1. Let \( X = [x_1 \ x_2 \ x_3 \ x_4]^T \in \mathbb{R}^4 \). Consider the following problem

\[
(1) : \begin{array}{l}
\text{Maximize} \\
\text{subject to}
\end{array} f(X) = -3x_1^2 - x_2^2 - 9x_3^2 - 6x_4^2 \\
x_1 + 3x_2 + x_3 + 3x_4 = 0 \\
3x_2 + 4x_3 + 2x_4 = 2 \\
x_1 + 6x_2 + 4x_3 + 3x_4 = 0
\]

Answer the following questions:

(a) Solve the problem (1) using “the method of Lagrange multipliers” and “the Jacobian method”.

(b) Is \( X^* = [-1/4 \ -1/2 \ 1/2 \ 3/4]^T \) an optimal point? Explain.

2. Solve the following problem by the method of Lagrange multipliers, and determine the character of the stationary point, if any:

(a) \[ \text{Maximize} \quad x_1x_2 \text{ subject to} \]
\[ x_1^2 + 2x_2^2 \leq 3 \]
\[ 2x_1^2 + x_2^2 \leq 3 \]

(b) \[ \text{Maximize} \quad 2x + y - \frac{1}{3}x^3 - xy - y^2 \text{ subject to} \]
\[ x \geq 1/4 \]
\[ x + y \leq 3 \]

3. Let \( X = \left[ \begin{array}{c} x \\ y \end{array} \right] \in \mathbb{R}^2 \). Solve the problem:

\[ \text{Minimize} \quad 4(x - 1)^2 + (y - 2)^2 \text{ subject to} \]
\[ x + y \leq 2 \]
\[ x \geq -1 \]
\[ y \geq -1 \]
4. (Optional) Solve the following problem by the generalized reduced gradient method. Start at the point \( x_0 = (2, 1, 3, 1) \) and have \( x_1 \) and \( x_4 \) be the basic or dependent variables and \( x_2 \) and \( x_3 \) the nonbasic or independent variables.

Minimize \( x_1^2 + 4x_2^2 \)

subject to

\[
\begin{align*}
  x_1 + 2x_2 - x_3 &= 1 \\
  -x_1 + x_2 + x_4 &= 0
\end{align*}
\]

5. (Optional) Use

(a) the method of Lagrange multipliers

(b) a penalty function

to solve the following problem:

Minimize \(-x_1^2 + x_2^2\)

subject to

\[
\begin{align*}
  x_1^2 + x_2^2 &= 4
\end{align*}
\]