Sample Solutions of Assignment 4 for MAT3270B: 3.1, 3.2, 3.3

Section 3.1

Find the general solution of the given differential equation

1. \( y'' + 2y' - 3y = 0 \)
4. \( 2y'' - 3y' + y = 0 \)
7. \( y'' - 9y' + 9y = 0 \)

**Answer:**

1. The characteristic equation is
   \[ r^2 + 2r - 3 = (r + 3)(r - 1) = 0 \]
   Thus the possible values of \( r \) are \( r_1 = -3 \) and \( r_2 = 1 \), and the general solution of the equation is
   \[ y(t) = c_1e^t + c_2e^{3t}. \]

4. The characteristic equation is
   \[ 2r^2 - 3r + 1 = (2r - 1)(r - 1) = 0 \]
   Thus the possible values of \( r \) are \( r_1 = \frac{1}{2} \) and \( r_2 = 1 \), and the general solution of the equation is
   \[ y(t) = c_1e^{\frac{t}{2}} + c_2e^t. \]

7. The characteristic equation is
   \[ r^2 - 9r + 9 = (r - 4)(r - 5) = 0 \]
   Thus the possible values of \( r \) are \( r_1 = 5 \) and \( r_2 = 4 \), and the general solution of the equation is
   \[ y(t) = c_1e^{5t} + c_2e^{3t}. \]
17. Find a differential equation whose general solution is \( y = c_1 e^{2t} + c_2 e^{-3t} \)

**Answer:** The characteristic equation is 
\[(r - 2)(r + 3) = r^2 + r - 6 = 0\]
So the equation is 
\[y'' + y' - 6y = 0.\]

21. Solve the initial value problem \( y'' - y' - 2y = 0, \ y(0) = \alpha, \ y'(0) = 2. \) Then find \( \alpha \) so that the solution approaches zero as \( t \to \infty. \)

**Answer:** The characteristic equation is 
\[r^2 - r - 2 = (r + 1)(r - 2) = 0\]
Thus the possible values of \( r \) are \( r_1 = -1 \) and \( r_2 = 2, \) and the general solution of the equation is 
\[y(t) = c_1 e^{2t} + c_2 e^{-t}.\]
Using the first initial condition, we obtain 
\[c_1 + c_2 = \alpha.\]
Using the second initial condition, we obtain 
\[2c_1 - c_2 = 2.\]
By solving above equations we find that \( c_1 = \frac{\alpha + 2}{3} \) and \( c_2 = \frac{2(\alpha + 1)}{3}. \)
Hence, 
\[y(t) = \frac{\alpha + 2}{3} e^{2t} + c_2 = \frac{2(\alpha + 1)}{3} e^{-t}.\]
From \( y(t) \to 0 \) as \( t \to \infty, \) we find \( \alpha = -2. \)
22. Solve the initial value problem \(4y'' - y = 0\), \(y(0) = 2\), \(y'(0) = \beta\). Then find \(\beta\) so that the solution approaches zero as \(t \to \infty\).

**Answer:** The characteristic equation is

\[4r^2 - 1 = (2r + 1)(2r - 1) = 0\]

Thus the possible values of \(r\) are \(r_1 = \frac{1}{2}\) and \(r_2 = \frac{-1}{2}\), and the general solution of the equation is

\[y(t) = c_1e^{\frac{1}{2}t} + c_2e^{-\frac{1}{2}t}.\]

Using the first initial condition, we obtain

\[c_1 + c_2 = 2.\]

Using the second initial condition, we obtain

\[2c_1 - c_2 = 2\beta.\]

By solving above equations we find that \(c_1 = \beta + 1\) and \(c_2 = 1 - \beta\). Hence,

\[y(t) = (\beta + 1)e^{\frac{1}{2}t} + (1 - \beta)e^{-\frac{1}{2}t}.\]

From \(y(t) \to 0\) as \(t \to \infty\), we find \(\beta = -1\).

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In each of the following problem determine the value of \(\alpha\), if any, for which all solutions tend to zero as \(t \to \infty\); Also determine the value of \(\alpha\), if any, for which all (nonzero) solutions become unbounded as \(t \to \infty\).

23. \(y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0\)

**Answer:** The characteristic equation is

\[r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = (r - \alpha)(r - (\alpha - 1)) = 0\]
Thus the possible values of $r$ are $r_1 = \alpha$ and $r_2 = \alpha - 1$, and the general solution of the equation is

$$y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha - 1)t}.$$ 

If we want $y(t) \to 0$ as $t \to \infty$, then $\alpha < 0$ and $\alpha - 1 < 0$. Hence, in this case $\alpha < 0$; If we want $y(t)$ become unbounded as $t \to \infty$, then $\alpha > 0$ and $\alpha - 1 > 0$. Hence, in this case $\alpha > 1$.

27. Find an equation of the form $ay'' - by' + cy = 0$ for which all solutions approach a multiple of $e^{-t}$ as $t \to \infty$.

**Answer:** We select $y_1 = e^{-t}$ and $y_2 = e^{-2t}$. Let $y(t) = c_1 y_1(t) + c_2 y_2(t)$ satisfy $ay'' - by' + cy = 0$, then the characteristic equation is

$$(r + 1)(r + 2) = (r^2 + 3r + 2) = 0.$$ 

Hence the equation is $y'' + 3y' + 2y = 0$.

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**Section 3.2**

Find the Wronskian of the given pair of functions.

1. $e^{2t}$, $e^{-3t}$
3. $e^{-2t}$, $te^{-2t}$
6. $\cos^2 \theta$, $1 + \cos 2\theta$

**Answer:** The computation is easy, so we just give the final result.

1. $W = \frac{-7}{2} e^\frac{t}{2}$
3. \( W = e^{-4t} \)
6. \( W = 0 \)

In the following problems determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

7. \( ty'' + 3y = t, y(1) = 1, y'(1) = 2 \)
11. \( (x - 3)y'' + xy' + (\ln |x|)y = 0, y(1) = 0, y'(1) = 1 \)

**Answer:** 7. The original solution can written as
\[
y'' + \frac{3}{t}y = 1.
\]
and \( p(t) = 0, \; q(t) = \frac{3}{t}, \; g(t) = 1. \) Then then only point of discontinuity of the coefficients is \( t = 0. \) Therefore, the longest open interval, containing the initial point \( t = 1, \) in which all the coefficients are continuous, is \( 0 < t < \infty. \)

**Answer:** 11. The original solution can written as
\[
y'' + \frac{x}{x-3}y' + \frac{\ln |x|}{x-3} = 0.
\]
and \( p(t) = \frac{x}{x-3}, \; q(t) = \frac{\ln |x|}{x-3}, \; g(t) = 0. \) Then the only points of discontinuity of the coefficients is \( t = 0, \) and \( t = 3. \) Therefore, the longest open interval, containing the initial point \( t = 1, \) in which all the coefficients are continuous, is \( 0 < t < 3. \)

14. Verify that \( y_1(t) = 1 \) and \( y_2(t) = t^\frac{1}{2} \) are solutions of the differential equation \( yy'' + (y')^2 = 0 \) for \( t > 0. \) Then show that \( c_1 + c_2 t^\frac{1}{2} \) is not, in
general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.

**Answer:** It is easy to verify $y_1$ and $y_2$ are solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$, and $y = c_1 + c_2 t^\frac{1}{2}$ is not a solution (in general) of this equation.

This result does not contradict Theorem 3.2.2 because this equation is nonlinear.

15. Show that if $y = \phi(t)$ is a solution of the differential equation $y'' + p(t)y' + q(t)y = g(t)$, where $g(t)$ is not always zero, the $y = c\phi(t)$, where $c$ is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.

**Answer:**

$$[c\phi(t)'' + p(t)c\phi(t)']' + q(t)c\phi(t)]$$

$$= c[\phi(t)'' + p(t)\phi(t)'] + q(t)\phi(t)]$$

$$= cg(t) \neq g(t)$$

if $c$ is a constant other than 1, and $g(t)$ is not always zero.

This result does not contradict Theorem 3.2.2 because this equation is not homogeneous.

17. If the Wronskian $W$ of $f$ and $g$ is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

**Answer:**

$$W = f(t)g'(t) - f'(t)g(t) = e^{2t}g'(t) - 2e^{2t}g(t)$$
Let $W = 3e^{4t}$, we get the following equation

$$g' - 2g(t) = 3e^{2t}.$$ 

From the above equation, $g(t) = te^{2t} + ce^{2t}$. 

19. If $W(f, g)$ is the Wronskian of $f$ and $g$, and if $u = 2f - g$, $v = f + 2g$, find the Wronskian $W(u, v)$ of $u$ and $v$ in term of $W(f, g)$.

**Answer:**

$$W(u, v) = uv' - u'v$$

$$= (2f - g)(f' - 2g') - (2f' - g')(f + 2g)$$

$$= 5fg' - 5f'g$$

$$= 5W(f, g).$$

20. If the Wronskian of $f$ and $g$ is $t \cos t - \sin t$ and if $u = f + 3g$, $v = f - g$, find the Wronskian of $u$ and $v$.

**Answer:**

$$W(u, v) = uv' - u'v$$

$$= (f + 3g)(f' - g') - (f' + 3g')(f - g)$$

$$= -4fg' + 4f'g$$

$$= -4W(f, g) = -4(t \cos t - \sin t).$$
In the following problems verify that the function \( y_1 \) and \( y_2 \) are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

23. \( y'' + 4y = 0 \), \( y_1(t) = \cos 2t \), \( y_2(t) = \sin 2t \)

25. \( x^2y'' - x(x + 2)y' + (x + 2)y = 0 \), \( x > 0 \), \( y_1(x) = x \), \( y_2(x) = xe^x \)

**Answer:** 23.

\[
y_1(t) = \cos 2t, \quad y'_1(t) = -2 \sin 2t, \quad y''_1(t) = -4 \cos 2t
\]

\[
y_2(t) = \sin 2t, \quad y'_2(t) = 2 \cos 2t, \quad y''_2(t) = -4 \sin 2t
\]

From above equation, we can verify that the function \( y_1 \) and \( y_2 \) are solutions of the given differential equation \( y'' + 4y = 0 \).

They constitute a fundamental set solutions because \( W(y_1, y_2) = 2. \)

25.

\[
y_1(x) = x, \quad y'_1(x) = 1, \quad y_1(x) = 0
\]

\[
y_2(x) = xe^x, \quad y'_2(x) = (1 + x)e^x, \quad y''_2(x) = (1 + x)e^x
\]

From above equation, we can verify that the function \( y_1 \) and \( y_2 \) are solutions of the given differential equation \( x^2y'' - x(x + 2)y' + (x + 2)y = 0. \)

They constitute a fundamental set solutions because \( W(y_1, y_2) = x^2e^x. \)

\[ \square \]

### Section 3.3

In the following problems determine whether the given pair of functions is linearly independent or linearly dependent.
3. \( f(t) = e^{\lambda t} \cos \mu t, \ g(t) = e^{\lambda t} \sin \mu t, \ \mu \neq 0 \)

4. \( f(x) = e^{3x}, \ g(x) = e^{3(x-1)} \)

**Answer:**

3.

\[
W(f, g) = fg' - f'g = e^{2\lambda t} \cos \mu t (\lambda \sin \mu t - \mu \sin \mu t) - e^{2\lambda t} \sin \mu t (\lambda \cos \mu t - \mu \sin \mu t) = -\mu e^{2\lambda t} \neq 0 \text{ for } \mu \neq 0
\]

Hence, the given pair of functions is linearly independent.

4.

\[
W(f, g) = fg' - f'g = e^{3x}3e^{3(x-1)} - 3e^{3x}e^{3(x-1)} = 0
\]

Hence, the given pair of functions is linearly dependent.

9. The Wronskian of two functions is \( W(t) = t \sin^2 t \). Are the functions linearly independent or linearly dependent? Why?

**Answer:** The functions is linearly independent because \( W \) is not always zero.

11. If the functions \( y_1 \) and \( y_2 \) are linearly independent solutions of \( y'' + p(t)y' + q(t)y = 0 \), prove that \( c_1 y_1 \) and \( c_2 y_2 \) are also linearly independent solutions, provided that neither \( c_1 \) nor \( c_2 \) is zero.
10

**Answer:** Obviously, \( W(c_1y_1, c_2y_2) = c_1c_2W(y_1, y_2) \). so \( c_1y_1 \) and \( c_2y_2 \) are also linearly independent solutions, provided that neither \( c_1 \) nor \( c_2 \) is zero.

\[13. \text{ If the functions } y_1 \text{ and } y_2 \text{ are linearly independent solutions of } y'' + p(t)y' + q(t)y = 0, \text{ determine under what conditions the function } y_3 = a_1y_1 + a_2y_2 \text{ and } y_4 = b_1y_1 + b_2y_2 \text{ also form a linearly independent set of solutions.} \]

**Answer:** \( W(y_3, y_4) = (a_1y_1 + a_2y_2)(b_1y_1' + b_2y_2') - (a_1y_1' + a_2y_2')(b_1y_1 + b_2y_2) = (a_1b_2 - a_2b_1)W(y_1, y_2). \) So if \( y_3 \) and \( y_4 \) also form a linearly independent set of solutions, then \( W(y_3, y_4) \) is not always zero. Hence \( (a_1b_2 - a_2b_1) \neq 0. \)

19. Show that if \( p \) is differentiable and \( p(t) > 0 \), then the Wronskian \( W(t) \) of two solutions of \([p(t)y']' + q(t)y = 0\) is \( W(t) = \frac{c}{p(t)} \), where \( c \) is constant.

**Answer:** The original equation can be written as
\[
p(t)y'' + p'(t)y' + q(t)y = 0
\]

\[
\Rightarrow
\]
\[
y'' + \frac{p'(t)}{p(t)}y' + \frac{q(t)}{p(t)}y = 0
\]

From Abel’s theorem \( W(t) = c \exp\left[-\int \frac{p'(t)}{p(t)} dt\right] = ce^{-\ln p(t)} = \frac{c}{p(t)}. \)
20. If \( y_1 \) and \( y_2 \) are linearly independent solutions \( ty'' + 2y' + te^t y = 0 \) and if \( W(y_1, y_2)(1) = 2 \), find the value of \( W(y_1, y_2)(5) \).

**Answer:** The original equation can be written as

\[
y'' + \frac{2}{t} y' + e^t y = 0
\]

From Abel’s theorem

\[
W(y_1, y_2)(t) = c \exp [- \int \frac{2}{t} dt] = \frac{c}{t^2}.
\]

We can find \( c = 2 \) by \( W(y_1, y_2)(1) = 2 \). Hence, \( W(y_1, y_2)(5) = \frac{2}{25} \).

In the following problems through 24 to 26 assume that \( p \) and \( q \) are continuous, and that the functions \( y_1 \) and \( y_2 \) are solutions of the differential equation \( y'' + p(t)y' + q(t)y = 0 \) on an open interval \( I \).

24. Prove that if \( y_1 \) and \( y_2 \) are zero at the same point in \( I \), then they cannot be a fundamental set of solutions on that interval.

**Answer:** \( W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0 \) at some point in \( I \) because \( y_1 \) and \( y_2 \) are zero at the same point in \( I \). Hence, from Theorem 3.3.3 they cannot be a fundamental set of solutions on \( I \).

25. Prove that if \( y_1 \) and \( y_2 \) have maxima or minima at the same point in \( I \), then they cannot be a fundamental set of solutions on that interval.

**Answer:** From \( y_1 \) and \( y_2 \) have maxima or minima at the same point in \( I \), saying \( t_0 \), we can get \( y_1'(t_0) = y_2'(t_0) = 0 \). Therefore \( W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 0 \) at \( t_0 \) in \( I \). Hence, from Theorem 3.3.3 they cannot be a fundamental set of solutions on \( I \).
26. Prove that if \( y_1 \) and \( y_2 \) have a common point of inflection \( t_0 \) in \( I \), then they cannot be a fundamental set of solutions on that interval unless both \( p \) and \( q \) are zero at \( t_0 \).

**Answer:** If \( y_1 \) and \( y_2 \) have a common point of inflection \( t_0 \) in \( I \), then 
\[
y_1'(t_0) = y_2'(t_0) = 0.
\]
Therefore, \( W(y_1, y_2) = y_1y_2' - y_1'y_2 = 0 \) at \( t_0 \) in \( I \). Hence, from Theorem 3.3.3 they cannot be a fundamental set of solutions on \( I \).

Supplement Problem: Consider the following two functions:

\[
y_1(t) = \begin{cases} t^2, & t \leq 0 \\ 0, & t > 0 \end{cases}
\]

(1)

\[
y_2(t) = \begin{cases} 0, & t \leq 0 \\ t^2, & t > 0 \end{cases}
\]

(2)

Show that \( y_1, y_2 \) is linearly independent but \( W[y_1, y_2] \equiv 0 \). What is wrong?

**Answer:** If there exist two constants \( k_1 \) and \( k_2 \) such that \( k_1y_1 + k_2y_2 = 0 \), then \( [k_1y_1 + k_2y_2](1) = k_2y_2(1) = k_2 = 0 \). Similarly, \( k_1 = 0 \). So \( y_1, y_2 \) is linearly independent. This result does not contradict Theorem 3.3.1 because Theorem 3.3.1 does not include this case.