1. For the following ODEs, (a) determine all critical points; (b) find the corresponding linear system near each critical point; (c) find the eigenvalues of each linear system; (d) state if the system is almost linear at each critical point; (e) conclude the types and stability of each critical point.

a). \( \frac{dx}{dt} = 1 - y \), \( \frac{dy}{dt} = x^2 - y^2 \)

b). \( \frac{dx}{dt} = 2x + y + xy^3 \), \( \frac{dy}{dt} = x - 2y - xy \)

c). \( \frac{dx}{dt} = (1 + x) \sin y \), \( \frac{dy}{dt} = 1 - x - \cos y \)

d). \( \frac{dx}{dt} = y + x(1 - x^2 - y^2) \), \( \frac{dy}{dt} = -x + y(1 - x^2 - y^2) \)

e). \( \frac{dx}{dt} = x(1.5 - x - 0.5y) \), \( \frac{dy}{dt} = y(2 - y - 0.75x) \)

f). \( \frac{dx}{dt} = x(1 - x - y) \), \( \frac{dy}{dt} = y(1.5 - y - x) \)

**Answer:** 
a). From

\[
\begin{align*}
F &= 1 - y = 0 \\
G &= x^2 - y^2 = 0
\end{align*}
\]

we know that all the critical points are \((-1,1), (1,1)\). Since the \(F,G\) are continuous differentiable, the system is almost linear near the critical points.

\[
\begin{pmatrix}
F_x & F_y \\
G_x & G_y
\end{pmatrix} = \begin{pmatrix} 0 & -1 \\
2x & -2y \end{pmatrix}
\]

The linear system near critical point \((-1,1)\) is

\[
\begin{align*}
\frac{du}{dt} &= -v \\
\frac{dv}{dt} &= -2u - 2v
\end{align*}
\]

where \(u = -1 + x, v = 1 + y\). The eigenvalues of the linear system are \(-1 - \sqrt{3}, -1 + \sqrt{3}\), hence the critical point \((-1,1)\) is saddle point and unstable. The linear system near critical point \((1,1)\) is

\[
\begin{align*}
\frac{du}{dt} &= -v \\
\frac{dv}{dt} &= 2u - 2v
\end{align*}
\]

where \(u = 1 + x, v = 1 + y\). The eigenvalues of the linear system are \(-1 - i, -1 + i\) and the real parts of the eigenvalues are negative, hence the critical point \((1,1)\) is spiral point and is asymptotically stable.
b). From

\[
F = 2x + y + xy^3 = 0 \\
G = x - 2y - xy = 0
\]

we know that all the critical points are \((0, 0), (-1.19345, -1.4797)\). Since the \(F, G\) are continuous differentiable, the system is almost linear near the critical points.

\[
\left(\begin{array}{cc}
F_x & F_y \\
G_x & G_y
\end{array}\right) = \left(\begin{array}{cc}
2 + y^3 & 1 + 3xy^2 \\
1 - y & -2 - x
\end{array}\right)
\]

The linear system near critical point \((0, 0)\) is

\[
\frac{du}{dt} = 2u + v \\
\frac{dv}{dt} = u - 2v
\]

where \(u = x, v = y\). The eigenvalues of the linear system are \(-\sqrt{5}, \sqrt{5}\), hence the critical point \((0, 0)\) is saddle point and unstable.

The linear system near critical point \((-1.19345, -1.4797)\) is

\[
\frac{du}{dt} = -1.2399u - 8.8393v \\
\frac{dv}{dt} = 2.4797u - 0.80655v
\]

where \(u = -1.19345 + x, v = -1.4797 + y\). The eigenvalues of the linear system are \(-1.0232 \pm 4.1125i\), hence the critical point \((-1.19345, -1.4797)\) is spiral point and asymptotically stable.

c). From

\[
F = (1 + x) \sin y = 0 \\
G = 1 - x - \cos y = 0
\]

we know that all the critical points are \((0, 2k\pi), (2, (2k + 1)\pi)\). Since the \(F, G\) are continuous differentiable, the system is almost linear near the critical points.

\[
\left(\begin{array}{cc}
F_x & F_y \\
G_x & G_y
\end{array}\right) = \left(\begin{array}{cc}
\sin y & (1 + x) \cos y \\
-1 & \sin y
\end{array}\right)
\]

The linear system near critical point \((2, (2k + 1)\pi)\) is

\[
\frac{du}{dt} = -3v \\
\frac{dv}{dt} = -u
\]
where \( u = 2 + x, \ v = (2k + 1)\pi + y \). The eigenvalues of the linear system are \(-\sqrt{3}, \sqrt{3}\), hence the critical point \((2, (2k + 1)\pi)\) is saddle point and unstable. The linear system near critical point \((0, 2k\pi)\) is

\[
\begin{align*}
\frac{du}{dt} &= v \\
\frac{dv}{dt} &= -u
\end{align*}
\]

where \( u = x, \ v = 2k\pi + y \). The eigenvalues of the linear system are \(-i, i\), hence the critical point \((0, 2k\pi)\) is center point and stable of the corresponding linear system. Then \((0, 2k\pi)\) maybe the spiral point or center point of the nonlinear system. The stability of \((0, 2k\pi)\) is indeterminate.

d). From

\[
\begin{align*}
F &= y + x(1 - x^2 - y^2) = 0 \\
G &= -x + y(1 - x^2 - y^2) = 0
\end{align*}
\]

we know that all the critical point is \((0, 0)\). Since the \(F, G\) are continuous differentiable, the system is almost linear near the critical points.

\[
\begin{pmatrix}
F_x & F_y \\
G_x & G_y
\end{pmatrix}
= 
\begin{pmatrix}
1 - 3x^2 - y^2 & 1 + 2xy \\
-1 - 2xy & 1 - x^2 - 3y^2
\end{pmatrix}
\]

The linear system near critical point \((2, (2k + 1)\pi)\) is

\[
\begin{align*}
\frac{du}{dt} &= u + v \\
\frac{dv}{dt} &= -u + v
\end{align*}
\]

where \( u = 2 + x, \ v = (2k + 1)\pi + y \). The eigenvalues of the linear system are \(1 \pm i\) and the real parts of the eigenvalues are positive, hence the critical point \((0, 0)\) is spiral point and unstable.

e). From

\[
\begin{align*}
F &= x(1.5 - x - 0.5y) = 0 \\
G &= y(2 - y - 0.75x) = 0
\end{align*}
\]

we know that all the critical point is \((0, 0), (1.5, 0), (0, 2), (0.8, 1.4)\). Since the \(F, G\) are continuous differentiable, the system is almost linear near the critical points.

\[
\begin{pmatrix}
F_x & F_y \\
G_x & G_y
\end{pmatrix}
= 
\begin{pmatrix}
1.5 - 2x - 0.5x & -0.5x \\
-0.75y & 2 - 2y - 0.75x
\end{pmatrix}
\]
The linear system near critical point \((0, 0)\) is

\[
\begin{align*}
\frac{du}{dt} &= 1.5u \\
\frac{dv}{dt} &= 2v
\end{align*}
\]

where \(u = x, \ v = y\). The eigenvalues of the linear system are 1.5, 2, hence the critical point \((0, 0)\) is node and unstable.

The linear system near critical point \((1.5, 0)\) is

\[
\begin{align*}
\frac{du}{dt} &= -1.5u - 0.75v \\
\frac{dv}{dt} &= 0.8725v
\end{align*}
\]

where \(u = 1.5 + x, \ v = y\). The eigenvalues of the linear system are \(-1.5, 0.8725\), hence the critical point \((1.5, 0)\) is saddle point and unstable.

f). From

\[
\begin{align*}
F &= x(1 - x - y) = 0 \\
G &= y(1.5 - y - x) = 0
\end{align*}
\]

we know that all the critical point are \((0, 0), (1, 0)\) and \((0, 1.5)\).

Since the \(F, G\) are continuous differentiable, the system is almost linear near the critical points.

\[
\begin{pmatrix}
F_x & F_y \\
G_x & G_y
\end{pmatrix} = \begin{pmatrix}
1 - 2x - y & -x \\
-y & 1.5 - x - 2y
\end{pmatrix}
\]

The linear system near critical point \((0, 0)\) is

\[
\begin{align*}
\frac{du}{dt} &= u \\
\frac{dv}{dt} &= 1.5v
\end{align*}
\]

where \(u = x, \ v = y\). The eigenvalues of the linear system are 1, 1.5, hence the critical point \((0, 0)\) is node and unstable.

The linear system near critical point \((1, 0)\) is

\[
\begin{align*}
\frac{du}{dt} &= -u - v \\
\frac{dv}{dt} &= 0.5v
\end{align*}
\]

where \(u = 1 + x, \ v = y\). The eigenvalues of the linear system are 1, \(-0.5\), hence the critical point \((1, 0)\) is saddle point and unstable.
The linear system near critical point \((0, 1.5)\) is
\[
\begin{align*}
\frac{du}{dt} &= -0.5u \\
\frac{dv}{dt} &= -1.5u - 1.5v
\end{align*}
\]
where \(u = x, \ v = 1.5 + y\). The eigenvalues of the linear system are \(-0.5, -1.5\), hence the critical point \((0,0)\) is node and asymptotically stable.

2. Consider the following two systems of ODEs:
(1). \(\frac{dx}{dt} = y + x(x^2 + y^2), \ \frac{dy}{dt} = -x + y(x^2 + y^2)\)
(2). \(\frac{dx}{dt} = y - x(x^2 + y^2), \ \frac{dy}{dt} = -x - y(x^2 + y^2)\)
(a). Show that \((0,0)\) is a critical point for (1) and (2) and furthermore, is a center of the corresponding linear system.
(b). Show that each system is almost linear at \((0,0)\).
(c). Using polar coordinates, \(x = r \cos \theta, \ y = r \sin \theta\), find out the corresponding ODEs for \((r, \theta)\). Then show that \((0,0)\) is asymptotical stable for (2) while \((0,0)\) is unstable for (1).

Answer:
(1) Let
\[
F(x, y) = y + x(x^2 + y^2) \quad G(x, y) = -x + y(x^2 + y^2)
\]
Obviously, \((0,0)\) is a critical point of the system.
The linear system near \((0,0)\) is
\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
The eigenvalues of this linear system are \(\lambda_1 = i, \ \lambda_2 = -i\). So \((0,0)\) is a center of the linear system.
Let \(x = r \cos \theta, \ y = r \sin \theta\) then
\[
\frac{x(x^2 + y^2)}{r} = r^2 \cos \theta \to 0, \ r \to 0
\]
\[
\frac{y(x^2 + y^2)}{r} = r^2 \sin \theta \to 0, \ r \to 0
\]
Hence, the system is almost linear.
Under above polar coordinate, the original system can be written as
\[ \frac{dr}{dt} = r^3 > 0 \]
\[ \frac{d\theta}{dt} = -1 \]
Hence, \((0,0)\) is an unstable critical point of the original system.

(2) Let
\[ F(x, y) = y - x(x^2 + y^2) \]
\[ G(x, y) = -x - y(x^2 + y^2) \]
Obviously, \((0,0)\) is a critical point of the system.
The linear system near \((0,0)\) is
\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
The eigenvalues of this linear system are \(\lambda_1 = i, \lambda_2 = -i\). So \((0,0)\) is a center of the linear system.
Let \(x = r \cos \theta,\ y = r \sin \theta\) then
\[ \frac{-x(x^2 + y^2)}{r} = -r^2 \cos \theta \to 0,\ r \to 0 \]
\[ \frac{-y(x^2 + y^2)}{r} = -r^2 \sin \theta \to 0,\ r \to 0 \]
Hence, the system is almost linear.
Under above polar coordinate, the original system can be written as
\[ \frac{dr}{dt} = -r^3 < 0 \]
\[ \frac{d\theta}{dt} = -1 \]
Hence, \((0,0)\) is an asymptotically stable critical point of the original system.

3. Consider the following undamped pendulum equation
\[ \frac{d^2 \theta}{dt^2} + \omega^2 \sin \theta = 0 \]
Let \(x = \theta,\ y = \frac{d\theta}{dt}\) to obtain the system of equations
\[ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\omega^2 \sin x. \]
(a). Find out all critical points, and their corresponding linear systems, and types and stability of the corresponding linear systems.

(b). Show that the system is almost linear. What can you say about the type and stabilities of the critical points?

(c). Solving the corresponding equation for \( \frac{dy}{dx} \), show that the trajectories are given by

\[
\omega^2 (1 - \cos x) + \frac{y^2}{2} = c.
\]

Find the trajectory passing through \((0, v)\).

**Answer:** Let

\[
x = \theta, \quad y = \frac{d\theta}{dt}
\]

then

\[
\frac{dx}{dt} = \frac{d\theta}{dt} = y
\]

\[
\frac{dy}{dt} = \frac{d^2\theta}{dt^2} = -\omega^2 \sin \theta = -\omega^2 \sin x
\]

Let

\[
F(x, y) = y
\]

\[
G(x, y) = -\omega^2 \sin x
\]

The critical points of the system are \((2k\pi, 0), ((2k+1)\pi, 0), \ k \ is \ integer.\)

The linear system near \((2k\pi, 0)\) is

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix}
\]

The eigenvalues of this linear system are \(\lambda_1 = i\omega, \ \lambda_2 = -i\omega.\) So \((2k\pi, 0)\) is a stable center of the linear system.

The linear system near \(((2k+1)\pi, 0)\) is

\[
\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix}
\]

The eigenvalues of this linear system are \(\lambda_1 = -\omega, \ \lambda_2 = \omega.\) So \(((2k+1)\pi, 0)\) is an unstable saddle of the linear system.

(b). The original system is almost linear at each critical point. From the stability of the linear system, \(((2k+1)\pi, 0)\) are unstable saddles of the original system. The stability of \((2k\pi, 0)\) is indeterminate.

(c). We get

\[
\frac{dy}{dx} = -\omega^2 \sin x
\]
Hence,
\[ \omega^2(1 - \cos x) + \frac{1}{2} y^2 = c. \]

From the initial conditions \((0, v)\), \(c = \frac{1}{2} v^2\) and the trajectory is
\[ \omega^2(1 - \cos x) + \frac{1}{2} y^2 = \frac{1}{2} v^2. \]

4. Consider the following Lienard equation
\[
\frac{d^2 x}{dt^2} + c(x) \frac{dx}{dt} + g(x) = 0
\]
where \(g(0) = 0\). (Note that damped or undamped pendulum equations are special case of Lienard equation.)

a). Write it as a system of two first order equations by introducing \(y = \frac{dx}{dt}\).

b). Show that \((0,0)\) is a critical point and that the system is almost linear in the neighborhood of \((0,0)\).

c). Show that if \(c(0) > 0\), \(g'(0) > 0\) then the critical point is asymptotically stable, and that if \(c(0) < 0\) or \(g'(0) < 0\), then the critical point is unstable.

**Answer:** a). Let
\[ y = \frac{dx}{dt} \]
then
\[ \frac{dy}{dt} = \frac{d^2 x}{dt^2} \]

From the original equation, we get
\[ \frac{dy}{dt} = -c(x)y - g(x). \]

Let
\[ X = \begin{pmatrix} x \\ y \end{pmatrix} \]

We get
\[ X' = \begin{pmatrix} 0 & 1 \\ 0 & -c(x) \end{pmatrix} X + \begin{pmatrix} 0 \\ -g(x) \end{pmatrix} \]
b). Since \( g(0) = 0 \), then \((0,0)\) is the critical point of the system.

Let 
\[
c(x) = c(0) + \eta_1(x)
\]

where \( \eta_1(x) \to 0, x \to 0 \)

\[
g(x) = g'(0)x + \eta_2(x)
\]

where \( \eta_1(x)/x \to 0, x \to 0 \)

The system can written as
\[
X' = \begin{pmatrix} 0 & 1 \\ -g'(0) & -c(0) \end{pmatrix} X + \begin{pmatrix} 0 \\ -\eta_1(x)y - \eta_2(x) \end{pmatrix}
\]

Take 
\[
G^T = (0, -\eta_1(x)y - \eta_2(x))
\]

Then \( ||G|| \to 0, (x, y) \to (0, 0) \). Hence the system is almost linear in the neighborhood of \((0,0)\).

c).

\[
det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -g'(0) & -c(0) - \lambda \end{vmatrix} = \lambda^2 + c(0)\lambda + g'(0) = 0
\]

Hence
\[
\lambda_1 = \frac{-c(0) + \sqrt{c^2(0) - 4g'(0)}}{2}
\]
\[
\lambda_1 = \frac{-c(0) - \sqrt{c^2(0) - 4g'(0)}}{2}
\]

Case A: \( c(0) > 0, g'(0) > 0 \)

If \( \Delta = c^2(0) - 4g'(0) > 0 \), then \( \lambda_2 < \lambda_1 < 0 \) and \((0,0)\) is an asymptotically stable critical point.

If \( \Delta = c^2(0) - 4g'(0) < 0 \), then \( \lambda_2 = \alpha + \beta i, \lambda_1 = \alpha - \beta i \) where \( \alpha = \frac{-c(0)}{2} < 0 \) and \((0,0)\) is an asymptotically stable critical point.

If \( \Delta = c^2(0) - 4g'(0) = 0 \), then \( \lambda_2 = \lambda_1 = \frac{-c(0)}{2} < 0 \) and \((0,0)\) is an asymptotically stable critical point.

Case B: \( c(0) < 0 \)

If \( \Delta = c^2(0) - 4g'(0) > 0 \), then \( \lambda_1 > \lambda_2 > 0 \) and \((0,0)\) is an unstable critical point.
If $\triangle = c^2(0) - 4g'(0) < 0$, then $\lambda_2 = \alpha + \beta i$, $\lambda_1 = \alpha - \beta i$ where $\alpha = -\frac{c(0)}{2} > 0$ and $(0,0)$ is an unstable critical point.

If $\triangle = c^2(0) - 4g'(0) = 0$, then $\lambda_2 = \lambda_1 = -\frac{c(0)}{2} > 0$ and $(0,0)$ is unstable critical point.

Case C: $g'(0) < 0$
In this case, $\lambda_1 > 0 > \lambda_2$ and $(0,0)$ is an unstable critical point.

5. In each of the following ODEs, construct a suitable Liapunov functions of the form $ax^2 + cy^2$, where $a, c$ are to be determined. Then show that the critical point $(0,0)$ is of the indicated type.

a. $\frac{dx}{dt} = -x^3 + xy^2$, $\frac{dy}{dt} = -2x^2y - y^3$, asymptotically stable.

b. $\frac{dx}{dt} = \frac{1}{2}x^3 + 2xy^2$, $\frac{dy}{dt} = -y^3$, asymptotic stable.

c. $\frac{dx}{dt} = -x^3 + 2y^3$, $\frac{dy}{dt} = -2xy^2$, stable.

d. $\frac{dx}{dt} = x^3 - y^3$, $\frac{dy}{dt} = 2xy^2 + 4x^2y + 2y^3$, unstable.

**Answer:**

a). $(0,0)$ is an isolated critical point of the system. Let $V(x, y) = ax^2 + cy^2$

then

$$\frac{dV}{dt} = -2ax^4 - 2cy^4 + (2a - 4c)x^2y^2$$

Let $a = 2$, $c = 1$, then $V(x, y) = 2x^2 + y^2$ is positive definite and

$$\frac{dV}{dt} = -4x^4 - 2y^4$$

is negative definite.
Hence, $(0,0)$ is an asymptotically stable critical point.

b). $(0,0)$ is an isolated critical point of the system. Let $V(x, y) = ax^2 + cy^2$

then

$$\frac{dV}{dt} = -ax^4 - 2cy^4 - 4ax^2y^2$$
Let $a = 1$, $c = 2$, then $V(x, y) = x^2 + 2y^2$ is positive definite and
\[
\frac{dV}{dt} = -x^4 - 4x^2y^2 - 4y^4 = -(x^2 + 2y^2)^2
\]
is negative definite.
Hence, $(0,0)$ is an asymptotically stable critical point.

c). $(0,0)$ is an isolated critical point of the system. Let
\[
V(x, y) = ax^2 + cy^2
\]
then
\[
\frac{dV}{dt} = -2ax^4 + 4axy^3 - 4cxy^3
\]
Let $a = 1$, $c = 1$, then $V(x, y) = x^2 + y^2$ is positive definite and
\[
\frac{dV}{dt} = -2x^4
\]
is negative semidefinite.
Hence, $(0,0)$ is a stable critical point.

d). $(0,0)$ is an isolated critical point of the system. Let
\[
V(x, y) = ax^2 + cy^2
\]
then
\[
\frac{dV}{dt} = 2ax^4 + 4cy^4 + (-2a + 4c)x^2y^2
\]
Let $a = 2$, $c = 1$, then $V(x, y) = 2x^2 + y^2$ is positive definite and
\[
\frac{dV}{dt} = 4x^4 + 8x^2y^2 + 4y^4 = (x^2 + y^2)^2
\]
is positive definite.
Hence, $(0,0)$ is an unstable critical point.

6. Consider the following system of the equations
\[
\frac{dx}{dt} = y - xf(x, y), \quad \frac{dy}{dt} = -x - yf(x, y)
\]
By construction Liapunov function of the type $c(x^2 + y^2)$, show that of $f(x, y) > 0$ in some neighborhood of $(0, 0)$, the $(0, 0)$ is asymptotically stable, and if $f(x, y) < 0$ in some neighborhood of $(0, 0)$, the $(0, 0)$ is an unstable critical point.

\qed
**Answer:**

(0,0) is an isolated critical point of the system. Let

\[ V(x, y) = cx^2 + cy^2 \]

then

\[ \frac{dV}{dt} = -2c(x^2 + y^2)f(x, y) \]

Let \( c = 1 \), then \( V(x, y) = x^2 + y^2 \) is positive definite and

\[ \frac{dV}{dt} = -2(x^2 + y^2)f(x, y) \]

If \( f(x, y) > 0 \) in some neighborhood of (0,0), then \( \frac{dV}{dt} \) is negative definite. Hence, (0,0) is an asymptotically stable critical point.

If \( f(x, y) < 0 \) in some neighborhood of (0,0), then \( \frac{dV}{dt} \) is positive definite. Hence, (0,0) is an unstable critical point.

\[ \square \]

7. Consider the following 2nd order of ODE

\[ \frac{d^2 x}{dt^2} + g(x) = 0 \]

where \( g(x) \) satisfies: \( g(0) = 0, \ xg(x) > 0 \) for \( x \neq 0, \ |x| < k \). (A typical example is \( g(x) = \sin x, \ k = \frac{\pi}{2}. \))

(a) Write the equation as system of two first order equations by introducing the variable \( y = \frac{dx}{dt} \).

(b) Show that (0,0) is a critical point.

(c) Show that

\[ V(x, y) = \frac{1}{2}y^2 + \int_0^x g(s)ds, \ -k < x < k \]

is positive definite, and use this result to show that (0,0) is stable.

**Answer:**

a). Let

\[ y = \frac{dx}{dt} \]

then

\[ \frac{dy}{dt} = \frac{d^2 x}{dt^2} = -g(x) \]
b). Since \( g(0) = 0 \), \((0,0)\) is a critical point of the system.

c). Let

\[
V(x, y) = \frac{1}{2} y^2 + \int_{0}^{x} g(s) ds, \quad -k < x < k
\]

Then \( V(0, 0) = 0 \).

If \(-k < x < 0\), from \( xg(x) > 0 \), we get \( g(x) < 0 \). So

\[
\int_{0}^{x} g(s) ds = -\int_{0}^{x} g(s) ds > 0
\]

If \( 0 < x < k \), from \( xg(x) > 0 \), we get \( g(x) > 0 \). So

\[
\int_{0}^{x} g(s) ds > 0
\]

Hence, \( V(x, y) \) is positive definite in \((-k, k) \times (-1, 1)\)

\[
d\frac{V}{dt} \equiv 0
\]

i.e. \( \frac{dV}{dt} \) is negative semidefinite.

From above argument, \((0,0)\) is a stable critical point of the system.

8. Do the same as in Problem 7 by consider the following Lienard equation:

\[
\frac{d^2x}{dt^2} + c(x) \frac{dx}{dt} + g(x) = 0
\]

where \( c(0) = 0, \ c'(0) \geq 0 \) and \( g(x) \) satisfies: \( g(0) = 0 \), \( xg(x) > 0 \) for \( x \neq 0 \), \( |x| < k \).

**Answer:**

a). Let

\[
y = \frac{dx}{dt}
\]

then

\[
\frac{dy}{dt} = \frac{d^2x}{dt^2} = -c(x)y - g(x)
\]

b). Since \( c(0) = 0 \), \( g(0) = 0 \), \((0,0)\) is a critical point of the system.

c). Let

\[
V(x, y) = \frac{1}{2} y^2 + \int_{0}^{x} g(s) ds, \quad -k < x < k
\]

Then \( V(0, 0) = 0 \).

If \(-k < x < 0\), from \( xg(x) > 0 \), we get \( g(x) < 0 \).
So
\[ \int_0^x g(s)ds = -\int_x^0 g(s)ds > 0 \]
If \( 0 < x < k \), from \( xg(x) > 0 \), we get \( g(x) > 0 \).
So
\[ \int_0^x g(s)ds > 0 \]
Hence, \( V(x, y) \) is positive definite in \((-k, k) \times (-1, 1)\)
\[ \frac{dV}{dt} = -c(x)y^2 \]
i.e. \( \frac{dV}{dt} \) is negative semidefinite.
From above argument, \((0,0)\) is a stable critical point of the system.

9. Consider the following undamped pendulum equation:
\[ \frac{d^2x}{dt^2} + \frac{dx}{dt} + \sin x = 0 \]
Show that \( V(x, y) = \frac{1}{2}(x + y)^2 + x^2 + \frac{1}{2}y^2 \) is a Liapunov function and conclude that \((0,0)\) is asymptotically stable.

**Answer:**
Let
\[ y = \frac{dx}{dt} \]
then
\[ \frac{dy}{dt} = \frac{d^2x}{dt^2} = -y - g(x) \]
Obviously, \((0,0)\) is a critical point of the system.
Let
\[ V(x, y) = \frac{1}{2}(x + y)^2 + x^2 + \frac{1}{2}y^2 \]
It is easy to check \( V(x, y) \) is positive definite and
\[ \frac{dV}{dt} = -y^2 + 2xy - 2y\sin x - x\sin x \]
Since \( \sin x = x - \frac{\alpha}{6}x^3 \), where \( \alpha \) depends on \( x \) and \( 0 < \alpha < 1 \),
\[ \frac{dV}{dt} = -y^2 - x^2 + \frac{\alpha}{3}x^3y + \frac{\alpha}{6}x^4 \]
Set $x = r \cos \theta, y = r \sin \theta$

$$\frac{dV}{dt} = -r^2[1 - r^2(\frac{\alpha}{3} \cos^3 \theta \sin \theta + \frac{\alpha}{6} \cos^4 \theta)] < -r^2(1 - r^2)$$

Hence, if $r = (x^2 + y^2)^{\frac{1}{2}} < 1$, then $\frac{dV}{dt}$ is negative definite.

From above argument, $(0,0)$ is an asymptotically stable critical point of the system.

10. Consider the following Lienard equation:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + g(x) = 0$$

where $g(x)$ satisfies: $g(0) = 0, xg(x) > 0$ for $x \neq 0, |x| < k$.

By choosing $A$ appropriately, show that

$$V(x, y) = \frac{1}{2}y^2 + Ayg(x) + \int_0^x g(s)ds$$

is a Liapunov function and conclude that $(0,0)$ is asymptotically stable.

**Answer:** We first note that, since $g(0) = 0, xg(x) > 0$ for $x \neq 0, |x| < k$,

$$\int_0^x g(s)ds > 0, \quad \text{if } x > 0$$

We also suppose that $g(x)$ is Lipschitz continuous, then,

$$|g'(x)| < M, \quad \text{if } |x| < r < \frac{k}{2}$$

Without lose generalization, we assume $M > 1$.

Let $y = x'$, then we get

$$x' = y$$
$$y' = -g(x) - y$$

Let $V(x, y) = \frac{1}{2}y^2 + Ayg(x) + \int_0^x g(s)ds$, here $A$ is a undetermined number. Then

$$V(x, y) = \frac{1}{2}y^2 + Ayg(x) + \int_0^x g(s)ds$$

$$\geq \frac{1}{4}y^2 - A^2g^2 + \int_0^x g(s)ds$$

$$= \frac{1}{4}y^2 + \int_0^x g(s)(1 - 2A^2g'(s))ds$$
Hence in a small ball $B(0, r)$, if we choose $|A|$ small enough say $|A| < \delta_1 = \frac{1}{2\sqrt{M}}$ then,
\[
1 - 2A^2 g'(x) > \frac{1}{2}
\]
Hence, in ball $B(0, r)$,
\[
V(x, y) > \frac{1}{4} y^2 + \frac{1}{2} \int_0^x g(s) ds > 0, \quad \text{if } (x, y) \neq (0, 0).
\]
On the other hand,
\[
W(x, y) = \dot{V}(x, y) = Ag'(x)y^2 - y^2 - Ag^2(x) - Ag(x)y \\
\leq Ag'(x)y^2 - y^2 - Ag^2(x) + \left(\frac{1}{2}Ag^2(x) + \frac{1}{2}Ay^2\right) \\
\leq -(1 - \frac{1}{2}A - Ag'(x))y^2 - \frac{1}{2}Ag^2(x)
\]
In the ball $B(0, r)$, if we choose $|A|$ small enough, say $|A| < \delta_2 = \frac{1}{2M+1}$, then
\[
W(x, y) = \dot{V}(x, y) \leq -\frac{1}{2} y^2 - \frac{1}{2}\delta_2 g^2(x) < 0, \quad \text{if } (x, y) \neq (0, 0)
\]
Then, if we choose $A$ such that $|A| < \min(\delta_1, \delta_2)$ then, in the ball $B(0, r)$, we have
\[
V(x, y) > 0, \quad \dot{V}(x, y) < 0
\]
If $(x, y) \neq (0, 0)$. Hence the critical point $(0, 0)$ is asymptotically stable.