Home Work and Quiz II, Friday, April 15

Problem 1

(1) Show that \( |x|^2 \) is convex where \( X \) is a normed space.

(2) \( \frac{1}{2} |x|^2 = F(x) \), where \( F \) is the duality map.

(3) Let \( X = W_0^{1,p}(\Omega) \) with norm \( |u|_X = (\int_\Omega |\nabla u|^p \, dx)^{1/p} \). Find \( F(x) \)

(4) Consider the minimization

\[
\min_{u} \int_{\Omega} (a(x) |\nabla u(x)|^p \, dx - f(x)u(x)) \, dx
\]

over \( u \in W_0^{1,p}(\Omega) \). Derive the necessary optimality.

(5) Consider

\[
\frac{\partial u}{\partial t} \in \partial \varphi(u)
\]

where \( \varphi \) is a convex and coercive functional on \( H_0^1(\Omega) \). Apply the Crandall-Liggett theory on \( X = L^2(\Omega) \).

Problem 2

Apply the Grandall-Ligget theory with \( x + L^2(\Omega) \) for the heat equation

\[
\frac{\partial u}{\partial t} = \Delta u, \quad -\frac{\partial u}{\partial n} \in \beta(u) \text{ at } \partial \Omega
\]

where \( b : R \to R \) is maximum monotone.