Ex7.2.1
(a) T. (b) T. (c) F. (d) T. (e) F. (f) F. (g) F. (h) T.

Ex7.2.3
(a) \((t - 2)^5(t - 3)^2\)

(b) The Jordan canonical form implies that the dog diagram corresponding to 
\[\lambda = 2\] is \[
\begin{bmatrix}
\bullet \\
\bullet
\end{bmatrix}
\] and dot diagram corresponding to \(\lambda = 3\) is \[
\begin{bmatrix}
\bullet \\
\bullet
\end{bmatrix}
\]

(c) \(E_3 = K_3\)

(d) The dot diagrams implies that \(p = 3\) for \(\lambda_1 = 2\) and \(p = 1\) for \(\lambda_2 = 3\).

(e) Let \(\beta\) be the basis such that \(T\) has the Jordan canonical form, then 
\[
[U_1]_\beta = 
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] and 
\[
[U_2]_\beta = 
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(i) \(\text{rank}(U_1) = 3, \text{rank}(U_2) = 0\);

(ii) \(\text{rank}(U_1^2) = 3, \text{rank}(U_2^2) = 0\);

(iii) \(\text{nullity}(U_1) = 2, \text{nullity}(U_2) = 2\);

(iv) \(\text{nullity}(U_1^2) = 2, \text{nullity}(U_2^2) = 2\).

Ex7.2.4(d)
Since \(\det(A - tI)) = t^2(t - 2)^2\), we get \(\lambda_1 = 0, \lambda_2 = 2\) and \(\dim K_{\lambda_1} = 2 = \dim K_{\lambda_2}\).

Now
\[
A - 0I = 
\begin{bmatrix}
0 & -3 & 1 & 2 \\
-2 & 1 & -1 & 2 \\
-2 & 1 & -1 & 2 \\
-2 & -3 & 1 & 4
\end{bmatrix}, \quad (A - 0I)^2 = 
\begin{bmatrix}
0 & -8 & 4 & 4 \\
-4 & 0 & 0 & 4 \\
-4 & 0 & 0 & 4 \\
-4 & -8 & 4 & 8
\end{bmatrix}.
\]

It is easily seen that \(\{(1, 1, 1)^T, (0, -1, -2, 0)^T\}\) is a basis for \(N((A - 0I)^2) = K_0\) and \(\{(1, 1, 1)^T\}\) is a basis for \(N((A - 0I)) = E_0\). Choose \(v_1 = (0, -1, -2, 0)^T\), then \((A - 0I)v_1 = (1, 1, 1)^T\).
We can check that \( \{(1, 0, 0, 1)^T, (0, 1, 1, 1)^T\} \) is a basis for \( E_2 = K_2 \).

Let \( Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \ J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \), then \( J = Q^{-1}AQ \).

**Ex7.2.7**

(a) Suppose \( \gamma = \{(T - \lambda I)^{-1}v, (T - \lambda I)^{-2}v, \cdots , v \} \) and \( W = \text{span}(\gamma) \), then \( W \) is \( \gamma \)-invariant and

\[
[T_w]_{\gamma} = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}
\]

Now \( \gamma' = \{v, (T - \lambda I)v, \cdots , (T - \lambda I)^{k-1}v\} \), so

\[
Tv = (T - \lambda I)v = \lambda v
\]

\[
\vdots
\]

\[
T(T - \lambda I)^{p-1}v = (T - \lambda I)^{p}v + \lambda(T - \lambda I)^{p-1}v, 1 < p \leq k
\]

\[
\vdots
\]

\[
T(T - \lambda I)^{k-1}v = \lambda(T - \lambda I)^{k-1}v
\]

Hence

\[
[T_w]_{\gamma'} = \begin{bmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & \lambda \end{bmatrix} = [T_w]_{\gamma}^T
\]

(b) Let \( \beta = \{\gamma_1, \gamma_2, \cdots , \gamma_m\} \) be a basis for \( V \) such that \( [T]_\beta = J \) where \( \gamma_i \) be a cycle of generalized eigenvectors corresponding to an eigenvalue \( \lambda_i \) for \( i = 1, 2, \cdots , m \). Then \( \beta' = \{\gamma'_1, \gamma'_2, \cdots , \gamma'_m\} \) is another basis for \( V \) and \( [T]_{\beta} = J^T \) where \( \gamma'_i \) is the ordered set obtained from \( \gamma \) by inversing the order of the vectors in \( \gamma \). Let \( Q \) be the change of coordinate matrix from \( \beta \) to \( \beta' \), then \( J^T = Q^{-1}JQ \).

(c) Since \( A \) is similar to \( J \), \( A^T \) is similar to \( J^T \) and \( J \) is similar to \( J^T \), \( A \) is similar to \( A^T \).
Ex7.2.12 Let $T = L_A$, then the only eigenvalue of $T$ is zero and $K_0 = V$. Hence for every $v \in V$, we have $T^n(v) = 0$, i.e. $T^N = T_0$. We get that $A^n = 0$, i.e. $A$ is nilpotent.

Ex7.2.13

(a) $\forall v \in N(T^i)$, we have $T^{i+1}(v) = T(T^i(v)) = 0$, hence $N(T^i) \subseteq N(T^{i+1})$

(b) Notice that $p$ is the smallest positive integer for which $T^p = T_0$, then

$$N(T^p) = N(T^2) \subseteq \cdots \subseteq N(T^p) = V.$$  

Now if $\beta_i$ is a basis for $N(T^i)$, then we can extent it to be a basis for $N(T^{i+1})$ for $1 \leq i \leq p - 1$.

(c) For all $e \in \beta_{i+1}$, i.e. $T^{i+1} e = T^i(T e) = 0$, hence $T(e) \in N(T^i)$ for $1 \leq i \leq p - 1$. So $[T]_{\beta}$ is an upper triangular matrix.

(d) It is implied by (c).

Ex7.2.17

(a) It is easy to check that $S$ is a linear operator on $V$. Now suppose $\gamma_i := \{v_{i1}, v_{i2}, \cdots, v_{im}\}$ be a basis for $K_{\lambda_i}$, $1 \leq i \leq k$, then $\gamma := \gamma_1 \cup \cdots \cup \gamma_k$ is a basis for $V$. $\forall v_{ij} \in \gamma_i$, $1 \leq j \leq n_i$, $S(v_{ij}) = \lambda_i v_{ij}$, $1 \leq i \leq k$. Hence

$$[S]_\gamma = diag(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2, \cdots, \lambda_k, \cdots, \lambda_k),$$

i.e. $S$ is diagonalizable.

(b) Now we choose $\gamma_i$ be the union of disjoint cycles in $K_{\lambda_i}$, then $[T]_\gamma$ is a Jordan matrix and $[S]_\gamma$ is diagonal matrix. So $[U]_\gamma$ is a upper triangle matrix, i.e $U$ is nilpotent by exercise(12). It is easy to check that $[T]_\gamma [S]_\gamma = [S]_\gamma [T]_\gamma$, hence $[U]_\gamma [S]_\gamma = [S]_\gamma [U]_\gamma$, i.e $SU = US$.

Ex7.2.20

(a) $A$ is a transition matrix, so as $A^m$. Hence $\|A^m\| \leq 1$.

(b) $\|J^m\| = \|(P^{-1}AP)^m\| = \|P^{-1}A^mP\| \leq n^2\|P^{-1}\|\|P\|$. Take $c = n^2\|P^{-1}\|\|P\|$, then result follows.

(c) Since $\lim_{m \to \infty} J^m = diag(\lim_{m \to \infty} J_1^m, \lim_{m \to \infty} J_2^m, \cdots, \lim_{m \to \infty} J_k^m)$ where $J_i$ is Jordan block, $1 \leq i \leq k$. By exercise (19.b), the results follows.

(c) Follows by exercise (19.b).

(e) From (c),each Jordan block corresponding to $\lambda = 1$ is $1 \times 1$ matrix, which implies $E_1 = K_1$. On the other hand, the multiplicity of 1 as an eigenvalue of $A$ is $dim(K_1) = dim(E_1) = 1$ by lemma 5.19.