Ex 1.2 Solution:

1 Optimal Value Function:
   \( S(x, y) = \) the value of the minimum cost path from \( A \) to node \( (x, y) \), for \( x = 0, 1, 2, 3, 4 \) and \( y = -x, -x + 2, \ldots, x \).

2 Recurrence Relation:
   \[
   S(x, y) = \min\{a_u(x, y) + S(x - 1, y + 1), a_d(x, y) + S(x - 1, y - 1)\}.
   \]
   where \( a_u(x, y) \) denotes the cost of arc that goes upward from \( (x, y) \), and \( a_d(x, y) \) denotes the cost of arc that goes downward.

3 Optimal policy function:
   \[
   P(x, y) = \begin{cases} 
   U(up) & a_u(x, y) + S(x - 1, y + 1) \leq a_d(x, y) + S(x - 1, y - 1) \\
   D(down) & \text{otherwise} 
   \end{cases}
   \]

4 Boundary Conditions:
   \[ S(0, 0) = 0. \]

5 Answer: \( \min\{S(4, y), y = -4, -2, 0, 2, 4\} \)
The solution to this DP is computed as following:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S(1,1)</strong></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td><strong>S(1,-1)</strong></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td><strong>S(2,2)</strong></td>
<td>2 + S(1,1) = 3</td>
<td></td>
</tr>
<tr>
<td><strong>S(2,0)</strong></td>
<td>(\min{3 + S(1,1), 5 + S(1,-1)}) = 4</td>
<td></td>
</tr>
<tr>
<td><strong>S(2,-2)</strong></td>
<td>4 + S(1,-1) = 4</td>
<td></td>
</tr>
<tr>
<td><strong>S(3,3)</strong></td>
<td>6 + S(2,2) = 9</td>
<td></td>
</tr>
<tr>
<td><strong>S(3,1)</strong></td>
<td>(\min{3 + S(2,2), 3 + S(2,0)}) = 6</td>
<td></td>
</tr>
<tr>
<td><strong>S(3,-1)</strong></td>
<td>(\min{2 + S(2,0), 4 + S(2,-2)}) = 6</td>
<td></td>
</tr>
<tr>
<td><strong>S(3,-3)</strong></td>
<td>2 + S(2,-2) = 6</td>
<td></td>
</tr>
<tr>
<td><strong>S(4,4)</strong></td>
<td>3 + S(3,3) = 12</td>
<td></td>
</tr>
<tr>
<td><strong>S(4,2)</strong></td>
<td>(\min{1 + S(3,3), 2 + S(3,1)}) = 8</td>
<td></td>
</tr>
<tr>
<td><strong>S(4,0)</strong></td>
<td>(\min{5 + S(3,1), 5 + S(3,-1)}) = 11</td>
<td></td>
</tr>
<tr>
<td><strong>S(4,-2)</strong></td>
<td>(\min{3 + S(3,-1), 6 + S(3,-3)}) = 9</td>
<td></td>
</tr>
<tr>
<td><strong>S(4,-4)</strong></td>
<td>5 + S(3,-3) = 11</td>
<td></td>
</tr>
</tbody>
</table>

Then we have \(\min\{S(4,y), y = -4, -2, 0, 2, 4\}\) = \(S(4,2) = 8\). And since \(P(4,2) = D, P(3,1) = U, P(2,2) = D, P(1,1) = D\), the optimal path is \((0,0) \rightarrow (1,1) \rightarrow (2,2) \rightarrow (3,1) \rightarrow (4,2)\).

\(\square\)

**Ex 1.3 Solution:** As the same method as previous solution, we have :

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S(_{1,3})(4,4)</strong></td>
<td>(-3 + S(_{1,2})(4,4)) = 9</td>
<td></td>
</tr>
<tr>
<td><strong>S(_{1,3})(4,2)</strong></td>
<td>(-4 + S(_{1,2})(4,2)) = 4</td>
<td></td>
</tr>
<tr>
<td><strong>S(_{1,3})(4,0)</strong></td>
<td>(-3 + S(_{1,2})(4,0)) = 8</td>
<td></td>
</tr>
<tr>
<td><strong>S(_{1,3})(4,-2)</strong></td>
<td>(-4 + S(_{1,2})(4,-2)) = 5</td>
<td></td>
</tr>
<tr>
<td><strong>S(_{1,3})(4,-4)</strong></td>
<td>(-3 + S(_{1,2})(4,-4)) = 8</td>
<td></td>
</tr>
</tbody>
</table>

It is easy to see that the optimal value is 4 and the optimal path is unchanged.

\(\square\)

**Ex 1.4 Solution:**

1 **Optimal Value Function:**

\(S(x, y)\) = the value of the minimum cost path from node \((x, y)\) to any point on line 

\(B.\) for \(x = 0, 1, 2, 3, 4, 5, 6.\) and

\[
y = \begin{cases} 
-1, & x \text{ is odd} \\
-2, 0, 2 & \text{otherwise} 
\end{cases}
\]

2 **Recurrence Relation:**

\[
S(x, y) = \min\{a_u(x, y) + S(x+1, y+1), a_d(x, y) + S(x+1, y-1)\}
\]

where \(a_u(x, y), a_d(x, y)\) denote the cost of the arc that goes upward and downward respectively.
3 Optimal Policy Function:
\[
P(x, y) = \begin{cases} 
U(up) & a_u(x, y) + S(x + 1, y + 1) \leq a_d(x, y) + S(x + 1, y - 1) \\
D(down) & \text{otherwise}
\end{cases}
\]

4 Boundary Conditions:
\[
S(6, 2) = 2, \quad S(6, 0) = 3, \quad S(6, -2) = 1.
\]

5 Answer: \( \min\{2 + S(0, 2), 3 + S(0, 0), 2 + S(0, -2)\} \)

The solution to the problem is computed as following:

<table>
<thead>
<tr>
<th>Node</th>
<th>Value</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>S(5, 1)</td>
<td>(\min{1 + S(6, 2), 2 + S(6, 0)}) = 3</td>
<td>(P(5, 1) = U)</td>
</tr>
<tr>
<td>S(5, -1)</td>
<td>(\min{5 + S(6, 0), 6 + S(6, -2)}) = 7</td>
<td>(P(5, -1) = D)</td>
</tr>
<tr>
<td>S(4, 2)</td>
<td>(7 + S(5, 1)) = 10</td>
<td>(P(4, 2) = D)</td>
</tr>
<tr>
<td>S(4, 0)</td>
<td>(\min{3 + S(5, 1), 3 + S(5, -1)}) = 6</td>
<td>(P(4, 0) = U)</td>
</tr>
<tr>
<td>S(4, -2)</td>
<td>(2 + S(5, -1)) = 9</td>
<td>(P(4, -2) = U)</td>
</tr>
<tr>
<td>S(3, 1)</td>
<td>(\min{5 + S(4, 2), 1 + S(4, 0)}) = 7</td>
<td>(P(3, 1) = D)</td>
</tr>
<tr>
<td>S(3, -1)</td>
<td>(\min{4 + S(4, 0), 1 + S(4, -2)}) = 10</td>
<td>(P(3, 1) = U)</td>
</tr>
<tr>
<td>S(2, 2)</td>
<td>(2 + S(3, 1)) = 9</td>
<td>(P(2, 2) = D)</td>
</tr>
<tr>
<td>S(2, 0)</td>
<td>(\min{6 + S(3, 1), 1 + s(3, -1)}) = 11</td>
<td>(P(2, 0) = D)</td>
</tr>
<tr>
<td>S(2, -2)</td>
<td>(3 + S(3, -1)) = 13</td>
<td>(P(2, -2) = U)</td>
</tr>
<tr>
<td>S(1, 1)</td>
<td>(\min{4 + S(2, 2), 5 + S(2, 0)}) = 13</td>
<td>(P(1, 1) = U)</td>
</tr>
<tr>
<td>S(1, -1)</td>
<td>(\min{2 + S(2, 0), 4 + S(2, -2)}) = 13</td>
<td>(P(1, -1) = U)</td>
</tr>
<tr>
<td>S(0, 2)</td>
<td>(1 + S(1, 1)) = 14</td>
<td>(P(0, 2) = D)</td>
</tr>
<tr>
<td>S(0, 0)</td>
<td>(\min{2 + S(1, 1), 3 + S(1, -1)}) = 15</td>
<td>(P(0, 0) = U)</td>
</tr>
<tr>
<td>S(0, -2)</td>
<td>(2 + S(1, -1)) = 15</td>
<td>(P(0, -2) = U)</td>
</tr>
</tbody>
</table>

Then
\[
\min\{2 + S(0, 2), 3 + S(0, 0), 2 + S(0, -2)\} = 2 + S(0, 2) = 16
\]

which is the minimal total cost. And since \(P(0, 2) = D, P(1, 1) = U, P(2, 2) = D, P(3, 1) = D, P(4, 0) = U, P(5, 1) = U\), the optimal path is \(0, 2 \to 1, 1 \to 2, 2 \to 3, 1 \to 4, 0 \to 5, 1 \to 6, 2\).

\[\square\]

Ex 1.7 Solution:

1 Optimal Value Function:
\[
f_n(s, d_n) = \text{the cost value of the shortest path from node } (n, s) \text{ to node B with starting direction } d_n, \text{ where } n = 0, 1, \ldots, N, s = -\min\{n, N - n\}, -\min\{n, N - n\} + 2, \ldots, \min\{n, N - n\}.
\]

2 Recurrence Relations:
\[
f_n(s, d_n) = \min_{d_n \in \{U, D\}} \{\delta(d_n, d_{n+1}) \cdot[R_{n+1}(T_n(s, d_n), d_{n+1}) + \delta(d_n, d_{n+1})] + f_{n+1}(T_n(s, d_n), d_{n+1})\}.
\]
where
\[
\delta(d_n, d_{n+1}) = \begin{cases} 
0 & \text{if } d_n \text{ and } d_{n+1} \text{ are the same direction.} \\
1 & \text{otherwise.}
\end{cases}
\]

\[
T_n(s, d_n) = \begin{cases} 
 s + 1 & d_n = U \\
 s - 1 & d_n = D 
\end{cases}
\]

and \(R_n(s, d_n)\) denotes the number of changes of direction in the optimal path from node \((s, d_n)\) to node B.

3 Optimal Policy Function:
\[
P_n(s, d_n) = \begin{cases} 
 U & f_n(s, d_n) \text{ takes the value when } d_n = U, \\
 D & \text{otherwise.}
\end{cases}
\]

4 Boundary Conditions:
\[
f_N(0, U) = 0, f_N(0, D) = 0.
\]

5 Answer: \(\min\{f_0(0, U), f_0(0, D)\}\).  □

Ex 1.8 Solution:

1 Optimal Value Function:
\(f_n(s)\) is the shortest cost value from node \((n, s)\) to node B, where \(n = 0, 1, \ldots, N\), \(s = -\min\{n, N - n\}, -\min\{n, N - n\} + 2, \ldots, \min\{n, N - n\}\).

2 Recurrence Relation:
\[
f_n(s) = \min\{\min\{a_n(s, D), V_{n+1}(s-1)\} + f_{n+1}(s-1), \min\{a_n(s, U), V_{n+1}(s+1)\} + f_{n+1}(s+1)\}
\]
where \(a_n(s, D)\) and \(a_n(s, U)\) denote the cost of the arcs that go diagonally downward and upward respectively, and \(V_n(s)\) is the maximal arc cost of the optimal path from \((n, s)\) to B.

3 Optimal Policy Function:
\[
P_n(s) = \begin{cases} 
 U & f_n(s) = \min\{a_n(s, U), V_{n+1}(s+1)\} + f_{n+1}(s + 1) \\
 D & \text{otherwise}
\end{cases}
\]

4 Boundary Condition: \(f_N(0) = 0\).

5 Answer: \(f_0(0)\).  □

Ex 1.9 Solution:
1 Optimal Value Function:
\[ f_n(s) = \text{the shortest cost value from node }(n, s)\text{ to node B, where } n = 0, 1, \cdots, N,\]
\[ s = -\min\{n, N - n\}, -\min\{n, N - n\} + 2, \cdots, \min\{n, N - n\}.\]

2 Recurrence Relation:
\[ f_n(s) = \min\{\max\{a_n(s, D), f_{n+1}(s-1)\}, \max\{a_n(s, U), f_{n+1}(s+1)\}\} \]
where \( a_n(s, D) \) and \( a_n(s, U) \) denote the cost of the arcs that go diagonally downward and upward respectively.

3 Optimal Policy Function:
\[ P_n(s) = \begin{cases} 
U & f_n(s) = \max\{a_n(s, U), f_{n+1}(s+1)\} \\
D & \text{otherwise} 
\end{cases} \]

4 Boundary Condition: \( f_N(0) = 0. \)

5 Answer: \( f_0(0). \)

\[ \square \]

Ex 1.10 Solution:

1 Optimal Value Function:
\[ f_n(s) = \text{the shortest cost value from node }(n, s)\text{ to node B, where } n = 0, 1, \cdots, N,\]
\[ s = -\min\{n, N - n\}, -\min\{n, N - n\} + 2, \cdots, \min\{n, N - n\}.\]

2 Recurrence Relation:
\[ f_n(s) = \min\{\min\{\max\{a_n(s, D), f_{n+1}(s-1)\}, V_{n+1}(s-1)\}, \min\{\max\{a_n(s, U), f_{n+1}(s+1)\}, V_{n+1}(s+1)\}\} \]
where \( a_n(s, D) \) and \( a_n(s, U) \) denote the cost of the arcs that go diagonally downward and upward respectively, and \( V_n(s) \) is the maximal arc cost of the optimal path from \((n, s)\) to B.

3 Optimal Policy Function:
\[ P_n(s) = \begin{cases} 
U & f_n(s) = \min\{\max\{a_n(s, U), f_{n+1}(s+1)\}, V_{n+1}(s+1)\} \\
D & \text{otherwise} 
\end{cases} \]

4 Boundary Condition: \( f_N(0) = 0. \)

5 Answer: \( f_0(0). \)
Ex 1.15 Solution: See Work Out Examples

Ex 1.16 Solution:

(a) In the approach described in knapsack problem, we know that: for any node \((k, y)\) such that \(k > 1, y \geq w_k\), an addition and a comparison will be taken (others we can write the result straightly). But in the approach described in Resource allocation problem, we must take \([\frac{y}{w_k}]\) additions and \([\frac{y}{w_k}]\) comparisons for each node \((k, y)\) as above. So the first approach is better.

(b) Suppose that \(r\) is the maximum index of the problem, and by the approach of Resource Allocation Problem, we have that:

\[
S(r, b) = \max_{j=1 \ldots \left\lfloor \frac{b}{w_r} \right\rfloor} \{v_k \ast j + S(r - 1, b - w_k \ast j); S(r - 1, b)\}
\]

by \(b \geq \frac{\rho_1}{\rho_1 - \rho_2} w_r\), we have \(\rho_2 b \leq \rho_1 (b - w_r)\), then

\[
S(r - 1, b) \leq \rho_2 \ast b \leq \rho_1 (b - w_r) \leq \rho_1 w_r \ast \left\lfloor \frac{b}{w_r} \right\rfloor \leq v_r \ast \left\lfloor \frac{b}{w_r} \right\rfloor + S(r - 1, b - w_r \ast \left\lfloor \frac{b}{w_r} \right\rfloor)
\]

that is

\[
S(r, b) = \max_{j=1 \ldots \left\lfloor \frac{b}{w_r} \right\rfloor} \{v_k \ast j + S(r - 1, b - w_k \ast j)\}
\]

so \(x_r\) must be positive.
Additional Problem 1 Solution:
First we rewrite the nodes 1, 2, · · · , 7 by (1, 1), (2, 2), (4, 2), (3, 3), (5, 4), (6, 4), (7, 5)

Backward Recursion.

1 Optimal Value Function:
\[ f_k(n) = \text{the minimum cost value from node } (n, k) \text{ to node } (7, 5), \text{ where } n = 1, 2, \cdots, 7; k = 1, 2, \cdots, 5. \]

2 Recurrence Relation:
\[ f_k(n) = \min \{ a(n, m) + f_{k+i}(m) \text{ where } i = 1, 2 \} \]
Here \( a(n, m) \) is the cost value of the arc from \( n \) to \( m \).

3 Optimal Policy Function:
\[ P_k(n) = \min \{ m; f_k(n) = a(n, m) + f_{k+i}(m) \} \]

4 Boundary Condition: \( f_5(7) = 0 \).

5 Answer: \( f_1(1) \). The solution to this problem is computed as following:

\[
\begin{array}{c|c}
  f_4(5) &= 4 \\
  f_4(6) &= 7 \\
  f_3(3) &= \min \{ 2 + f_4(5), 3 + f_4(6) \} = 6 \\
  f_2(2) &= \min \{ 12 + f_4(5), 6 + f_3(3), 12 + f_4(6) \} = 12 \\
  f_2(4) &= \min \{ 7 + f_3(3), 9 + f_4(6) \} = 13 \\
  f_1(1) &= \min \{ 5 + f_2(2), 14 + f_3(3), 6 + f_2(4) \} = 17 \\
\end{array}
\]

since \( P(1) = 2, P(2) = 3, P(3) = 5, P(5) = 7 \), the optimal path is \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 7 \).

Forward Recursion.

1 Optimal Value Function:
\[ g_k(n) = \text{the minimum cost value from node } (1, 1) \text{ to node } (n, k), \text{ where } n = 1, 2, \cdots, 7; k = 1, 2, \cdots, 5. \]

2 Recurrence Relation:
\[ g_k(n) = \min \{ a(m, n) + f_{k-i}(m) \text{ where } i = 1, 2 \} \]
Here \( a(m, n) \) is the cost value of the arc from \( m \) to \( n \).

3 Optimal Policy Function:
\[ P'_k(n) = \min \{ m; g_k(n) = a(m, n) + f_{k-i}(m) \} \]
4 Boundary Condition: \( g_1(1) = 0 \).

5 Answer: \( g_5(7) \). The solution to this problem is computed as following:

\[
\begin{align*}
  g_2(2) &= 5 \\
  g_2(4) &= 6 \\
  g_3(3) &= \min\{14, 6 + g_2(2), 7 + g_2(4)\} = 11 \\
  g_4(5) &= \min\{12 + g_2(2), 2 + g_3(3)\} = 13 \\
  g_4(6) &= \min\{12 + g_2(2), 3 + g_3(3), 9 + g_2(4)\} = 14 \\
  g_5(7) &= \min\{4 + g_4(5), 7 + g_4(6)\} = 17
\end{align*}
\]

Since \( P'(7) = 5, P'(5) = 3, P'(3) = 2, P'(2) = 1 \), the optimal path is \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 7 \).

\(\square\)

Additional Problem 2 Solution:

(a) A formulation of this problem is:

1 Optimal Value Function:
\( f_k(x) = \) the maximum revenue from year \( k \) to year \( n \) with an age-\( x \) machine at the start of year \( k \), for \( k = 1, 2, \cdots, n \), and \( x = 1, 2, \cdots, k-1, y+k-1 \), where \( y \) is the age of the starting machine.

2 Recurrence Relation:
\[
f_k(x) = \max\{s(x) - p + r(0) + f_{k+1}(1), r(x) + f_{k+1}(x+1)\}
\]

3 Optimal Policy Function:
\[
P_k(x) = \begin{cases} 
  B_{(buy)} & f_k(x) = s(x) - p + r(0) + f_{k+1}(1) \\
  K_{(keep)} & \text{otherwise}
\end{cases}
\]

4 Boundary Conditions:
\[
f_{n+1}(x) = s(x), \quad x = 1, 2, \cdots, n, y+n.
\]

5 Answer: \( f_1(y) \).

(b) Remark: \( s(x) \) and \( r(x) \) are in thousand dollar units. Now since \( p = 10 \) (thousand dollars), \( n = 5, y = 2 \), the boundary conditions become

\[
f_6(1) = 8, f_6(2) = 6, f_6(3) = 4, f_6(4) = 2, f_6(5) = f_6(7) = 0.
\]

and the recurrence relation is
\[
f_k(x) = \max\{s(x) + 15 + f_{k+1}(1), r(x) + f_{k+1}(x+1)\}
\]
we solve this DP problem as following:

<table>
<thead>
<tr>
<th>$f_5(1)$</th>
<th>$= \max{8 + 15 + f_6(1), 24 + f_6(2)}$</th>
<th>$= 31$</th>
<th>$P_5(1) = B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_5(2)$</td>
<td>$= \max{6 + 15 + f_6(1), 21 + f_6(3)}$</td>
<td>$= 29$</td>
<td>$P_5(2) = B$</td>
</tr>
<tr>
<td>$f_5(3)$</td>
<td>$= \max{4 + 15 + f_6(1), 16 + f_6(4)}$</td>
<td>$= 27$</td>
<td>$P_5(3) = B$</td>
</tr>
<tr>
<td>$f_5(4)$</td>
<td>$= \max{2 + 15 + f_6(1), 9 + f_6(5)}$</td>
<td>$= 25$</td>
<td>$P_5(4) = B$</td>
</tr>
<tr>
<td>$f_5(6)$</td>
<td>$= \max{0 + 15 + f_6(1), 0 + f_6(7)}$</td>
<td>$= 23$</td>
<td>$P_5(6) = B$</td>
</tr>
</tbody>
</table>

| $f_4(1)$ | $= \max\{8 + 15 + f_5(1), 24 + f_5(2)\}$ | $= 54$ | $P_4(1) = B$ |
| $f_4(2)$ | $= \max\{6 + 15 + f_5(1), 21 + f_5(3)\}$ | $= 52$ | $P_4(2) = B$ |
| $f_4(3)$ | $= \max\{4 + 15 + f_5(1), 16 + f_5(4)\}$ | $= 50$ | $P_4(3) = B$ |
| $f_4(5)$ | $= \max\{0 + 15 + f_5(1), 0 + f_5(6)\}$ | $= 46$ | $P_4(5) = B$ |
| $f_3(1)$ | $= \max\{8 + 15 + f_4(1), 24 + f_4(2)\}$ | $= 77$ | $P_3(1) = B$ |
| $f_3(2)$ | $= \max\{6 + 15 + f_4(1), 21 + f_4(3)\}$ | $= 75$ | $P_3(2) = B$ |
| $f_3(4)$ | $= \max\{2 + 15 + f_4(1), 9 + f_4(5)\}$ | $= 71$ | $P_3(4) = B$ |
| $f_2(1)$ | $= \max\{8 + 15 + f_3(1), 24 + f_3(2)\}$ | $= 100$ | $P_2(1) = B$ |
| $f_2(3)$ | $= \max\{4 + 15 + f_3(1), 16 + f_3(4)\}$ | $= 96$ | $P_2(3) = B$ |
| $f_1(2)$ | $= \max\{6 + 15 + f_2(1), 21 + f_2(3)\}$ | $= 121$ | $P_1(2) = B$ |

So the optimal revenue is 121, and we easily know that the optimal path is BBBBB.

\[\square\]

**Additional Problem 3 Solution:**

1. **Optimal Value Function:**
   
   \[f_k(x) = \text{the maximum return obtainable by distributing } x \text{ units of clothes among activities } k \text{ to 3, where } k = 1, 2, 3 \text{ and } x = 0, 1, 2, 3, 4, 5.\]

2. **Recurrence relation:**

   \[f_k(x) = \max_{j=0,1,\ldots,x} \{r_k(j) + f_{k+1}(x-j)\}\]

3. **Optimal Policy Function:**

   \[P_k(x) = \min_{j=0,1,\ldots,x} \{j; f_k(x) = r_k(j) + f_{k+1}(x-j)\}\]

4. **Boundary Conditions:**

   \[f_3(x) = r_3(x), \quad x = 0, 1, 2, 3, 4, 5.\]

5. **Answer:** \(f_3(5)\). We compute this problem as following:

<table>
<thead>
<tr>
<th>$f_2(0)$</th>
<th>$= 0$</th>
<th>$P_2(0) = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_2(1)$</td>
<td>$= \max{f_2(1), 1 + f_2(0)}$</td>
<td>$= 1$</td>
</tr>
<tr>
<td>$f_2(2)$</td>
<td>$= \max{f_2(2), 1 + f_2(1), 2 + f_2(0)}$</td>
<td>$= 2$</td>
</tr>
<tr>
<td>$f_2(3)$</td>
<td>$= \max{f_2(3), 1 + f_2(2), 2 + f_2(1), 2 + f_2(0)}$</td>
<td>$= 2$</td>
</tr>
<tr>
<td>$f_2(4)$</td>
<td>$= \max{f_2(4), 1 + f_2(3), 2 + f_2(2), 2 + f_2(1), 3 + f_2(0)}$</td>
<td>$= 3$</td>
</tr>
<tr>
<td>$f_2(5)$</td>
<td>$= \max{f_2(5), 1 + f_2(4), 2 + f_2(3), 2 + f_2(2), 3 + f_2(1), 3 + f_2(0)}$</td>
<td>$= 4$</td>
</tr>
<tr>
<td>$f_1(5)$</td>
<td>$= \max{f_2(5), 1 + f_2(4), 1 + f_2(3), 2 + f_2(2), 2 + f_2(1), 3 + f_2(0)}$</td>
<td>$= 4$</td>
</tr>
</tbody>
</table>

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So the optimal distribution is $(0, 1, 4)$ and the maximal return is $4$. 

**Additional Problem 4 Solution:**

1. **Optimal Value Function:**
   
   
   $f_k(x) =$ the maximal knowledge obtainable by selecting $x$ courses from dept. $k$ to dept. 4, for $k = 1, 2, 3, 4$ and $x = 5 - k, \ldots, 11 - k$.

   **Remark:** Since from dept. $k$ to dept. 4, there are $5 - k$ departments, and we must select at least one course in each department, so the number of courses $x \geq 5 - k$. Similarly, the number of the rest courses $10 - x \geq k - 1$, which is the number of the rest departments: dept. 1 to dept. $k - 1$, that is, $x \leq 11 - k$.

2. **Recurrence Relation:**

   
   
   $f_k(x) = \max_{j = 1, \ldots, x-(4-k)} \{r_k(j) + f_{k+1}(x - j)\}$

   where $r_k(j)$ denotes the knowledge value of $j$ courses in dept. $k$.

3. **Optimal Policy Function:**

   
   
   $P_k(x) = \min_{j = 1, \ldots, x-(4-k)} \{j; f_k(x) = r_k(j) + f_{k+1}(x - j)\}$

4. **Boundary Conditions:**

   
   
   $f_4(1) = 10, f_4(2) = 20, f_4(3) = 30, f_4(4) = 40, f_4(5) = 50, f_4(6) = 60, f_4(7) = 70$.

5. **Answer:** $f_1(10)$. We compute this problem as following

<table>
<thead>
<tr>
<th>$f_3(2)$</th>
<th>40 + $f_3(1)$ = 50</th>
<th>$P_3(2)$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_3(3)$</td>
<td>max{40 + $f_4(2)$, 60 + $f_4(1)$} = 70</td>
<td>$P_3(3)$</td>
<td>2</td>
</tr>
<tr>
<td>$f_3(4)$</td>
<td>max{40 + $f_4(3)$, 60 + $f_4(2)$, 80 + $f_4(1)$} = 90</td>
<td>$P_3(4)$</td>
<td>3</td>
</tr>
<tr>
<td>$f_3(5)$</td>
<td>110</td>
<td>$P_3(5)$</td>
<td>4</td>
</tr>
<tr>
<td>$f_3(6)$</td>
<td>120</td>
<td>$P_3(6)$</td>
<td>4</td>
</tr>
<tr>
<td>$f_3(7)$</td>
<td>130</td>
<td>$P_3(7)$</td>
<td>4</td>
</tr>
<tr>
<td>$f_3(8)$</td>
<td>140</td>
<td>$P_3(8)$</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$f_2(3)$</th>
<th>20 + $f_3(2)$ = 70</th>
<th>$P_2(3)$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_2(4)$</td>
<td>max{20 + $f_3(3)$, 70 + $f_3(2)$} = 120</td>
<td>$P_2(4)$</td>
<td>1</td>
</tr>
<tr>
<td>$f_2(5)$</td>
<td>max{20 + $f_3(4)$, 70 + $f_3(3)$, 90 + $f_3(2)$} = 140</td>
<td>$P_2(5)$</td>
<td>2</td>
</tr>
<tr>
<td>$f_2(6)$</td>
<td>160</td>
<td>$P_2(6)$</td>
<td>2</td>
</tr>
<tr>
<td>$f_2(7)$</td>
<td>180</td>
<td>$P_2(7)$</td>
<td>2</td>
</tr>
<tr>
<td>$f_2(8)$</td>
<td>200</td>
<td>$P_2(8)$</td>
<td>3</td>
</tr>
<tr>
<td>$f_2(9)$</td>
<td>210</td>
<td>$P_2(9)$</td>
<td>3</td>
</tr>
</tbody>
</table>

| $f_1(10)$ | max $\{25 + f_2(9), 50 + f_2(8), 60 + f_2(7), 80 + f_2(6), 100 + f_2(5), 100 + f_2(4), 100 + f_2(3)\}$ = 250 | $P_1(10)$ | 2 |

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Since $P_1(10) = 2$, $P_2(8) = 3$, $P_3(5) = 4$, the optimal policy is $(2, 3, 4, 1)$. □