1. Determine the definiteness of the following constrained quadratics:

(a) \( Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2 \)
   subject to
   \( x_1 + x_2 = 0. \)

(b) \( Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_3 - 2x_1x_2 \)
   subject to
   \( x_1 + x_2 + x_3 = 0. \)

(c) (Optional) \( Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4, \)
   subject to
   \( x_1 + x_2 - x_3 + x_4 = 0; \)
   \( x_1 - 9x_2 + x_4 = 0. \)

2. Use the method of Lagrange multipliers to maximize the following problem:

Maximize \( x^{1/5}y^{2/5}z^{1/5} \)
subject to \( px + qy + rz = m \)

where all the constants \( p, q, r \) and \( m \) are positive. Verify that

\( px^* = \frac{1}{4}m, \ qy^* = \frac{1}{2}m, \ rz^* = \frac{1}{4}m \)

or not.

3. Find the distance from the origin to the plane

\( x + 2y + 2z = 10. \)

(a) using a geometric argument (no calculus).
(b) by reducing the problem to an unconstrained problem in two variables, and
(c) using the method of Lagrange multipliers.
4. Answer the following questions:

(a) Solve the problem:

\[
\text{Maximize} \quad x + 4y + z \\
\text{subject to} \\
x + 2y + 3z = 0 \\
x^2 + y^2 + z^2 = 216.
\]

(b) Change the first constraint to \( x + 2y + 3z = 0.1 \) and the second to \( x^2 + y^2 + z^2 = 215 \). What is the approximate change in optimal value of the objective function?

5. Consider the following problem:

\[
\text{Minimize} \quad x_1^2 + x_2^2 + \cdots + x_n^2 \\
\text{subject to} \\
x_1 + x_2 + \cdots + x_n = 1.
\]

Show whether or not the candidate point, obtained via the Lagrangian method, is indeed a constrained minimum.

6. Use the second-order conditions for a local minimum to classify the candidates for optimality in the following problem:

\[
\text{Minimize} \quad x_1^2 + x_2^2 + x_3^2 \\
\text{subject to} \\
x_1 + x_2 + x_3 = 0 \\
x_1 + 2x_2 + 3x_3 = 1.
\]

Hint: It may be useful to use the results of Question 8.

7. (Bonus) Find and classify the three critical points for the Lagrangian function

\[
L(x, y, u, v, \lambda, \nu) = S + \lambda(y - x^2) + \nu(v - 2u^2 - 1)
\]

corresponding to the problem:

\[
\text{Extremize} \quad S = (x - u)^2 + (y - v)^2 \\
\text{subject to} \\
y = x^2 \\
v = 2u^2 + 1.
\]

Which is the minimum distance between the curves \( y = x^2 \) and \( y = 2x^2 + 1 \)? Justify your answers.
8. (Optional) Answer the following questions:

(a) Consider the following model:

Minimize \[ J = \frac{1}{2} (z^T P z + u^T Q u) \]
subject to \[ A z + B u + C = 0 \]

Here, the dimensions of \( z, u, P, Q, A, B \) and \( C \) are \( p \times 1, m \times 1, p \times p, m \times m, p \times p, p \times m \) and \( p \times 1 \), respectively. Also \( P \) and \( Q \) are positive definite and symmetric, and \( P, Q, A, B, C \) are constants. The variable \( z \) represents the dependent variables, and the variable \( u \) represents the independent variables assuming that \( A \) is invertible. Show that the optimal point is defined by

\[ u = -Q^{-1} B^T (AP^{-1} A^T + BQ^{-1} B^T)^{-1} C \]
\[ z = -P^{-1} A^T (AP^{-1} A^T + BQ^{-1} B^T)^{-1} C \]
\[ \lambda = (AP^{-1} A^T + BQ^{-1} B^T)^{-1} C \]

and that the condition for a minimum is given by

\[ B^T A^{-T} P A^{-1} B + Q \geq 0. \]

(Hint: the method of Lagrange multipliers)

(b) Apply the results of Question 8 to the following problem:

Minimize \[ J = \frac{1}{2} (x^2 + y^2 + z^2) \]
subject to \[ x + 2y - z = 3 \]
\[ x - y + 2z = 12 \]

Show that \( x = 5, \ y = 1, \ z = 4, \ \lambda_1 = -2, \ \lambda_2 = -3. \)

9. (Optional) Find the value of the constant \( a \) for which the function \( f(x) = ax^2 \) best approximates the function \( g(x) = x^3 \) on the interval \([0,1]\), in the sense that the integral:

\[ I = \int_0^1 (f(x) - g(x))^2 \, dx \]

is minimized. What is the minimum value of \( I \)?