Sample Solutions of Quiz 4 for MAT3270B

1. State the type and stability of critical point \((0,0)\) for the following system.

\[
\frac{dX}{dt} = \begin{pmatrix} 2 & 5 \\ -2 & -3 \end{pmatrix} X
\]

**Answer:**

\[
\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & 5 \\ -2 & -3 - \lambda \end{vmatrix} = \lambda^2 + \lambda + 4
\]

The eigenvalues are

\[
\lambda_1 = -\frac{1}{2} + \frac{\sqrt{15}}{2}i, \quad \lambda_2 = -\frac{1}{2} - \frac{\sqrt{15}}{2}i
\]

Hence, \((0,0)\) is an asymptotically stable spiral sink.

2. Prove that if a trajectory \((x(t), y(t))\) starts at a noncritical point of the system

\[
\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y)
\]

then it can not reach a critical point in a finite length of time.

**Answer:** Assume the contrary. we suppose that \((x_0, y_0)\) is a critical point of the system, and the solution \(x = \phi(t), \ y = \psi(t)\) satisfies \(\phi(a) = x_0, \ \psi(a) = y_0\).

On the other hand, \(x = x_0, \ y = y_0\) is a solution of the given system satisfying the initial condition \(x = x_0, \ y = y_0\) at \(t = a\). By the uniqueness, \(\phi(t) = x_0, \ \psi(t) = y_0, \forall t \geq 0\), hence, \(\phi(0) = x_0, \ \psi(0) = y_0\).

This contradict the assumption of the problem.