Q.1) Solution:

(a) The corresponding LP is

\[
\begin{align*}
\text{Max} & \quad f(t, s) \\
\text{subject to} & \quad f(s, 2) + f(s, 3) - f(t, s) = 0 \\
& \quad f(2, 3) + f(2, 4) + f(2, 5) - f(s, 2) = 0 \\
& \quad f(3, 5) - f(s, 3) - f(2, 3) = 0 \\
& \quad f(4, t) - f(2, 4) - f(5, 4) = 0 \\
& \quad f(5, 4) + f(5, t) - f(2, 5) - f(3, 5) = 0 \\
& \quad f(t, s) - f(4, t) - f(5, t) = 0 \\
& \quad 0 \leq f(s, 2) \leq 10 \\
& \quad 0 \leq f(s, 3) \leq 8 \\
& \quad 0 \leq f(2, 3) \leq 2 \\
& \quad 0 \leq f(2, 4) \leq 6 \\
& \quad 0 \leq f(2, 5) \leq 3 \\
& \quad 0 \leq f(3, 5) \leq 4 \\
& \quad 0 \leq f(5, 4) \leq 4 \\
& \quad 0 \leq f(4, t) \leq 4 \\
& \quad 0 \leq f(5, t) \leq 10 
\end{align*}
\]
(b) The dual problem is

\[
\begin{align*}
\text{Min} & \quad 10w_{s2} + 8w_{s3} + 2w_{24} + 6w_{24} + 3w_{25} + 4w_{35} + 4w_{4t} + 10w_{5t} \\
\text{subject to} & \quad u_s - u_2 + w_{s2} \geq 0 \\
& \quad u_s - u_3 + w_{s3} \geq 0 \\
& \quad u_2 - u_3 + w_{23} \geq 0 \\
& \quad u_2 - u_4 + w_{24} \geq 0 \\
& \quad u_2 - u_5 + w_{s5} \geq 0 \\
& \quad u_3 - u_5 + w_{35} \geq 0 \\
& \quad u_5 - u_4 + w_{54} \geq 0 \\
& \quad u_4 - u_t + w_{4t} \geq 0 \\
& \quad u_5 - u_t + w_{5t} \geq 0 \\
& \quad u_t - u_s \geq 1 \\
& \quad u_s, u_2, u_3, u_4, u_5, u_t \text{ is free}, \\
& \quad w_{s2}, w_{s3}, w_{23}, w_{24}, w_{25}, w_{35}, w_{54}, w_{4t}, w_{5t} \geq 0
\end{align*}
\]

(c) First, we rewrite the network as following:

Iteration 1. Assign a flow of 4 to \( s \rightarrow 2 \rightarrow 4 \rightarrow t \) and we get:

Iteration 2. Assign a flow of 4 to \( s \rightarrow 3 \rightarrow 5 \rightarrow t \) and we get:

Iteration 3. Assign a flow of 3 to \( s \rightarrow 2 \rightarrow 5 \rightarrow t \) and we get:
Since there are no paths with positive capacity, the current flow pattern is optimal. Comparing it with the original network, we have that: $v(f)^* = 11$ and $f(s, 2) = 7, f(s, 3) = 4, f(2, 3) = 0, f(2, 4) = 4, f(2, 5) = 3, f(3, 5) = 4, f(5, 4) = 0, f(4, t) = 4, f(5, t) = 7$.

(d) By the proof of Theorem 3.3, we define that:

$$X = \{i \in V; \text{there exists an augmenting path from } s \text{ to } i\} \cup \{s\} = \{s, 2, 3, 4\}$$

and $\bar{X} = V \setminus X = \{5, t\}$. Then $(X, \bar{X})$ is an $s - t$ cut, and its capacity is

$$C(X, \bar{X}) = c_{25} + c_{35} + c_{4t} = 11$$

which is minimal.

(e) By the proof of Lemma 3.5 and based on the above cut we set the corresponding solution to the dual problem as:

$$u_i = \begin{cases} 
0, & i \in X \\
1, & i \in \bar{X}
\end{cases}$$

and

$$w_{ij} = \begin{cases} 
1, & (i, j) \in (X, \bar{X}) \\
0, & (i, j) \notin (X, \bar{X})
\end{cases}$$

that is

$$\begin{cases} 
u_s = u_2 = u_3 = u_4 = 0, u_5 = u_t = 1, \\
w_{s2} = w_{s3} = w_{23} = w_{24} = w_{54} = w_{5t} = 0, w_{25} = w_{35} = w_{4t} = 1.
\end{cases}$$

Obviously, this solution is also optimal for the dual.

Q.2) Solution: Since there exists a directed circuit $2 \rightarrow 3 \rightarrow 5 \rightarrow 2$, this network is NOT acyclic.

Q.3) Solution: By the method in Section 5 (chapter 3), we have the following table:
0

Since the last connection for Node 7 is (5, 7), for Node 5 is (2, 5) and for Node 2 is (1, 2), the shortest path from Node 1 to Node 7 is

\[ 1 \rightarrow 2 \rightarrow 5 \rightarrow 7 \]

with the total length 13.

\[ \square \]

Q.4) **Solution:** By the method of Section 6(Chapter 3), we get the following table:

<table>
<thead>
<tr>
<th>n</th>
<th>solved nodes set</th>
<th>closest unsolved node</th>
<th>length of new tree</th>
<th>added edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{1}</td>
<td>Node 1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>{1,2}</td>
<td>Node 2</td>
<td>1</td>
<td>(1,2)</td>
</tr>
<tr>
<td>2</td>
<td>{1,2,5}</td>
<td>Node 4</td>
<td>4+3=7</td>
<td>(2,4)</td>
</tr>
<tr>
<td>3</td>
<td>{1,2,4,5}</td>
<td>Node 6</td>
<td>8+3=11</td>
<td>(4,6)</td>
</tr>
<tr>
<td>4</td>
<td>{1,2,3,4,5,6}</td>
<td>The END.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

So the minimal spanning tree is \{(1,2), (2,5), (2,4), (4,6), (1,3)\} or \{(1,2), (2,5), (2,4), (4,6), (3,4)\}, with total length 16.

\[ \square \]