6.4.1
(a) $T$
(b) $F$
(c) $F$
(d) $T$
(e) $T$
(f) $F$
(g) $T$
(h) $T$

Hint of (b): If $V$ is infinite-dimensional, we not sure.

6.4.2 (f)
\[ A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ is a basis for } \mathbb{M}_2(\mathbb{F}) \]

\[ T_A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = [T]_A = [T^*]_A \]

- $T$ is self-adjoint and thus normal.
- $\det (T - \lambda I) = (\lambda - 1)^4 (\lambda - 1)^2$

\[ E_1 = \text{span} \{ v_1 = [1 \ 0 \ 0 \ 0], v_2 = [0 \ 1 \ 0 \ 0], v_3 = [0 \ 0 \ 1 \ 0], v_4 = [0 \ 0 \ 0 \ 1] \} \]

Let $\langle A, B \rangle = \text{tr} (B^T A)$ be an inner product on $\mathbb{M}_2(\mathbb{F})$.

Then:
\[ \langle v_i, v_j \rangle = 2 \delta_{ij}, \quad i, j = 1, 2, 3, 4 \]

Hence:
\[ \{ v_1, v_2, v_3, v_4 \} \text{ is an orthonormal basis of eigenvectors of } T \text{ for } V \text{ and the corresponding eigenvalues are } 1, 1, -1, -1. \]

If we define the inner product as
\[ \langle A, B \rangle = \sum_{i,j=1}^4 a_{i,j} b_{i,j} \]

then:
\[ \langle v_i, v_j \rangle = 2 \delta_{ij}, \quad i, j = 1, 2, 3, 4. \]
\( T = T^* \) and \( W \) is \( T \)-invariant, then for all \( \forall X, Y \in W \),
\[
\langle TX, Y \rangle = \langle X, TY \rangle = \langle X, T^*Y \rangle = \langle X, Y \rangle
\]

\( T \) is self-adjoint.

(a) \( \forall X \in W, \forall Y \in W^\perp \), we have
\[
\langle X, T^*Y \rangle = \langle TX, Y \rangle \quad \text{for } w = 0
\]

(b) \( T^*Y \in W^\perp \)

(c) \( W \) is \( T \)-and \( T^* \)-invariant, for all \( \forall X, Y \in W \), we have
\[
\langle (T^*X)Y, Y \rangle = \langle X, TY \rangle = \langle T^*X, Y \rangle = \langle X, T^*Y \rangle
\]

\( (T^*)^2 = T \).

(d) \( \forall X, Y \in W \)
\[
\langle (T^*)^2X, Y \rangle = \langle TX, T^*Y \rangle = \langle TX, TY \rangle = \langle T^*X, Y \rangle = \langle X, T^*Y \rangle
\]

\( (T^*)^2 = (T^*)W \), i.e. \( T \) is normal.

6.9.9

\( T = T^* \) \( T \) is normal) and \( V \) is finite-dimensional.

\( \forall X \in \mathcal{N}(T) \Leftrightarrow TX = 0 \Leftrightarrow \|TX\| = 0 \Leftrightarrow \|T^*X\| = 0 \) by Thm 6.15 (a).

\( \|T^*X\| = 0 \Leftrightarrow T^*X = 0 \Leftrightarrow TX \in \mathcal{N}(T) \)

\( \mathcal{N}(T) = \mathcal{N}(T^*) \)

By 9.19 (b) of 6.3, we have \( R(T^*) = (\mathcal{N}(T))^\perp \)

\( (N(T))^\perp = R(T^*) = R(T) \)

\( R(T^*) = R(T) \)
1.4.12

\[ T \text{ is normal and the characteristic polynomial splits.} \]

\[ \text{by Schur's thm. } \exists \text{ an orthonormal basis } \beta = \{v_1, v_2, \ldots, v_n\} \text{ for } V \text{ } \]

\[ \text{such that } [T]_\beta = H \text{ is upper triangular.} \]

We use induction on \( k \), \( 1 \leq k \leq n \), being the index of vectors in \( \beta \).

For \( k = 1 \), \( T \, v_1 = A_{11} \, v_1 = \gamma \, v_1 \) is an eigenvector.

Assume that \( v_1, v_2, \ldots, v_{k-1} \) are eigenvectors. Now consider \( v_k \).

Let \( \lambda \) denote the corresponding eigenvalue of \( T \) corresponding to \( v_k \). (\( 1 \leq k \leq n \)).

\[ \text{by def 6.15 } T \, v_k = \lambda_1 \, v_k. \]

Since \( T \, v_k \in V \), \( T \, v_k = A_{kk} \, v_k + A_{kk} \, v_k + \cdots + A_{kk} \, v_k \)

where \( A_{kk} = A_{kk} \, v_k \), \( v_k \neq 0 \).

\[ v_k \neq 0 \]

\[ \lambda \neq \lambda_1 \]

\[ \lambda \neq \lambda_2 \]

\[ \vdots \]

\[ \lambda \neq \lambda_k \]

\[ T \, v_k = A_{kk} \, v_k \]. i.e. \( v_k \) is an eigenvector.

Hence \( \beta \) is an orthonormal basis of eigenvectors of \( T \).

Hence \( T^* \beta = \beta \) is diagonal and \( T^* T = T T^* \).

Thus \( T \) is diagonal and all the entries are real.

\[ T^* T = T T^* \]

\[ \beta = T^* \beta \text{ i.e. } T \text{ is self-adjoint.} \]

4.18

(a) \[ \lambda T X, X > = < TX, TX > = 0 \]

\[ < T^* Y, Y > = < Y^* T, Y > \geq 0 \]

\[ T^* \text{ and } T \text{ are positive semidefinite.} \]

(b) \[ \text{For } X \in N(T), X = 0 \Rightarrow T^* T X = 0 \Rightarrow X \in N(T^* T) \]

\[ \lambda T X, X > = 0 \Rightarrow |1| \lambda^2 = 0 \Rightarrow T X = 0 \Rightarrow \lambda \in N(T) \]

\[ \Rightarrow N(T^* T) = N(T) \]

As \( \text{ dim } V = \text{ rank } T + \text{ null } (T) = \text{ rank } (T^* T) + \text{ null } (T^* T) \) is finite.

\[ \Rightarrow \text{ rank } T = \text{ rank } (T^* T) \]

Similarly \( \text{ rank } T^* = \text{ rank } (T T^*) \Rightarrow \text{ rank } (T T^*) = \text{ rank } (T^* T) = \text{ rank } (T^* T) = \text{ rank } (T T^*) \).

By 6.15 (c), 6.3, we have \( \text{ rank } (T^* T) = \text{ rank } T \).