generalized) Laplacian eigenfunction.

First, we introduce some notifications for the subsequent use. Let Π be a flat plane in $\mathbb{R}^3$. For any non-empty connected open subset $\Sigma \Subset \Pi$, it is said to be a cell of Π. Denote Π_Σ to be the connected component of $\Pi \cap \Omega$ which contains Σ.

Definition 2.1. Let Σ be a cell of Π such that $\Sigma \subset \Omega$ and $\eta \in L^\infty(\Pi)$. Consider a Laplacian eigenfunction u to (2.1). If $\partial_\nu u + \eta u = 0$ on Σ, where ν denotes any unit normal vector that is perpendicular to Σ, then Σ is said to be the generalized singular cell in Ω and Π is called the generalized singular plane. In particular, if $\eta \equiv 0$ on Σ (i.e. $\partial u = 0$ on Σ), a generalized singular plane is called a singular plane and if $\eta = \infty$ on Σ (i.e. $u = 0$ on Σ), a generalized singular plane is also called a nodal plane.

Next, we follow Definition 1.4 and Definition 1.5 in [13] to present the precise definition for vanishing orders of the Laplacian eigenfunction u at a given point associated with an edge corner or a vertex corner.

Definition 2.2. Let u be a Laplacian eigenfunction to (2.1). For a given point $\mathbf{x}_0 \in \Omega$, if there exists a number $N \in \mathbb{N} \cup \{0\}$ such that

$$\lim_{\rho\to 0+} \frac{1}{\rho^m} \int_{B_\rho(\mathbf{x}_0)} |u(\mathbf{x})| \, \mathrm{d}\mathbf{x} = 0 \quad \text{for} \quad m = 0, 1, \cdots, N+2, \tag{2.2}$$

where $B_\rho(\mathbf{x}_0)$ is a ball centered at $\mathbf{x}_0$ with radius $\rho \in \mathbb{R}_+$, we say that u vanishes at $\mathbf{x}_0$ up to the order N. The largest possible N such that (2.2) holds is called the vanishing order of u at $\mathbf{x}_0$ and we denote

$$\mathrm{Vani}(u; \mathbf{x}_0) = N.$$

If (2.2) holds for any $N \in \mathbb{N}$ at $\mathbf{x}_0 \in \Omega$, then we say that the vanishing order of u at $\mathbf{x}_0$ is infinity, i.e. $\mathrm{Vani}(u; \mathbf{x}_0) = \infty$.

In particular, for any two intersecting generalized singular planes $\Pi_1, \Pi_2 \subset \Omega$ such that $\Pi_1 \cap \Pi_2 = \boldsymbol{l}$, if (2.2) is fulfilled for a point $\mathbf{x}_0 \in \boldsymbol{l}$ associated with the edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$, then we say that u vanishes at $\mathbf{x}_0$ associated with $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ up to the order N which is denoted as

$$\mathrm{Vani}(\mathrm{u}; \mathbf{x}_0, \Pi_1, \Pi_2) = \mathrm{N}.$$

For a vertex corner $\mathbf{x}_0 \in \Omega$ which is intersected by $\Pi_i \subset \Omega$, $i = 1, 2, ...n$, the vanishing order of u at $\mathbf{x}_0$ is defined by

$$\mathrm{Vani}(u; \mathbf{x}_0) := \max\Big\{ \max_{i=1,2,\ldots n-1} \mathrm{Vani}(u; \mathbf{x}_0, \Pi_i, \Pi_{i+1}), \mathrm{Vani}(u; \mathbf{x}_0, \Pi_n, \Pi_1) \Big\}.$$

With the above definitions, we are now in a position to investigate the vanishing properties of the Laplacian eigenfunction at corners intersected by at least two generalized singular planes. Since $-\Delta$ is invariant under rigid motions, we can assume that the edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l})$ fulfills

$$\boldsymbol{l} = \big\{ \ \mathbf{x} = (\mathbf{x}', x_3) \in \mathbb{R}^3; \mathbf{x}' = 0, \ x_3 \in (-H, H) \big\} \Subset \Omega, \tag{2.3}$$

for $H \in \mathbb{R}^+$ throughout the rest of our paper. Indeed, this indicates that the edge corner coincides with the x_3-axis. For simplification, we further assume that Π_1 coincides with the $(x_1, x_3)-$plane and Π_2 possesses a dihedral angle of $\alpha \cdot \pi$ away from Π_1 in the anti-clockwise direction. The considering point $\mathbf{x}_0 \in \boldsymbol{l}$ is assumed to be located at the origin $\mathbf{x}_0 = \mathbf{0}$. Similar to [13], with the help of analytic continuation property for u, we can restrict our discussion to $\alpha \in (0, 1)$.

In order to utilize the spherical wave expansion method to discuss the vanishing properties of u, we first introduce several propositions and lemmas based on the spherical coordinates in $\mathbb{R}^3$. For any point $\mathbf{x} \in \mathbb{R}^3$, we denote

$$\mathbf{x} = (x_1, x_2, x_3) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta) := (r, \theta, \phi), \tag{2.4}$$

where $r \geq 0, \theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$.

It is known that the Laplacian eigenfunction u possesses the spherical wave expansion as follows.

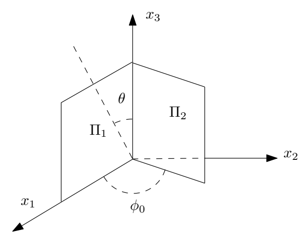

FIGURE 1. A schematic illustration for an edge corner with the dihedral angle ϕ_0.

Lemma 2.3. *[16, Theorem 2.8]The solution u to (2.1) has the spherical wave expansion in spherical coordinates around the origin:*

$$u(\mathbf{x}) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) Y_n^m(\theta, \phi), \tag{2.5}$$

where $j_n(\sqrt{\lambda} r)$ is the spherical Bessel function of order n, and $Y_n^m(\theta,\phi)$ is the spherical harmonics given by

$$Y_n^m(\theta,\phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\phi}$$

with $P_n^m(\cos\theta)$ being the associated Legendre functions.

See also [16, equation (3.92)] for detailed representations. In particular, the associated Legendre polynomials P_n^m fulfill the following orthogonal relation.

Lemma 2.4. *[46, Theorem 2.4.4] In the spherical coordinate system, the associated Legendre functions fulfill the following orthogonality condition for any fixed $n \in \mathbb{N}$, and any two integers $m \geq 0$ and $l \leq n$:*

$$\int_0^{\pi} \frac{P_n^m(\cos\theta) P_n^l(\cos\theta)}{\sin\theta}\, d\theta = \begin{cases} 0 & \text{if } l \neq m, \\ \frac{(n+m)!}{m(n-m)!} & \text{if } \quad l = m \neq 0, \\ \infty & \text{if } \quad m = l = 0. \end{cases}$$

Furthermore, we know that the following recursive relations hold for P_n^m [8] and for the spherical Bessel function $j_n(\sqrt{\lambda} r)$ [4].

Lemma 2.5. *In the spherical coordinate system, the associated Legendre functions P_n^m fulfills the following recursive equations for any fixed $n, m \in \mathbb{Z}$:*

$$\frac{dP_n^{|m|}(\cos\theta)}{d\theta} = \frac{1}{2}\Big((n+|m|)(n-|m|+1)P_n^{|m|-1}(\cos\theta) - P_n^{|m|+1}(\cos\theta)\Big), \tag{2.6}$$

and

$$|m|\frac{P_n^{|m|}(\cos\theta)}{\sin\theta} = -\frac{1}{2}\Big(P_{n+1}^{|m|+1}(\cos\theta) + (n-|m|+1)(n-|m|+2)P_{n+1}^{|m|-1}(\cos\theta)\Big), \tag{2.7}$$

which can also be represented as

$$|m|\frac{P_n^{|m|}(\cos\theta)}{\sin\theta} = -\frac{1}{2}\Big(P_{n-1}^{|m|+1}(\cos\theta) + (n+|m|-1)(n+|m|)P_{n-1}^{|m|-1}(\cos\theta)\Big), \tag{2.8}$$

with slight changes of the lower index. Besides, the spherical Bessel function $j_n(\sqrt{\lambda}r)$ *satisfies that*

$$\frac{j_n(\sqrt{\lambda}r)}{r} = \frac{\sqrt{\lambda}}{2n+1}\left(j_{n-1}(\sqrt{\lambda}r) + j_{n+1}(\sqrt{\lambda}r)\right). \tag{2.9}$$

We proceed to present our main theorem regarding the theoretical study of the vanishing properties of Laplacian eigenfunctions at any edge corner intersected by two generalized singular planes. For simplification in the subsequent study, we assume the impedance boundary parameter η to be a nonzero constant. Nevertheless, it is pointed out that the same results hold for any real analytic function η satisfying the admissible conditions (a) to (c) in Definition 1.1.; see Corollary 2.10.

Theorem 2.6. *Let u be a Laplacian eigenfunction to (2.1). Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ associated with two generalized singular planes Π_1 and Π_2, where the corresponding surface parameter on Π_1 and Π_2 are two non-zero constants η_1 and η_2. If the corresponding dihedral angle can be written as*

$$\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha \cdot \pi, \quad \alpha \in (0,1),$$

where α satisfies that for any $N \in \mathbb{N}$, $N \geq 2$,

$$\alpha \neq \frac{q}{p}, \quad p = 1, 2, \cdots, N-1, \; q = 0, 1, 2, \cdots, p-1, \tag{2.10}$$

and the corresponding surface parameters η_i associated with Π_i, $i = 1, 2$, fulfill that

$$2\eta_1 \cos\phi_0 + \eta_2(1 + \cos 2\phi_0) \neq 0, \tag{2.11}$$

then u vanishes up to the order at least N at the edge corner $\mathbf{0}$.

In order to prove Theorem 2.6 more systematically and rigorously, we first give the following lemmas in terms of the expressions of generalized singular planes and the corresponding edge corner by recursive form.

Lemma 2.7. *Let u be a Laplacian eigenfunction to (2.1). Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ associated with two generalized singular planes Π_1 and Π_2. Then there holds the following recursive equations with respect to P_n^m and $j_n(\sqrt{\lambda}r)$ ($n, m \in \mathbb{N}$) on Π_1 and Π_2, respectively:*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{\substack{m=-n\\ m\neq 0}}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}}{2n+1}\left(j_{n-1}(\sqrt{\lambda}r) + j_{n+1}(\sqrt{\lambda}r)\right)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\\
&\times \frac{1}{2|m|}\left(P_{n-1}^{|m|+1}(\cos\theta) + (n+|m|-1)(n+|m|)P_{n-1}^{|m|-1}(\cos\theta)\right)\\
&+\eta_1\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}P_n^{|m|}(\cos\theta) = 0 \quad \text{on} \quad \Pi_1,
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&-\sum_{n=1}^{\infty}\sum_{\substack{m=-n\\ m\neq 0}}^{n} \mathrm{i}^{n+1} m a_n^m \frac{\sqrt{\lambda}}{2n+1}\left(j_{n-1}(\sqrt{\lambda}r) + j_{n+1}(\sqrt{\lambda}r)\right)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}e^{\mathrm{i}m\phi_0}\\
&\times \frac{1}{2|m|}\left(P_{n-1}^{|m|+1}(\cos\theta) + (n+|m|-1)(n+|m|)P_{n-1}^{|m|-1}(\cos\theta)\right)
\end{aligned}$$

$$+\eta_2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\phi_0} = 0 \quad \textit{on} \quad \Pi_2. \tag{2.13}$$

Proof. Since Π_1 and Π_2 are two generalized singular planes such that $\Pi_1 \cap \Pi_2 = \boldsymbol{l}$, it can be seen that one has

$$\frac{\partial u}{\partial \nu_1} + \eta_1 u = 0 \ \text{ on } \Pi_1, \quad \frac{\partial u}{\partial \nu_2} + \eta_2 u = 0 \ \text{ on } \Pi_2, \tag{2.14}$$

where ν_i, $i = 1, 2$, are the unit normal vectors perpendicular to Π_1 and Π_2, respectively. η_1 and η_2 are the corresponding impedance parameters. Recall the spherical wave expansion (2.5) of the Laplacian eigenfunction u in Lemma 2.3. Since we assume that Π_1 coincides with the $(x_1, x_3)-$plane, by taking $\phi = 0$, we know that on Π_1, it holds that

$$\begin{aligned} &-\frac{1}{r\sin\theta} 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m j_n \sqrt{\lambda} r \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) \\ &+\eta_1 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m \sqrt{\lambda} r \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) = 0 \quad \text{on } \Pi_1. \end{aligned} \tag{2.15}$$

Combining with the recursive equations (2.8) and (2.9) in Lemma 2.5 with respect to P_n^m and $j_n(\sqrt{\lambda} r)$, we can deduce (2.12) from (2.15) directly.

Similarly, since Π_2 possesses a dihedral angle of ϕ_0 away from Π_1 in the anticlockwise direction, rewriting $\frac{\partial u}{\partial \nu_2} + \eta_2 u = 0$ in terms of (2.5) with $\phi = \phi_0$, it yields that

$$\begin{aligned} &\frac{1}{r\sin\theta} 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^{n+1} m a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\phi_0} \\ &+\eta_2 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{i}^n a_n^m j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\phi_0} \quad \text{on} \quad \Pi_2. \end{aligned} \tag{2.16}$$

Combining with (2.8) and (2.9), we can obtain from (2.16) the recursive equation (2.13) in the same manner. $\square$

Lemma 2.8. *Let u be a Laplacian eigenfunction to (2.1). Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ associated with two generalized singular planes Π_1 and Π_2. Then there holds the following recursive equations with respect to P_n^m and $j_n(\sqrt{\lambda} r)$ $(n, m \in \mathbb{N})$ on $\boldsymbol{l}$ that*

$$\begin{aligned} &-\sum_{n=1}^{\infty} \mathrm{i}^n \frac{\sqrt{\lambda}}{2n+1} \left(j_{n-1}(\sqrt{\lambda} r) + j_{n+1}(\sqrt{\lambda} r) \right) \sqrt{\frac{2n+1}{4\pi}} \frac{1}{2} n(n+1) \sqrt{\frac{(n-1)!}{(n+1)!}} \sin\phi_0 \\ &\times (a_n^1 e^{\mathrm{i}2\phi_0} + a_n^{-1} e^{-\mathrm{i}2\phi_0}) + (\eta_1 \cos\phi_0 + \eta_2) \sum_{n=0}^{\infty} \mathrm{i}^n a_n^0 j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} = 0, \quad \textit{on} \quad \boldsymbol{l}. \end{aligned} \tag{2.17}$$

Proof. From Lemma 2.7, we know that

$$\frac{\partial u}{\partial \nu_1} + \eta_1 u = 0 \ \text{ on } \Pi_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu_2} + \eta_2 u = 0 \ \text{ on } \Pi_2.$$

Since Π_1 locates in the $(x_1, x_3)-$plane and Π_2 possesse a dihedral angle of ϕ_0 away from Π_1 in the anti-clockwise direction, without loss of generality, we can denote $\nu_1 = (0, -1, 0)$ and $\nu_2 = (-\sin\phi_0, \cos\phi_0, 0)$; seeing Figure 1 for the schematic illustration in spherical coordinate system.

Thus, we can derive

$$\frac{\partial u}{\partial \nu_1} + \eta_1 u = -\frac{\partial u}{\partial x_2} + \eta_1 u = 0, \qquad \text{on } \Pi_1, \tag{2.18}$$

$$\frac{\partial u}{\partial \nu_2} + \eta_2 u = -\frac{\partial u}{\partial x_1}\sin\phi_0 + \frac{\partial u}{\partial x_2}\cos\phi_0 + \eta_2 u = 0, \qquad \text{on } \Pi_2. \tag{2.19}$$

Substituting (2.19) into (2.18), we have

$$-\frac{\partial u}{\partial x_1}\sin\phi_0 + (\eta_1\cos\phi_0 + \eta_2)u = 0 \quad \text{on} \quad \boldsymbol{l} = \Pi_1 \cap \Pi_2. \tag{2.20}$$

By utilizing the chain rule of spherical coordinate system, there holds

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x_1} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x_1} + \frac{\partial u}{\partial \phi}\cdot\frac{\partial \phi}{\partial x_1} \\ &= \frac{\partial u}{\partial r}\sin\theta\cos\phi + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta\cos\phi}{r} - \frac{\partial u}{\partial \phi}\cdot\frac{\sin\phi}{r\sin\theta}.\end{aligned} \tag{2.21}$$

Restricting (2.21) on $\boldsymbol{l}$, by taking $\theta = 0$ and $\phi = \phi_0$, we know that on $\boldsymbol{l}$

$$\left.\frac{\partial u}{\partial x_1}\right|_{\boldsymbol{l}} = \left.\frac{\partial u}{\partial \theta}\right|_{\theta=0}\frac{\cos\phi_0}{r} - \lim_{\theta\to 0}\left.\frac{\partial u}{\partial \phi}\frac{\sin\phi_0}{r\sin\theta}\right|_{\phi=\phi_0}.$$

We first prove that

$$\begin{aligned}&\lim_{\theta\to 0}\left.\frac{\partial u}{\partial \phi}\frac{\sin\phi_0}{r\sin\theta}\right|_{\phi=\phi_0} \\ =& -2\pi\sum_{n=0}^{\infty}\mathrm{i}^{n+1}(a_n^1 - a_n^{-1})\frac{j_n(\sqrt{\lambda}r)}{r}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-1)!}{(n+1)!}}n(n+1) \\ &\times e^{\mathrm{i}m\phi_0}\sin\phi_0\end{aligned}$$

exists. Indeed, by the spherical wave expansion (2.5) of u in Lemma 2.3, we have

$$\begin{aligned}\left.\frac{\partial u}{\partial \phi}\frac{\sin\phi_0}{r\sin\theta}\right|_{\phi=\phi_0} =& 4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}ma_n^m\frac{j_n(\sqrt{\lambda}r)}{r}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\frac{P_n^{|m|}(\cos\theta)}{\sin\theta} \\ &\times e^{\mathrm{i}m\phi_0}\sin\phi_0.\end{aligned}$$

Recall the recursive equation (2.7). Since

$$\lim_{\theta\to 0}\frac{P_n^{|m|}(\cos\theta)}{\sin\theta} = -\frac{1}{2|m|}\left(P_{n+1}^{|m|+1}(1) + (n-|m|+1)(n-|m|+2)P_{n+1}^{|m|-1}(1)\right), \tag{2.22}$$

where we can know from [8] that

$$P_n^{|m|}(\pm 1) \equiv 0, \text{ for } m \neq 0, \quad P_n^0(1) \equiv 1, \tag{2.23}$$

it is easy to obtain

$$\lim_{\theta\to 0}\frac{P_n^{|m|}(\cos\theta)}{\sin\theta} = \begin{cases} 0 & \text{if } |m| \neq 1, \\ -\frac{1}{2}n(n+1) & \text{if } |m| = 1.\end{cases}$$

Thus we can derive

$$\begin{aligned}\lim_{\theta\to 0}\frac{\partial u}{\partial \phi}\frac{\sin\phi_0}{\sin\theta}\bigg|_{\phi=\phi_0} &= 4\pi\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}m a_n^m\frac{j_n(\sqrt{\lambda}r)}{r}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\\ &\quad\times\lim_{\theta\to 0}\frac{P_n^{|m|}(\cos\theta)}{\sin\theta}e^{\mathrm{i}m\phi_0}\sin\phi_0\\ &= -2\pi\sum_{n=0}^{\infty}\mathrm{i}^{n+1}(a_n^1-a_n^{-1})\frac{j_n(\sqrt{\lambda}r)}{r}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-1)!}{(n+1)!}}n(n+1)\\ &\quad\times e^{\mathrm{i}m\phi_0}\sin\phi_0. \end{aligned}\tag{2.24}$$

Then, by restricting (2.21) on $\boldsymbol{l}$, (2.20) can be rewritten in terms of (2.21) as

$$\left(-\frac{\partial u}{\partial\theta}\bigg|_{\theta=0}\frac{\cos\phi_0}{r}+\lim_{\theta\to 0}\frac{\partial u}{\partial\phi}\frac{\sin\phi_0}{r\sin\theta}\right)\cdot\sin\phi_0+(\eta_1\cos\phi_0+\eta_2)u=0\quad\text{on }\boldsymbol{l}.\tag{2.25}$$

Substituting the spherical wave expansion (2.5) of the Laplacian eigenfunction u into (2.25) again, with the help of the recursive equations (2.6) and (2.7) associated with $P_n^{|m|}$ in Lemma 2.5, we can obtain that

$$\begin{aligned}&\Big(-\sum_{n=0}^{\infty}\sum_{m=-n}^{m}\mathrm{i}^n a_n^m\frac{j_n\sqrt{\lambda}r}{r}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\frac{dP_n^{|m|}(\cos\theta)}{d\theta}\bigg|_{\theta=0}e^{\mathrm{i}m\phi_0}\cos\phi_0\\ &+\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^{n+1}m a_n^m\frac{j_n\sqrt{\lambda}r}{r}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}e^{\mathrm{i}m\phi_0}\sin\phi_0\frac{P_n^{|m|}(\cos\phi_0)}{\sin\phi_0}\bigg|_{\theta=0}\Big)\\ &\times\sin\phi_0+(\eta_1\cos\phi_0+\eta_2)\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathrm{i}^n a_n^m j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-|m|)!}{(n+|m|)!}}\\ &\times P_n^{|m|}(1)e^{\mathrm{i}m\phi_0}=0,\end{aligned}\tag{2.26}$$

where

$$\frac{dP_n^{|m|}(\cos\theta)}{d\theta}\bigg|_{\theta=0}=\frac{1}{2}\Big((n+|m|)(n-|m|+1)P_n^{|m|-1}(1)-P_n^{|m|+1}(1)\Big),\tag{2.27}$$

and $\frac{P_n^{|m|}(\cos\theta)}{\sin\theta}\Big|_{\theta=0}$ is given by (2.22). Therefore, (2.26) can be further simplified as the following equation by virtue of (2.23) as

$$\begin{aligned}&\Big(-\sum_{n=0}^{\infty}\mathrm{i}^n\frac{j_n(\sqrt{\lambda}r)}{r}\sqrt{\frac{2n+1}{4\pi}}(a_n^0\frac{1}{2}n(n+1)P_n^{-1}(1)+(a_n^1e^{\mathrm{i}\phi_0}+a_n^{-1}e^{-\mathrm{i}\phi_0})\sqrt{\frac{(n-1)!}{(n+1)!}}\\ &\times\frac{(n+1)n}{2})\cos\phi_0-\sum_{n=0}^{\infty}\mathrm{i}^{n+1}\frac{j_n(\sqrt{\lambda}r)}{r}\sqrt{\frac{2n+1}{4\pi}}\sqrt{\frac{(n-1)!}{(n+1)!}}(a_n^1e^{\mathrm{i}\phi_0}-a_n^{-1}e^{-\mathrm{i}\phi_0})\\ &\times\sin\phi_0\frac{n(n+1)}{2}\Big)\sin\phi_0+(\eta_1\cos\phi_0+\eta_2)\sum_{n=0}^{\infty}\mathrm{i}^n a_n^0 j_n(\sqrt{\lambda}r)\sqrt{\frac{2n+1}{4\pi}}=0,\end{aligned}\tag{2.28}$$

where

$$P_n^{-1}(1)=(-1)^1\frac{(n-1)!}{(n+1)!}P_n^1(1)=-\frac{1}{n(n+1)}P_n^1(1)=0.$$

By substituting the recursive expression (2.9) of $j_n(\sqrt{\lambda})$ into (2.28), we can obtain (2.17). $\square$

With the help of Lemma 2.7 and Lemma 2.8, we are now in a position to give the rigorous proof of Theorem 2.6 as follows.

Proof of Theorem 2.6. We apply mathematical induction by investigating the coefficients with respect to r^n for $n=0,1,2,\cdots$ in (2.17) on $\boldsymbol{l}$,(2.12) on Π_1 and (2.13) on Π_2, separately, to prove our main results. For simplification of notations, we denote $\mathcal{O}(r^n), n=0,1,\cdots$, by the n-th order term of r.

(I). Comparing the coefficients of $\mathcal{O}(r^0)$.

From (2.17), we can know that the coefficient of r^0 on $\boldsymbol{l}$ fulfills

$$-\mathrm{i}\frac{\sqrt{\lambda}}{3}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{1}{2}}\sin\phi_0(a_1^1e^{\mathrm{i}2\phi_0}+a_1^{-1}e^{-\mathrm{i}2\phi_0})+(\eta_1\cos\phi_0+\eta_2)a_0^0\sqrt{\frac{1}{4\pi}}=0 \quad \text{on } \boldsymbol{l}, \tag{2.29}$$

by taking $r\to 0$ in (2.17) under (1.2). Similarly, we can show that the coefficients of r^0 in (2.12) and (2.13) respectively satisfy

$$-(a_1^1-a_1^{-1})\frac{\sqrt{\lambda}}{3}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{1}{2}}P_0^0(\cos\theta)+\eta_1a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta)=0 \quad \text{on } \Pi_1, \tag{2.30}$$

and

$$(a_1^1e^{\mathrm{i}\phi_0}-a_1^{-1}e^{-\mathrm{i}\phi_0})\frac{\sqrt{\lambda}}{3}\sqrt{\frac{3}{4\pi}}\sqrt{\frac{1}{2}}P_0^0(\cos\theta)+\eta_2a_0^0\sqrt{\frac{1}{4\pi}}P_0^0(\cos\theta)=0 \quad \text{on } \Pi_2. \tag{2.31}$$

For simplification, we further rewrite (2.29), (2.30) and (2.31) more briefly as the following system:

$$\begin{pmatrix} -\mathrm{i}\sqrt{\frac{\lambda}{6}}\sin\phi_0e^{\mathrm{i}2\phi_0} & -\mathrm{i}\sqrt{\frac{\lambda}{6}}\sin\phi_0e^{-\mathrm{i}2\phi_0} & \eta_1\cos\phi_0+\eta_2 \\ \sqrt{\frac{\lambda}{6}} & -\sqrt{\frac{\lambda}{6}} & -\eta_1 \\ \sqrt{\frac{\lambda}{6}}e^{\mathrm{i}\phi_0} & -\sqrt{\frac{\lambda}{6}}e^{-\mathrm{i}\phi_0} & \eta_2 \end{pmatrix}\begin{pmatrix} a_1^1 \\ a_1^{-1} \\ a_0^0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.32}$$

It is easy to see that the determinant of the coefficient matrix of (2.32) with respect to $a_1^{\pm1}$ and a_0 fulfills

$$\begin{aligned}&\begin{vmatrix} -\mathrm{i}\sqrt{\frac{\lambda}{6}}\sin\phi_0e^{\mathrm{i}2\phi_0} & -\mathrm{i}\sqrt{\frac{\lambda}{6}}\sin\phi_0e^{-\mathrm{i}2\phi_0} & \eta_1\cos\phi_0+\eta_2 \\ \sqrt{\frac{\lambda}{6}} & -\sqrt{\frac{\lambda}{6}} & -\eta_1 \\ \sqrt{\frac{\lambda}{6}}e^{\mathrm{i}\phi_0} & -\sqrt{\frac{\lambda}{6}}e^{-\mathrm{i}\phi_0} & \eta_2 \end{vmatrix}\\ &=\frac{\lambda}{6}\left((\eta_1\cos\phi_0+\eta_2)2\mathrm{i}\sin\phi_0+\eta_1\mathrm{i}\sin 2\phi_0+\eta_2 2\mathrm{i}\sin\phi_0\cos 2\phi_0\right)\\ &=\frac{\mathrm{i}\lambda}{3}\sin\phi_0(2\eta_1\cos\phi_0+\eta_2(\cos 2\phi_0+1)).\end{aligned} \tag{2.33}$$

By the condition (2.55), we can know that $\phi_0\neq 0,\pi$. Due to (2.11), one has

$$a_1^{\pm1}=a_0^0=0. \tag{2.34}$$

(II).Comparing the coefficients of $\mathcal{O}(r^1)$.

We proceed to compare the coefficients of r^1 on the both sides of (2.17), (2.12) and (2.13), where we can derive that

$$\frac{\sqrt{\lambda}}{5}\frac{\sqrt{\lambda}}{3}\sqrt{\frac{5}{4\pi}}3\sqrt{\frac{1}{6}}\sin\phi_0(a_2^1e^{\mathrm{i}2\phi_0}+a_2^{-1}e^{-\mathrm{i}2\phi_0})+(\eta_1\cos\phi_0+\eta_2)\mathrm{i}a_1^0\frac{\sqrt{\lambda}}{3}\sqrt{\frac{3}{4\pi}}=0 \quad \text{on } \boldsymbol{l},$$

$$\begin{aligned}&-\mathrm{i}\left(2(a_2^2-a_2^{-2})\sqrt{\frac{1}{4!}}3P_1^1(\cos\theta)+(a_2^1-a_2^{-1})\sqrt{\frac{1}{3!}}3P_1^0(\cos\theta)\right)\frac{\sqrt{\lambda}}{5}\frac{\sqrt{\lambda}}{3}\sqrt{\frac{5}{4\pi}}\\&+\eta_1\mathrm{i}\frac{\sqrt{\lambda}}{3}\sqrt{\frac{3}{4\pi}}\left((a_1^1+a_1^{-1})\sqrt{\frac{1}{2!}}P_1^1(\cos\theta)+a_1^0P_1^0(\cos\theta)\right)=0 \quad \text{on } \Pi_1,\end{aligned} \tag{2.35}$$

and

$$\begin{aligned}&\mathrm{i}\frac{\sqrt{\lambda}}{5}\frac{\sqrt{\lambda}}{3}\sqrt{\frac{5}{4\pi}}\left(2(a_2^2e^{\mathrm{i}2\phi_0}-a_2^{-2}e^{-\mathrm{i}2\phi_0})\sqrt{\frac{1}{4!}}3P_1^1(\cos\theta)+(a_2^1e^{\mathrm{i}\phi_0}-a_2^{-1}e^{-\mathrm{i}\phi_0})\right.\\&\left.\times\sqrt{\frac{1}{3!}}3P_1^0(\cos\theta)\right)+\eta_2\mathrm{i}\frac{\sqrt{\lambda}}{3}\sqrt{\frac{3}{4\pi}}\Big((a_1^1e^{\mathrm{i}\phi_0}+a_1^{-1}e^{-\mathrm{i}\phi_0})\sqrt{\frac{1}{2!}}P_1^1(\cos\theta)\\&+a_1^0P_1^0(\cos\theta)\Big)=0 \quad \text{on } \Pi_2.\end{aligned} \tag{2.36}$$

From (2.34), by substituting $a_1^{\pm1}=a_0^0=0$ into (2.35) and (2.36), we have

$$\begin{aligned}&\sqrt{\frac{3\lambda}{10}}\sin\phi_0(a_2^1e^{\mathrm{i}2\phi_0}+a_2^{-1}e^{-\mathrm{i}2\phi_0})+(\eta_1\cos\phi_0+\eta_2)\mathrm{i}a_1^0\sqrt{3}=0 \quad \text{on } \boldsymbol{l},\\&-\mathrm{i}\left(6(a_2^2-a_2^{-2})\sqrt{\frac{1}{4!}}P_1^1(\cos\theta)+3(a_2^1-a_2^{-1})\sqrt{\frac{1}{3!}}P_1^0(\cos\theta)\right)\sqrt{\frac{\lambda}{5}}\\&+\eta_1\mathrm{i}\sqrt{3}a_1^0P_1^0(\cos\theta)=0 \quad \text{on } \Pi_1,\\&\mathrm{i}\sqrt{\frac{\lambda}{5}}\left(6(a_2^2e^{\mathrm{i}2\phi_0}-a_2^{-2}e^{-\mathrm{i}2\phi_0})\sqrt{\frac{1}{4!}}P_1^1(\cos\theta)+3(a_2^1e^{\mathrm{i}\phi_0}-a_2^{-1}e^{-\mathrm{i}\phi_0})\sqrt{\frac{1}{3!}}P_1^0(\cos\theta)\right)\\&+\eta_2\mathrm{i}\sqrt{3}a_1^0P_1^0(\cos\theta)=0 \quad \text{on } \Pi_2.\end{aligned} \tag{2.37}$$

After rearranging (2.37) in terms of $P_1^1(\cos\theta)$ and $P_1^0(\cos\theta)$, we can know that there holds respectively on $\boldsymbol{l}$, Π_1 and Π_2 the following equations:

$$\sqrt{\frac{\lambda}{10}}\sin\phi_0(a_2^1e^{\mathrm{i}2\phi_0}+a_2^{-1}e^{-\mathrm{i}2\phi_0})+(\eta_1\cos\phi_0+\eta_2)\mathrm{i}a_1^0=0, \tag{2.38}$$

$$-\mathrm{i}6\sqrt{\frac{1}{4!}}\sqrt{\frac{\lambda}{5}}(a_2^2-a_2^{-2})P_1^1(\cos\theta)+\mathrm{i}\left(\sqrt{3}\eta_1a_1^0-3(a_2^1-a_2^{-1})\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}}\right)P_1^0(\cos\theta)=0, \tag{2.39}$$

and

$$\begin{aligned}&\mathrm{i}6\sqrt{\frac{1}{4!}}\sqrt{\frac{\lambda}{5}}(a_2^2e^{\mathrm{i}2\phi_0}-a_2^{-2}e^{-\mathrm{i}2\phi_0})P_1^1(\cos\theta)+\mathrm{i}\Big(\sqrt{3}\eta_2a_1^0\\&+3(a_2^1e^{\mathrm{i}\phi_0}-a_2^{-1}e^{-\mathrm{i}\phi_0})\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}}\Big)P_1^0(\cos\theta)=0.\end{aligned} \tag{2.40}$$

In (2.39) and (2.40), using the orthogonal property of the associated Legendre polynomials in Lemma 2.4, we can obtain that

$$\begin{cases} a_2^2 - a_2^{-2} = 0, \\ \eta_1\sqrt{3}a_1^0 - 3\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}}(a_2^1 - a_2^{-1}) = 0, \end{cases} \tag{2.41}$$

and

$$\begin{cases} a_2^2 e^{\mathrm{i}2\phi_0} - a_2^{-2}e^{-\mathrm{i}2\phi_0} = 0, \\ \eta_2\sqrt{3}a_1^0 + 3\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}}(a_2^1 e^{\mathrm{i}\phi_0} - a_2^{-1}e^{-\mathrm{i}\phi_0}) = 0. \end{cases} \tag{2.42}$$

From the first equations in (2.41) and (2.42), it is easy to see that under condition (2.55), which indicates $\phi_0 \neq 0, \frac{\pi}{2}, \pi$, the determinant of the coefficient matrix of $a_2^{\pm 2}$ fulfills that

$$\begin{vmatrix} 1 & -1 \\ e^{\mathrm{i}\phi_0} & e^{-\mathrm{i}\phi_0} \end{vmatrix} = 2\mathrm{i}\sin 2\phi_0 \neq 0,$$

and therefore we have $a_2^{\pm 2} = 0$. Now combining with (2.38), (2.41) and (2.42), we can derive the deteminant of the coefficient matrix with respect to a_1^0, $a_2^{\pm 1}$ fulfills

$$\begin{aligned} &\begin{vmatrix} \mathrm{i}(\eta_1\cos\phi_0 + \eta_2) & \sqrt{\frac{\lambda}{10}}\sin\phi_0 e^{\mathrm{i}2\phi_0} & \sqrt{\frac{\lambda}{10}}\sin\phi_0 e^{-\mathrm{i}2\phi_0} \\ \sqrt{3}\eta_1 & -3\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}} & 3\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}} \\ \sqrt{3}\eta_2 & 3\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}}e^{\mathrm{i}\phi_0} & -3\sqrt{\frac{1}{3!}}\sqrt{\frac{\lambda}{5}}e^{-\mathrm{i}\phi_0} \end{vmatrix} \\ &= \frac{3\lambda}{5}\sin\phi_0(2\eta_1\cos\phi_0 + \eta_2 + \eta_2\cos 2\phi_0) \neq 0, \end{aligned}$$

due to the fact that $\phi_0 \neq 0, \pi$. Under the condition (2.11), we derive $a_1^0 = a_2^{\pm 1} = 0$.

(III). Comparing the coefficients of $\mathcal{O}(r^{N-1})$ by mathematical induction.

By utilizing mathematical induction method, we assume that $a_{N-2}^0 = a_{N-1}^{\pm m} = 0$ for $m = 1, 2, \cdots, N-1$, where $N \geq 3$. Following the fact that

$$\lim_{z\to 0}\frac{j_n(z)}{z^n} = \frac{1}{(2n+1)!!},$$

by comparing the coefficients of r^{N-1} on the both sides of (2.17), (2.12) and (2.13), respectively, we can know that on $\boldsymbol{l}$ there holds:

$$\begin{aligned} &-\mathrm{i}\sqrt{\frac{\lambda}{2N+1}}\frac{1}{2}N(N+1)\sqrt{\frac{(N-1)!}{(N+1)!}}\frac{(\sqrt{\lambda})^{N-1}}{(2N-1)!!}\sin\phi_0(a_N^1 e^{\mathrm{i}2\phi_0} + a_N^{-1}e^{-\mathrm{i}2\phi_0}) \\ &+ (\eta_1\cos\phi_0 + \eta_2)a_{N-1}^0\sqrt{2N-1}\frac{(\sqrt{\lambda})^{N-1}}{(2N-1)!!} = 0, \end{aligned} \tag{2.43}$$

and on Π_1,

$$\begin{aligned} &-\sqrt{\frac{\lambda}{2N+1}}\sum_{\substack{m=-N\\ m\neq 0}}^{N} m a_N^m\sqrt{\frac{(N-|m|)!}{(N+|m|)!}}\frac{1}{2|m|}(P_{N-1}^{|m|+1}(\cos\theta) + (N+|m|-1) \\ &\times (N+|m|)P_{N-1}^{|m|-1}(\cos\theta)) + \eta_1\sqrt{2N-1}a_{N-1}^0 P_{N-1}^0(\cos\theta) = 0, \end{aligned} \tag{2.44}$$

as well as on Π_2, we have

$$\sqrt{\frac{\lambda}{2N+1}}\sum_{\substack{m=-N\\ m\neq 0}}^{N} m a_N^m e^{\mathrm{i}m\phi_0}\sqrt{\frac{(N-|m|)!}{(N+|m|)!}}\frac{1}{2|m|}(P_{N-1}^{|m|+1}(\cos\theta)$$

$$+ (N+|m|-1)(N+|m|)P_{N-1}^{|m|-1}(\cos\theta)) + \eta_2\sqrt{2N-1}a_{N-1}^0 P_{N-1}^0(\cos\theta) = 0. \tag{2.45}$$

From the fact that

$$P_n^m = 0 \quad \text{for} \quad |m| > n,$$

we can further deduce from (2.44) that

$$\begin{aligned}
&-\sqrt{\frac{\lambda}{2N+1}}[N(a_N^N - a_N^{-N})\sqrt{\frac{1}{(2N)!}}(2N-1)P_{N-1}^{N-1}(\cos\theta) + (N-1)\\
&\times (a_N^{N-1} - a_N^{-(N-1)})\sqrt{\frac{1}{(2N-1)!}}(2N-1)P_{N-1}^{N-2}(\cos\theta)\\
&+ \sum_{\substack{m=-(N-2)\\ m\neq 0}}^{N-2} m a_N^m \sqrt{\frac{(N-|m|)!}{(N+|m|)!}}\frac{1}{2|m|}(P_{N-1}^{|m|+1}(\cos\theta) + (N+|m|-1)(N+|m|)\\
&\times P_{N-1}^{|m|-1}(\cos\theta))] + \eta_1\sqrt{2N-1}a_{N-1}^0 P_{N-1}^0(\cos\theta) = 0 \quad \text{on } \Pi_1.
\end{aligned} \tag{2.46}$$

Rearranging terms in (2.46) by virtue of $P_{N-1}^{|m|}$, we have

$$\begin{aligned}
&\sum_{m=2}^{N-1}\big[-\sqrt{\frac{\lambda}{2N+1}}\,(\,(m-1)(a_N^{m-1} - a_N^{-(m-1)})\sqrt{\frac{(N-m+1)!}{(N+m-1)!}}\frac{1}{2(m-1)} + (m+1)\\
&\times (a_N^{m+1} - a_N^{-(m+1)})\sqrt{\frac{(N-m+1)!}{(N+m-1)!}}\frac{1}{2(m+1)}(N+m)(N+m+1)\,)\,\big]P_{N-1}^m(\cos\theta)\\
&+\big[-\sqrt{\frac{\lambda}{2N+1}}2(a_N^2 - a_N^{-2})\sqrt{\frac{(N-2)!}{(N+2)!}}\frac{1}{4}(N+1)(N+2)\big]P_{N-1}^1(\cos\theta)\\
&+\big[-\sqrt{\frac{\lambda}{2N+1}}(a_N^1 - a_N^{-1})\sqrt{\frac{(N-1)!}{(N+1)}}\frac{1}{2}N(N+1) + \eta_1\sqrt{2N-1}a_{N-1}^0\big]\\
&\times P_{N-1}^0(\cos\theta) = 0,
\end{aligned} \tag{2.47}$$

where $N \geq 3$. With the help of the orthogonal property of the associated Legendre polynomials in Lemma 2.4, we can derive from (2.47) that

$$\begin{aligned}
&-\sqrt{\frac{\lambda}{2N+1}}\frac{1}{2}\big[\sqrt{\frac{(N-m+1)!}{(N+m-1)!}}(a_N^{m-1} - a_N^{-(m-1)})\\
&+\sqrt{\frac{(N-m-1)!}{(N+m+1)!}}(N+m)(N+m+1)(a_N^{m+1} - a_N^{-(m+1)})\big] = 0,
\end{aligned} \tag{2.48}$$

where $m = 2, 3, \cdots, N-1$ $(N \geq 3)$, and

$$-\sqrt{\frac{\lambda}{2N+1}}\frac{1}{2}\sqrt{\frac{(N-2)!}{(N+2)!}}(N+1)(N+2)(a_N^2 - a_N^{-2}) = 0, \tag{2.49}$$

$$-\sqrt{\frac{\lambda}{2N+1}}\frac{1}{2}\sqrt{\frac{(N-1)!}{(N+1)!}}(N+1)N(a_N^1 - a_N^{-1}) + \eta_1\sqrt{2N-1}a_{N-1}^0 = 0. \tag{2.50}$$

For further simplification of the expressions, we can deduce from (2.48), (2.49) and (2.50) the following system

$$
\begin{aligned}
&\sqrt{\frac{(N-m+1)!}{(N+m-1)!}}(a_N^{m-1}-a_N^{-(m-1)})+\sqrt{\frac{(N-m-1)!}{(N+m+1)!}}(N+m)(N+m+1)\\
&\times(a_N^{m+1}-a_N^{-(m+1)})=0,\\
&a_N^2-a_N^{-2}=0,\\
&-\sqrt{\frac{\lambda}{2N+1}}\frac{1}{2}\sqrt{\frac{(N-1)!}{(N+1)!}}(N+1)N(a_N^1-a_N^{-1})+\eta_1\sqrt{2N-1}a_{N-1}^0=0,
\end{aligned}\tag{2.51}
$$

where $m=2,3,\cdots,N-1$ $(N\geq 3)$.

Following the similar arguments above in the analyses of (2.44) on Π_1, from (2.45), we can deduce that on Π_2, there holds

$$
\begin{aligned}
&(a_N^{m-1}e^{\mathrm{i}(m-1)\phi_0}-a_N^{-(m-1)}e^{-\mathrm{i}(m-1)\phi_0})\sqrt{\frac{(N-m+1)!}{(N+m-1)!}}+(N+m)(N+m+1)\\
&\times\sqrt{\frac{(N-m-1)!}{(N+m-1)!}}(a_N^{m+1}e^{\mathrm{i}(m+1)\phi_0}-a_N^{-(m+1)}e^{-\mathrm{i}(m+1)\phi_0})=0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for } m=2,\cdots,N-1,\\
&a_N^2e^{\mathrm{i}2\phi_0}-a_N^{-2}e^{-\mathrm{i}2\phi_0}=0,\\
&\sqrt{\frac{\lambda}{2N+1}}\frac{1}{2}N(N+1)\sqrt{\frac{(N-1)!}{(N+1)!}}(a_N^1e^{\mathrm{i}\phi_0}-a_N^{-1}e^{-\mathrm{i}\phi_0})+\eta_2\sqrt{2N-1}a_{N-1}^0=0.
\end{aligned}\tag{2.52}
$$

Combining with (2.43), (2.51) and (2.52), we can directly see that since $\phi_0\neq 0,\frac{\pi}{2},\pi$, the determinant of the coefficient matrix with respect to $a_N^{\pm 2}$ satisfies

$$
\begin{vmatrix} 1 & -1\\ e^{\mathrm{i}2\phi_0} & e^{-\mathrm{i}2\phi_0}\end{vmatrix}=2\mathrm{i}\sin 2\phi_0\neq 0,
$$

which indicates that $a_N^{\pm 2}=0$. Moreover, denoting

$$
c(N,\lambda):=\sqrt{\frac{\lambda}{2N+1}}\frac{N}{2}(N+1)\sqrt{\frac{(N-1)!}{(N+1)!}},
$$

the determinant of the coefficient matrix associated with $a_N^{\pm 1}$, and a_{N-1}^0 fulfills

$$
\begin{aligned}
&\begin{vmatrix}
-\mathrm{i}c(N,\lambda)\sin\phi_0 e^{\mathrm{i}2\phi_0} & -\mathrm{i}c(N,\lambda)\sin\phi_0 e^{-\mathrm{i}2\phi_0} & (\eta_1\cos\phi_0+\eta_2)\sqrt{2N-1}\\
-c(N,\lambda) & c(N,\lambda) & \eta_1\sqrt{2N-1}\\
c(N,\lambda)e^{\mathrm{i}\phi_0} & -c(N,\lambda)e^{-\mathrm{i}\phi_0} & \eta_2\sqrt{2N-1}
\end{vmatrix}\\
&=-2\mathrm{i}\sin\phi_0\sqrt{2N-1}\frac{\lambda}{2N+1}\frac{N^2}{4}(N+1)^2\frac{(N-1)!}{(N+1)!}\left(2\eta_1\cos\phi_0+\eta_2(1+\cos 2\phi_0)\right).
\end{aligned}\tag{2.53}
$$

Therefore, under the condition (2.11), in view of $\phi_0\neq 0,\pi$, we can obtain that $a_N^{\pm 1}=a_{N-1}^0=0$.

Now, by substituting $a_{N-1}^0 = a_N^{\pm 1} = a_N^{\pm 2} = 0$ into (2.51) and (2.52), respectively, we can deduce that for $m = 2$, there holds

$$\begin{pmatrix} 1 & -1 \\ e^{\mathrm{i}3\phi_0} & -e^{-\mathrm{i}3\phi_0} \end{pmatrix} \begin{pmatrix} a_N^3 \\ a_N^{-3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\phi_0 \neq \frac{q\pi}{3}$ $(q = 0, 1, 2,)$ under condition (2.55), we have $a_N^{\pm 3} = 0$. Following the similar argument, by taking $a_N^{\pm 3} = 0$ into (2.51) and (2.52), we can generally derive that for $m = 2, 3, \cdots, N$, since $\phi_0 \neq \frac{q\pi}{m}$ $(q = 0, 1, 2, \cdots, m-1)$, there holds $a_N^{\pm m} = 0$ for $m = 1, 2, \cdots, N$.

The proof is complete. □

Remark 2.9. If the impedance parameters η_i associated with Π_i $(i = 1, 2)$ fulfill

$$\eta_1 = \eta_2 = \eta \neq 0,$$

where $\eta \in \mathbb{C}$ is a constant, then the condition (2.11) can be directly satisfied when $\phi_0 \neq 0, \pi/2$, and π. Indeed, we have

$$2\eta_1 \cos\phi_0 + \eta_2(\cos 2\phi_0 + 1) = 2\eta \cos\phi_0 (1 + \cos\phi_0) \neq 0.$$

Corollary 2.10. *Let u be a Laplacian eigenfunction to (2.1). Consider an edge corner $\mathcal{E}(\Pi_1, \Pi_2, \boldsymbol{l}) \Subset \Omega$ associated with two generalized singular planes Π_1 and Π_2, where the corresponding surface parameters on Π_1 and Π_2 have the following absolute convergence series expansion*

$$\eta_1 = \sum_{\ell=0}^{\infty} \alpha_\ell^{(1)}(\theta, \phi) r^\ell \quad \text{and} \quad \eta_2 = \sum_{\ell=0}^{\infty} \alpha_\ell^{(2)}(\theta, \phi) r^\ell,\ \alpha_\ell^{(i)} \in \mathbb{C},\ \alpha_0^{(i)} \neq 0,\ i = 1, 2. \tag{2.54}$$

If the corresponding dihedral angle can be written as

$$\angle(\Pi_1, \Pi_2) = \phi_0 = \alpha \cdot \pi, \quad \alpha \in (0, 1),$$

where α satisfies that for any $N \in \mathbb{N}$, $N \geq 2$,

$$\alpha \neq \frac{q}{p}, \quad p = 1, 2, \cdots, N-1,\ q = 0, 1, 2, \cdots, p-1, \tag{2.55}$$

and the corresponding surface parameters η_i associated with Π_i, $i = 1, 2$, fulfill that

$$2\alpha_0^{(1)} \cos\phi_0 + \alpha_0^{(2)} (\cos 2\phi_0 + 1) \neq 0, \tag{2.56}$$

then u vanishes up to the order at least N at the edge corner $\mathbf{0}$.

Proof. Since the proof this corollary is similar to the proof of Theorem 2.6, in the following we only give a sketched proof of this corollary. Recall the equations (2.17), (2.12) and (2.13). Substituting (2.54) into (2.17), we can obtain that on $\boldsymbol{l}$, there holds

$$\begin{aligned}
&-\sum_{n=1}^{\infty} \mathrm{i}^n \frac{\sqrt{\lambda}}{2n+1} \left(j_{n-1}(\sqrt{\lambda} r) + j_{n+1}(\sqrt{\lambda} r) \right) \sqrt{\frac{2n+1}{4\pi}} \frac{1}{2} n(n+1) \sqrt{\frac{(n-1)!}{(n+1)!}} \sin\phi_0 \\
&\times (a_n^1 e^{\mathrm{i}2\phi_0} + a_n^{-1} e^{-\mathrm{i}2\phi_0}) + \left((\alpha_0^{(1)} + \sum_{\ell=1}^{\infty} \alpha_\ell^{(1)} r^\ell) \cos\phi_0 + (\alpha_0^{(2)} + \sum_{\ell=1}^{\infty} \alpha_\ell^{(2)} r^\ell) \right) \\
&\times \sum_{n=0}^{\infty} \mathrm{i}^n a_n^0 j_n(\sqrt{\lambda} r) \sqrt{\frac{2n+1}{4\pi}} = 0,
\end{aligned}$$

which can be written more explicitly as

$$
\begin{aligned}
&-\sum_{n=1}^{\infty} \mathrm{i}^n \frac{\sqrt{\lambda}}{2n+1} \left(j_{n-1}(\sqrt{\lambda}r) + j_{n+1}(\sqrt{\lambda}r)\right) \frac{1}{2} n(n+1) \sqrt{\frac{(n-1)!}{(n+1)!}} \sin\phi_0 \\
&\times (a_n^1 e^{\mathrm{i}2\phi_0} + a_n^{-1} e^{-\mathrm{i}2\phi_0}) + \left((\alpha_0^{(1)} \cos\phi_0 + \alpha_0^{(2)}) + \sum_{\ell=1}^{\infty} (\alpha_\ell^{(1)} \cos\phi_0 + \alpha_\ell^{(2)}) r^\ell \right) \\
&\times \sum_{n=1}^{\infty} \mathrm{i}^{n-1} a_{n-1}^0 j_{n-1}(\sqrt{\lambda}r) = 0. \qquad (2.57)
\end{aligned}
$$

Assume that

$$
a_{N-2}^0 = a_{N-1}^{\pm m} = 0 \text{ for } m = 1, 2, \cdots, N-1. \qquad (2.58)
$$

The lowest order of r on the left side of (2.57) is $N-1$. Comparing the coefficients of r^{N-1} on both sides of (2.57), it yields that

$$
\begin{aligned}
&-\mathrm{i}\sqrt{\frac{\lambda}{2N+1}} \frac{1}{2} N(N+1) \sqrt{\frac{(N-1)!}{(N+1)!}} \frac{(\sqrt{\lambda})^{N-1}}{(2N-1)!!} \sin\phi_0 (a_N^1 e^{\mathrm{i}2\phi_0} + a_n^{-1} e^{-\mathrm{i}2\phi_0}) \\
&+ (\alpha_0^{(1)} \cos\phi_0 + \alpha_0^{(2)}) a_{N-1}^0 \sqrt{2N-1} \frac{(\sqrt{\lambda})^{N-1}}{(2N-1)!!} = 0. \qquad (2.59)
\end{aligned}
$$

Similarly, substituting (2.54) into (2.12) and (2.13), under the assumption (2.58), comparing the coefficients of r^{N-1} on both sides of the resulting equations, one has

$$
\begin{aligned}
&-\sqrt{\frac{\lambda}{2N+1}} \sum_{\substack{m=-N \\ m\neq 0}}^{N} m a_N^m \sqrt{\frac{(N-|m|)!}{(N+|m|)!}} \frac{1}{2|m|} (P_{N-1}^{|m|+1}(\cos\theta) \\
&+ (N+|m|-1)(N+|m|) \times P_{N-1}^{|m|-1}(\cos\theta)) + \alpha_0^{(1)} \sqrt{2N-1} a_{N-1}^0 P_{N-1}^0(\cos\theta) = 0, \qquad (2.60)
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{\frac{\lambda}{2N+1}} \sum_{\substack{m=-N \\ m\neq 0}}^{N} m a_N^m e^{\mathrm{i}m\phi_0} \sqrt{\frac{(N-|m|)!}{(N+|m|)!}} \frac{1}{2|m|} (P_{N-1}^{|m|+1}(\cos\theta) + (N+|m|-1) \\
&\times (N+|m|) P_{N-1}^{|m|-1}(\cos\theta)) + \alpha_0^{(2)} \sqrt{2N-1} a_{N-1}^0 P_{N-1}^0(\cos\theta) = 0. \qquad (2.61)
\end{aligned}
$$

By virtue of (2.59) , (2.60) and (2.61) , under the assumption (2.56) and $\alpha_0^{(i)} \neq 0$ ($i = 1, 2$), following the similar mathematical argument in the proof of Theorem 2.6, we can prove this corollary. ☐

Remark 2.11. Compared with the study on the geometric structure of Laplacian eigenfunctions in [13, Theorem 2.11], it is direct to see that our results in Corollary 2.10 are more general by relaxing the technical condition $u|_{B_\epsilon(\mathbf{0})\cap \boldsymbol{l}} \equiv 0$, which is relatively hard to be fulfilled in the study on the application of inverse problems for an edge corner. Moreover, the condition (2.11) implies that the rationality on the dihedral angle of two intersecting adjacent planes is sufficient to determine the vanishing orders of Laplacian eigenfunctions. Indeed, we need certain assumption on the roots of the associated Legendre polynomials to study the vanishing order of the underlying Laplacian eigenfunction at a vertex corner; see [13, Theorem 3.1] for more details.

3. **Unique identifiability for inverse problems.** In this section, we shall establish the unique identifiability results for the inverse scattering problems by using the geometric results established in the previous section.

3.1. **Unique recovery for the inverse obstacle problem.** Recalling the mathematical setup for the inverse obstacle problem in Subsection 1.1, we are going to present the proofs of the uniqueness results for the inverse problem (1.4) for certain admissible complex polyhedral obstacles defined by Definitions 1.1 and 1.6.

Proof of Theorem 1.7(Irrational case). We prove the theorem by contradiction. Assume that there exists a corner $\mathbf{x}_c$ on $\partial\mathbf{G}$, which is either located at $\partial\Omega_1\backslash\partial\Omega_2$ or $\partial\Omega_2\backslash\partial\Omega_1$. Without loss of generality, we assume that $\mathbf{x}_c$ is a corner on $\partial\Omega_2$, i.e. $\mathbf{x}_c \in \overline{\Omega}_2\backslash\overline{\Omega}_1$. Suppose that $B_h(\mathbf{x}_c)$ is an open ball centered at $\mathbf{x}_c$ with sufficiently small $h \in \mathbb{R}_+$ fulfilling that $B_h(\mathbf{x}_c) \Subset \mathbb{R}^3\backslash\overline{\Omega}_1$. Let

$$B_h(\mathbf{x}_c) \cap \partial\Omega_2 = \bigcup_i \Pi_i, \quad i = 1, 2, \cdots, n, \tag{3.1}$$

for $n \geq 2$.

Recall that $\mathbf{G}$ is the unbounded connected component of $\mathbb{R}^3\backslash\overline{(\Omega_1 \cup \Omega_2)}$. From (1.6), by the Rellich theorem (cf. [16]), we have

$$u^1(\mathbf{x}; k, \mathbf{d}) = u^2(\mathbf{x}; k, \mathbf{d}), \quad \mathbf{x} \in \mathbf{G}. \tag{3.2}$$

Since $B_h(\mathbf{x}_c) \Subset \mathbb{R}^3\backslash\overline{\Omega}_1$, in view of (3.1), we can claim that $\Pi_i \subset \partial\mathbf{G}$. Furthermore, it can be directly to see that

$$-\Delta u^1 = k^2 u^1 \text{ in } B_h(\mathbf{x}_c). \tag{3.3}$$

Combining with (3.2) and the generalized singular boundary condition defined on $\partial\Omega_2$, it is easy to know that

$$\partial_\nu u^1 + \eta_2 u^1 = \partial_\nu u^2 + \eta_2 u^2 = 0 \quad \text{on } \Pi_i, \ i = 1, 2, \cdots, n. \tag{3.4}$$

Pick up any fixed point $\mathbf{x}_0 \in \Pi_1 \cap \Pi_2$. Since $-\Delta$ is invariant under rigid motion, without loss of generality, we assume that $\mathbf{x}_0 = \mathbf{0}$. Recall that Ω_2 is an admissible complex polyhedral obstacle, according to Definitions 1.1 and 1.6, we know that the surface impedance parameter η_2 on Π_i $(i = 1, 2)$ is a real-analytic function with an absolutely convergent series

$$\eta(\mathbf{x})\big|_{\Pi_i} = \alpha_0^{(i)} + \sum_{\ell=1}^{\infty} \alpha_\ell^{(i)} r^\ell, \quad \alpha_m^{(i)} \in \mathbb{C}, \quad m = 0, 1, 2, \ldots, \tag{3.5}$$

where $r = |\mathbf{x} - \mathbf{x}_0|$ with $\mathbf{x} \in \Pi_i$. By the assumptions (a) and (c) in Definition 1.1, we know that $\alpha_0^{(1)} = \alpha_0^{(2)} = \alpha_0 \neq 0$. Therefore the condition (2.11) in Corollary 2.10 is satisfied by Remark 2.9. It is clear that the impedance parameter with the form (3.5) satisfies the assumption (2.54) in Corollary 2.10. Utilizing Corollary 2.10 for the Laplacian eigenfunction u^1 to (3.3) in $B_h(\mathbf{x}_c)$, since the dihedral angle of two adjacent plane associated with $\mathbf{x}_c$ is irrational, we can obtain that $\mathrm{Vani}(u^1; \mathbf{0}) = \infty$, which implies

$$u^1(\mathbf{x}; k, \mathbf{d}) \equiv 0 \text{ in } B_h(\mathbf{x}_c). \tag{3.6}$$

Here we use the fact that u^1 is analytic in $B_h(\mathbf{x}_c)$. According to (3.6), each component of Ω_2 is simply connected and disjoint pairwisely, it yields by the analytic continuation that

$$u^1(\mathbf{x}; k, \mathbf{d}) = 0 \quad \text{in} \quad \mathbb{R}^3\backslash\overline{\Omega}. \tag{3.7}$$

In particular, one has from (3.7) that

$$\lim_{|\mathbf{x}|\to\infty} \left|u^1(\mathbf{x}; k, \mathbf{d})\right| = 0. \tag{3.8}$$

However, we know from the formulations of the forward scattering problem that

$$\lim_{|\mathbf{x}|\to\infty} \left|u^1(\mathbf{x}; k, \mathbf{d})\right| = \lim_{|\mathbf{x}|\to\infty} \left|e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} + u^s(\mathbf{x}; k, \mathbf{d})\right| = 1, \tag{3.9}$$

which induce the contradiction to (3.8).

Now, we show that $\eta_1 = \eta_2$. Suppose that $\Gamma \subset \partial\Omega_1 \cap \partial\Omega_2$ is an open subset such that $\eta_1 \neq \eta_2$ on Γ. From (3.2), we have already known that $u^1 = u^2$ in $\mathbb{R}^3\backslash\overline{(\Omega_1 \cup \Omega_2)}$, from which we can directly derive that

$$\partial_\nu u^1 + \eta_1 u^1 = 0, \quad \partial_\nu u^2 + \eta_2 u^2 = 0, \quad u^1 = u^2, \quad \partial_\nu u^1 = \partial_\nu u^2 \quad \text{on} \quad \Gamma. \tag{3.10}$$

After rearranging terms in (3.10), we have

$$(\eta_1 - \eta_2)u^1 = 0 \quad \text{on} \quad \Gamma. \tag{3.11}$$

Since $\eta_1 \neq \eta_2$ on Γ, we can deduce by direct computing that

$$u^1 = \partial_\nu u^1 = 0 \quad \text{on} \quad \Gamma.$$

By the classcial Holmgren's uniqueness result (cf. [41]), it is easy to obtain that $u^1 = 0$ in $\mathbb{R}^3\backslash\Omega$. Therefore, we derive the same contradiction as in (3.8), which leads to the conslusion. □

In the following, we shall give the detailed proof of Theorem 1.8 regarding the unique determination for an admissible rational complex obstacle by a single far-field measurement.

Proof of Theorem 1.8(Rational case). We prove the theorem by contradiction. Assume that there exists a corner $\mathbf{x}_c$ on $\partial\mathbf{G}$. Without loss of generality, we still assume that $\mathbf{x}_c$ is a corner on $\partial\Omega_2$, i.e. $\mathbf{x}_c \in \overline{\Omega}_2\backslash\overline{\Omega}_1$. Following the same notation of $B_h(\mathbf{x}_c)$ in Theorem 1.7 such that $B_h(\mathbf{x}_c) \cap \partial\Omega_2 = \bigcup_i \Pi_i$, $i = 1, 2, \cdots, n$, for $n \geq 2$. With the help of the condition (1.6) and the Rellich lemma, it is direct to verify that (3.2) and (3.4) still hold. Moreover, we can know by Theorem 2.6 that u^1 fulfills (3.6) and

$$u^1(\mathbf{x}_c) = 0, \quad \nabla u^1(\mathbf{x}_c) \neq 0. \tag{3.12}$$

under the condition (1.10). However, for the admissible rational complex obstacle Ω_2, under the assumption (1.9), we can know that $\mathbf{x}_c$ is either an irrational corner or a rational corner of degree $p \geq 3$. In either of the above case, by Theorem 2.6 we can obtain that u^1 vanishes at least to the second order, which implies that there holds $\nabla u^1(\mathbf{x}_c) = 0$, and this contradicts to (3.12). Similar to the proof of Theorem 1.7, if we further assume that $\eta_1 \neq \eta_2$, then the uniqueness for the impedance parameter η can be deduced immediately by the Holmgren's uniqueness principle. □

Remark 3.1. The uniqueness results with respect to the admissible impedance obstacles and the corresponding argument in Theorem 1.7 and Theorem 1.8 are "localized" in the neighborhood of $B_h(\mathbf{x}_c)$ based on the generalized Holmgren's principle obtained from Section 2. Therefore, the results are also applicable to other different types of wave incidences such as the point source.

Remark 3.2. In Theorem 1.7 and Theorem 1.8, if the considering adimissible polyhedral obstacles are convex, then we can achieve the global unique identifiaility results which indicate that $\Omega_1 = \Omega_2$ and also $\eta_1 = \eta_2$ simultaneously by a single far-field measurement. The detailed proof of Corollary 1.10 is omitted.

3.2. **Unique recovery for the inverse diffraction grating problem.** In this subsection, we consider the inverse diffraction grating problem in determining a diffraction grating profile as well as its surface parameter in $\mathbb{R}^3$ by a single far-field pattern.

Lemma 3.3. *[12, Lemma 8.1] Let $\boldsymbol{\xi}_\ell \in \mathbb{R}^3$, $\ell = 1, 2, \cdots, n$, be n vectors which are distinct from each other. Let $U \subset \mathbb{R}^3$ be any open subset. Then all the functions in the following set are linearly independent:*

$$\{e^{\mathrm{i}\boldsymbol{\xi}_\ell\cdot\mathbf{x}}; \mathbf{x} \in U, \ell = 1, 2, \cdots, n\}.$$

Proof of Theorem 1.14(Irrational case). We prove this theorem by contradiction. Without loss of generality, we assume that there exists a corner point $\mathbf{x}_c$ of Λ_f lies on $\partial\mathbf{G}\backslash\Lambda_g$. By the wellposedness of the diffraction grating problem (1.14) and the unique continuation property, we can know from (1.19) that

$$u_f(\mathbf{x}; k, \mathbf{d}) = u_g(\mathbf{x}; k, \mathbf{d}) \quad \text{for} \quad \mathbf{x} \in \mathbf{G}.$$

Indeed, define

$$v(\mathbf{x}; k, \mathbf{d}) := u_f(\mathbf{x}; k, \mathbf{d}) - u_g(\mathbf{x}; k, \mathbf{d}).$$

Denote $\boldsymbol{\Sigma} := \mathbf{G} \cap \{\mathbf{x} \in \mathbb{R}^3; \mathbf{x}' \in \mathbb{R}^2, x_3 > b\} \subset \mathbb{R}^3$, then it is obvious that $v(\mathbf{x}; k, \mathbf{d})$ fulfills

$$\Delta v + k^2 v = 0 \quad \text{in } \boldsymbol{\Sigma}; \quad v = 0 \text{ on } \Gamma_b,$$

and the Rayleigh series expansion (1.15), where Γ_b is the boundary of $\boldsymbol{\Sigma}$. Thus, from the uniqueness of the diffraction grating scattering problem (1.14), we can know that $v = 0$ in $\boldsymbol{\Sigma}$. Since $u_f(\mathbf{x}; k, \mathbf{d})$ and $u_g(\mathbf{x}; k, \mathbf{d})$ are analytic in $\mathbf{G}$, it is direct to derive that $v(\mathbf{x}; k, \mathbf{d})$ is analytic in $\mathbf{G}$, which implies $v = 0$ in $\mathbf{G}$. Therefore, we have $u_f(\mathbf{x}; k, \mathbf{d}) = u_g(\mathbf{x}; k, \mathbf{d})$ in $\mathbf{G}$.

Since $\mathbf{x}_c \in \Lambda_f$ lying on $\partial\mathbf{G}\backslash\Lambda_g$, for sufficiently small $h \in \mathbb{R}^+$, suppose $B_h(\mathbf{x}_c) \Subset \Omega_g$ such that

$$B_h(\mathbf{x}_c) \cap \Lambda_f = \bigcup_i \Pi_i, \quad i = 1, 2, \cdots, n, \quad n \geq 2. \tag{3.13}$$

It is clear that $\Pi_i \subset \Lambda_f\backslash\Lambda_g \subset \partial\mathbf{G}$, $i = 1, 2, \cdots, n$. Following a similar argument in the proof of Theorem 1.7, we can obtain that

$$u_g(\mathbf{x}; k, \mathbf{d}) = 0 \ \text{ for } \ x_3 > \max_{\mathbf{x}' \in [0,2\pi)^2} g(\mathbf{x}'),$$

by utilizing Corollary 2.10. Moreover, we know that $u_g(\mathbf{x}; k, \mathbf{d})$ satisfies the Rayleigh series expansion as follows

$$u_g(\mathbf{x}; k, \mathbf{d}) = e^{\mathrm{i}k\mathbf{d}\cdot\mathbf{x}} + \sum_{\mathbf{n}\in\mathbb{Z}^2} u_{\mathbf{n}} e^{\mathrm{i}\boldsymbol{\xi}_{\mathbf{n}}\cdot\mathbf{x}} \ \text{ for } \ x_3 > \max_{\mathbf{x}' \in [0,2\pi)^2} g(\mathbf{x}'), \tag{3.14}$$

where $\boldsymbol{\xi}_{\mathbf{n}}$ is defined in (1.16).

Combining with (1.13) and (1.16), it is easy to calculate that in (3.14),

$$k\mathbf{d} = (k\sin\phi_d\cos\theta_d, k\sin\phi\sin\theta, -k\cos\phi_d) = (\boldsymbol{\alpha}_0, -\boldsymbol{\beta}_0),$$

with

$$\boldsymbol{\alpha}_0 = \boldsymbol{\alpha} := (k\sin\phi_d\cos\theta_d, k\sin\phi_d\sin\theta_d).$$

Clearly, $k\mathbf{d} \notin \{\boldsymbol{\xi}_{\mathbf{n}} | \mathbf{n} \in \mathbb{Z}^2\}$ since $\phi \in (-\pi/2, \pi/2)$ and $\theta \in [0, 2\pi)$. Besides, from (1.15) and (1.16), we can know that any two vectors of $\{\boldsymbol{\xi}_{\mathbf{n}} | \mathbf{n} \in \mathbb{Z}^2\}$ are distinct from each other. Therefore, we can deduce the contradiction in view of (3.14) by Lemma 3.3.

The proof of the uniqueness of η is similar to the proof of Theorem 1.7 and we skip the details here to avoid repetition. □

Finally, we sketch the proof of the unique determination results for admissible rational polyhedral diffraction gratings as follows.

Proof of Theorem 1.15 (Rational case). We prove by absurdity. Following the same notations and assumptions above in the proof of Theorem 1.14, we suppose that there exists a corner point $\mathbf{x}_c \in \Lambda_f$ which lies on $\partial \mathbf{G} \backslash \Lambda_g$ such that $B_h(\mathbf{x}_c) \Subset \Omega_g$ and (3.13) holds. Using (1.20), by Theorem 2.6, we know that $u_g(\mathbf{x}; k, \mathbf{d})$ satisfies

$$u_g(\mathbf{x}_c) = 0, \quad \nabla u_g(\mathbf{x}_c) = 0,$$

which contradicts with (1.21). The uniqueness result can now be attained by a similar absurdity as stated in the proof of Theorem 1.8. □

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REFERENCES

[1] H.-D. Alber, A quasi-periodic boundary value problem for the Laplacian and the continuation of its resolvent, *Proc. Roy. Soc. Edinburgh Sect. A*, **82** (1978/79), 251–272.

[2] G. Alessandrini and L. Rondi, Determining a sound-soft polyhedral scatterer by a single far-field measurement, *Proc. Amer. Math. Soc.*, **133** (2005), 1685–1691.

[3] H. Ammari, Uniqueness theorems for an inverse problem in a doubly periodic structure, *Inverse Problems*, **11** (1995), 823–833.

[4] G. B. Arfken, H. J. Weber and F. E. Harris, *Mathematical Methods for Physicists (Seventh Edition)*, Academic Press, New York-London 1966.

[5] E. Blåsten and H. Liu, Recovering piecewise constant refractive indices by a single far-field pattern, *Inverse Problems*, **36** (2020), 085005, 16 pp.

[6] E. Blåsten and H. Liu, Scattering by curvatures, radiationless sources, transmission eigenfunctions, and inverse scattering problems, *SIAM J. Math. Anal.*, **53** (2021), 3801–3837.

[7] E. Blåsten, H. Liu and J. Xiao, On an electromagnetic problem in a corner and its applications, *Anal. PDE*, **14** (2021), 2207–2224.

[8] W. Bosch, On the computation of derivatives of Legendre functions, *Phys. Chem. Earth*, **25** (2000), 655–659.

[9] M Cadilhac, Some mathematical aspects of the grating theory, *Electromagnetic Theory of Gratings*, Springer, (1980), 53–62.

[10] X. Cao, H. Diao and J. Li, Some recent progress on inverse scattering problems within general polyhedral geometry, *Electron. Res. Arch.*, **29** (2021), 1753–1782.

[11] X. Cao, H. Diao and H. Liu, Determining a piecewise conductive medium body by a single far-field measurement, *CSIAM Trans. Appl. Math.*, **1** (2020), 740–765.

[12] X. Cao, H. Diao, H. Liu and J. Zou, On nodal and generalized singular structures of Laplacian eigenfunctions and applications to inverse scattering problems, *J. Math. Pures Appl.*, **143** (2020), 116–161.

[13] X. Cao, H. Diao, H. Liu and J. Zou, On novel geometric structures of Laplacian eigenfunctions in $\mathbb{R}^3$ and applications to inverse problems, *SIAM J. Math. Anal.*, **53** (2021), 1263–1294.

[14] X. Cao, Y.-H. Lin and H. Liu, Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators, *Inverse Probl. Imaging*, **13** (2019), 197–210.

[15] J. Cheng and M. Yamamoto, Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves, *Inverse Problems*, **19** (2003), 1361–1384.

[16] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 4^{th} edition, volume 93 of Applied Mathematical Sciences, Springer, Cham, 2019.

[17] D. Colton and R. Kress, Looking back on inverse scattering theory, *SIAM Rev.*, **60** (2018), 779–807.

[18] H. Diao, X. Cao and H. Liu, On the geometric structures of transmission eigenfunctions with a conductive boundary condition and applications, *Comm. Partial Differential Equations*, **46** (2021), 630–679.

[19] H. Diao, X. Fei, H. Liu and K. Yang, Visibility, invisibility and unique recovery of inverse electromagnetic problems with conical singularities, `arXiv:2204.02835`, 2022.

[20] H. Diao, H. Liu and B. Sun, On a local geometric property of the generalized elastic transmission eigenfunctions and application, *Inverse Problems*, **37** (2021), Paper No. 105015, 36 pp.

[21] H. Diao, H. Liu and L. Wang, On generalized Holmgren's principle to the Lamé operator with applications to inverse elastic problems, *Calc. Var. Partial Differential Equations*, **59** (2020), Paper No. 179, 50 pp.

[22] H. Diao, H. Liu and L. Wang, Further results on generalized Holmgren's principle to the Lamé operator and applications, *J. Differential Equations*, **309** (2022), 841–882.

[23] H. Diao, H. Liu, L. Zhang and J. Zou, Unique continuation from a generalized impedance edge-corner for Maxwell's system and applications to inverse problems, *Inverse Problems*, **37** (2021), Paper No. 035004, 32 pp.

[24] J. Elschner and M. Yamamoto, Uniqueness in determining polygonal periodic structures, *Z. Anal. Anwend.*, **26** (2007), 165–177.

[25] J. Elschner and M. Yamamoto, Uniqueness in determining polyhedral sound-hard obstacles with a single incoming wave, *Inverse Problems*, **24** (2008), 035004, 7 pp.

[26] F. Hettlich and A. Kirsch, Schiffer's theorem in inverse scattering for periodic structures, *Inverse Problems*, **13** (1997), 351–361.

[27] V. Isakov, *Inverse Problems for Partial Differential Equations*, 2^{nd} edition, Springer-Verlag, New York, 2006.

[28] A. Kirsch, Uniqueness theorems in inverse scattering theory for periodic structures, *Inverse Problems*, **10** (1994), 145–152.

[29] J. Li and H. Liu, Recovering a polyhedral obstacle by a few backscattering measurements, *J. Differential Equations*, **259** (2015), 2101–2120.

[30] J. Li, H. Liu and Y. Wang, Recovering an electromagnetic obstacle by a few phaseless backscattering measurements, *Inverse Problems*, **33** (2017), 035011, 20 pp.

[31] J. Li, H. Liu, Z. Shang and H. Sun, Two single-shot methods for locating multiple electromagnetic scatterers, *SIAM J. Appl. Math.*, **73** (2013), 1721–1746.

[32] J. Li, H. Liu and J. Zou, Locating multiple multiscale acoustic scatterers, *Multiscale Model. Simul.*, **12** (2014), 927–952.

[33] J. Li, H. Liu and Q. Wang, Locating multiple multiscale electromagnetic scatterers by a single far-field measurement, *SIAM J. Imaging Sci.*, **6** (2013), 2285–2309.

[34] H. Liu, A global uniqueness for formally determined inverse electromagnetic obstacle scattering, *Inverse Problems*, **24** (2008), 035018, 13 pp.

[35] H. Liu, On local and global structures of transmission eigenfunctions and beyond, *J. Inverse Ill-Posed Probl.*, **30** (2022), 287–305.

[36] H. Liu, M. Petrini, L. Rondi and J. Xiao, Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements, *J. Differential Equations*, **262** (2017), 1631–1670.

[37] H. Liu, L. Rondi and J. Xiao, Mosco convergence for $H(curl)$ spaces, higher integrability for Maxwell's equations, and stability in direct and inverse EM scattering problems, *J. Eur. Math. Soc. (JEMS)*, **21** (2019), 2945–2993.

[38] H. Liu and C.-H. Tsou, Stable determination of polygonal inclusions in Calderón's problem by a single partial boundary measurement, *Inverse Problems*, **36** (2020), 085010, 23 pp.

[39] H. Liu and J. Xiao, Decoupling elastic waves and its applications, *J. Differential Equations*, **263** (2017), 4442–4480.

[40] H. Liu, M. Yamamoto and J. Zou, Reflection principle for the Maxwell equations and its application to inverse electromagnetic scattering, *Inverse Problems*, **23** (2007), 2357–2366.

[41] H. Liu and J. Zou, Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers, *Inverse Problems*, **22** (2006), 515–524.

[42] H. Liu and J. Zou, On unique determination of partially coated polyhedral scatterers with far field measurements, *Inverse Problems*, **23** (2007), 297–308.

[43] H. Liu and J. Zou, On uniqueness in inverse acoustic and electromagnetic obstacle scattering problems, *J. Phys.: Conf. Ser.*, **124** (2008), 012006.

[44] H. Liu and J. Zou, Zeros of the Bessel and spherical Bessel functions and their applications for uniqueness in inverse acoustic obstacle scattering, *IMA J. Appl. Math.*, **72** (2007), 817–831.

[45] W. Mclean, *Strongly Elliptic Systems and Boundary Integral Equation*, Cambridge University Press, Cambridge, 2000.

[46] J.-C. Nédélec, *Acoustic and Electromagnetic Equations*, Springer-Verlag, New York, 2001.

[47] W. Yin, W. Yang and H. Liu, *A neural network scheme for recovering scattering obstacles with limited phaseless far-field data*, J. Comput. Phys., **417** (2020), 109594, 18pp.

[48] L. Rondi, Unique determination of non-smooth sound-soft scatterers by finitely many far-field measurements, *Indiana Univ. Math. J.*, **52** (2003), 1631–1662.

[49] L. Rondi, Stable determination of sound-soft polyhedral scatterers by a single measurement, *Indiana Univ. Math. J.*, **57** (2008), 1377–1408.

[50] L. Rondi, E. Sincich and M. Sini, Stable determination of a rigid scatterer in elastodynamics, *SIAM J. Math. Anal.*, **53** (2021), 2660–2689.

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