Assignment 5 of MAT2060B

1. Do all the exercises in Section 7.3 and 7.4

2. Prove the following integral mean-value theorem: Suppose that $f$ is continuous in $[a, b]$. Then exists a $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

3. Suppose that $f \in R[a,b]$ and $f$ is convex. Show that

$$(b-a)f\left(\frac{b+a}{2}\right) \leq \int_a^b f(x)dx$$

4. Let

$$f^+(x) = \max(f(x), 0), \quad f^-(x) = \min(f(x), 0)$$

Show that $f \in R[a,b]$ if and only if $f^+, f^- \in R[a,b]$. 

5. Suppose that $\max(f(x), g(x)) \in R[a,b], \min(f(x), g(x)) \in R[a,b]$. Show that $f(x)g(x) \in R[a,b]$. 

6. Show that if $f \in R[0,1]$, then

$$e^{\int_a^b f(x)dx} \leq \int_0^1 e^{f(x)}dx$$

7. Suppose $f \in R[0,1], f \geq \alpha > 0$. Show that

$$\int_0^1 \frac{1}{f(x)}dx \int_0^1 f(x)dx \geq 1.$$ 

8. Let

$$f(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \text{ for } x \neq 0; = 0, \text{ for } x = 0$$

Is $f \in R[-1,1]$?

9. Construct an example such that $f \in R[a,b]$ and $f$ is discontinuous at $x_0$ but 

$$(\int_a^b f(t)dt)'_{x=x_0} = f(x_0).$$

10. Suppose that $f' \in R[a,b], f(a) = 0$. Show that

$$f(x) \leq \int_a^x f'(t)dt$$

11. Use definite integrals to compute the following limits

$$\lim_{n \to +\infty} \frac{1}{n} \left( \sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \ldots + \sin \frac{(n-1)\pi}{n} \right)$$

$$\lim_{n \to +\infty} \frac{\Gamma \left( \frac{1}{n} \right)}{n}$$

$$\lim_{n \to +\infty} \sin \frac{\pi}{n} \sum_{k=1}^{n} \frac{1}{2 + \cos \frac{2\pi}{n}}$$

12. Compute the following integrals:

$$\int_0^1 x\sqrt{1-x}dx, \quad \int_0^1 \ln(1+\sqrt{x})dx$$

$$\int_0^a \sqrt{a^2-x^2}dx, \quad \int_0^a x^2 \sqrt{a^2-x^2}dx$$

13. Show that

$$\int_0^{2\pi} \frac{1-r^2}{1-2r\cos \theta + r^2}dr = 2\pi, \quad 0 < r < 1$$
14. Suppose \( f \) is monotone increasing in \([0, +\infty)\). Show that 
\[
F(x) = \frac{1}{x} \int_0^x f(t) \, dt
\]
is also monotone increasing.
15. Suppose \( f \) is convex in \([0, +\infty)\). Show that 
\[
F(x) = \frac{1}{x} \int_0^x f(t) \, dt
\]
is also convex.
16. Using integration by parts to compute the following integrals:
\[
(1) \int_0^1 x^n (\ln x)^n \, dx, \quad (2) \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \\
(3) \int_0^1 x^2 e^{\sqrt{x}} \, dx, \quad (4) \int_0^1 \frac{dx}{(2-x^2)^2} \, dx
\]
17. Let 
\[
a_n = \int_0^1 x^n \sqrt{1-x^2} \, dx
\]
Show that \( a_n = \frac{n+1}{n+2} a_{n-2} \) and \( a_n \leq a_{n-1} \leq a_{n-2} \)
18. Let \( f' \in C[0,1] \). Show that 
\[
\int_0^1 x^n f(x) \, dx = \frac{f(1)}{n} + o\left(\frac{1}{n}\right)
\]
19. Let \( f'' \in C[0,1] \). Show that 
\[
\int_0^1 x^n f'(x) \, dx = \frac{f(1)}{n} - \frac{f(1) + f'(1)}{n^2} + o\left(\frac{1}{n^2}\right)
\]
20. Let \( f'' \in C[a,b] \), \( f(a) = f(b) = 0 \). Show that 
\[
\int_a^b f'(x) \, dx = \frac{1}{2} \int_a^b f''(x-a)(x-b) \, dx \\
As a consequence, show that 
\[
|\int_a^b f(x) \, dx| \leq \frac{(b-a)^3}{12} \max_{a \leq x \leq b} |f''(x)|
\]
21. Use Simpson’s formula to compute 
\[
\int_0^\pi \sqrt{3+\cos x} \, dx, \quad n = 6; \quad \int_0^\pi \frac{\sin x}{x} \, dx, \quad n = 6
\]