Homework 5 for MAT6041S  
Due Date: Dec. 8

1. (20pts) Show that the problem $\Delta u = -1$ for $|x| < 1, |y| < 1, u = 0$ for $|x| = 1$ and $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0$ for $|y| = 1$ has at least one solution.
   Hint: try $u(x, y) = f(x)g(y)$.

2. (20pts) Let $L[u] = u_{xx} + u_{yy} + 2u$ and $D$ be the square $|x| < b, |y| < b$. Construct a function $w(x, y) = 1 - \beta e^{\alpha(x+y)}$ such that the generalized maximum principle holds.

3. (20pts) Let $D$ be the square $|x| < 1, |y| < 1$, and let $u$ be the solution of

$$\Delta u = 0 \text{ in } D, \quad u = |x| \text{ for } |y| = 1, \quad u = |y| \text{ for } |x| = 1$$

Select a harmonic polynomial of second degree in $x$ and $y$ whose boundary values approximate those of $u$ on $\partial D$, and thereby obtain bounds for $u(0,0)$.

Hint: there are two kinds of harmonic polynomials of second degree in $x$ and $y$: $x^2 - y^2, xy$.

4. (20pts) Verify that

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \frac{(x^2 + y^2)(\xi^2 + \eta^2) - 2(x\xi + y\eta) + 1}{(x - \xi)^2 + (y - \eta)^2}$$

is the Green’s function for the problem

$$\Delta u = f \text{ in } D : x^2 + y^2 < 1, \quad u = g \text{ for } x^2 + y^2 = 1$$

5. (20pts) For the Green’s function in Exercise 4, verify that the outer normal derivatvie is negative at each point of $\partial D$. 

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