(a) F, (b) T, (c) F, (d) F, (e) T, (f) T.

Hint of (d): Let $T(x, y) = (x, 0)$ be an operator on $\mathbb{R}^2$ and $v = (1, 1)$.

Since $W$ is $T$-invariant, it is $T^k$-invariant too for $\forall k \in \mathbb{N}$. Now $g \in \mathbb{P}$, we suppose $g(t) = \sum_{k=0}^{n} a_k t^k$, where $n = \text{deg} g$. For all $w \in W$, we have

$$g(T)w = a_0 w + a_1 (Tw) + \cdots + a_n (T^n w).$$

Since $T^k w \in W$ for $1 \leq k \leq n$ and $W$ is a subspace of $V$, $g(T)w \in W$, i.e. $W$ is $g(T)$-invariant.

Now we know $W = \text{span}\{v, Tv, T^2v, \cdots, T^m v\}$, $v \neq 0$, $m + 1 = \text{dim}W$.

$\Rightarrow$ Suppose $w \in W$, we have $a_i \in F$, s.t.

$$w = \sum_{i=0}^{m} a_i T^i (v) = (\sum_{i=0}^{m} a_i T^i)(v) = g(T)(v),$$

where $T^0 = \text{Id}$, $g = \sum_{i=0}^{m} a_i t^i \in \mathbb{P}$.

$\Leftarrow$ By exercise 5.4.4, it is trivial.

Since $W$ is $T$-invariant, $T^k(v) \in W = \text{span}\{v, Tv, \cdots, T^m v\}$, i.e. $\exists a_{ki} \in \mathbb{F}$, s.t.

$$T^k = \sum_{i=0}^{m} a_{ki} T^i v, \quad \forall k \in \mathbb{N}, \text{ where } m + 1 = \text{dim}W.$$ Now suppose

$$g(t) = a_0 + a_1 t + \cdots + a_n t^n.$$

Hence $g(t)$ of Ex.13 can always be chosen s.t. $\text{deg}(g) \leq \text{dim}W$.

By Carley-Hamilton Thm, the exists $a_0^n, a_1^n, \cdots, a_{n-1}^n$, s.t.

$$A^n = a_0^n I + a_1^n A + \cdots + a_{n-1}^n A^{n-1}.$$
Hence

\[ A^{n+1} = A \cdot A^n \]
\[ = a_0^n A + a_1^n A^2 + \cdots + a_{n-1}^n A^n \]
\[ = a_0^n A + a_1^n A^2 + \cdots + a_{n-2}^n A^{n-1} + (a_0^n I + a_1^n A + \cdots + a_{n-1}^n A^{n-1}) \]
\[ := a_0^{n+1} I + a_1^{n+1} A + \cdots + a_{n-1}^{n+1} A^{n-1}. \]

Now suppose

\[ A^{n+k} = a_0^{n+k} I + a_1^{n+k} A + \cdots + a_{n-1}^{n+k} A^{n-1}, \]

then

\[ A^{n+k+1} = A \cdot A^{n+k} \]
\[ = a_0^{n+k} A + a_1^{n+k} A^2 + \cdots + a_{n-1}^{n+k} A^n \]
\[ = a_0^{n+k} A + a_1^{n+k} A^2 + \cdots + a_{n-2}^{n+k} A^{n-1} + (a_0^{n+k} I + a_1^{n+k} A + \cdots + a_{n-1}^{n+k} A^{n-1}) \]
\[ := a_0^{n+k+1} I + a_1^{n+k+1} A + \cdots + a_{n-1}^{n+k+1} A^{n-1}. \]

Hence

\[ \dim(\text{span}(\{I_n, A, A^2, \cdots \})) = \dim(\text{span}(\{I_n, A, A^2, \cdots, A^{n-1}\})) \leq n. \]

**Ex5.4.17**

For \( k = 1, A = (-a_0) \Rightarrow |A - tI| = -a_0 - t = (-1)(a_0 + t), \) hence it is true for \( k = 1. \)

Now assume it is true for \( k = m, \) then for \( k = m + 1 \)

\[ |A - tI| = (-t)(-1)^m(a_1 + a_2 t + \cdots + a_m t^{m-1} + t^m) + (-1)^{m+1} a_0 \]
\[ = (-1)^{m+1} a_0 + a_1 t + \cdots a_m t^m + t^{m+1} \]

It is true for \( k = m + 1, \) which implies it is true for \( \forall k \in \mathbb{N}. \)

**Ex5.4.20**

\( \Leftarrow \) Suppose \( U = g(T) \) for some polynomial \( g(t), \) then for all \( v \in V \)

\[ TU(v) = T(g(T)(v)), \]
\[ UT(v) = g(T)(T(v)) = T(g(T)(v)). \]

Hence \( UT = TU. \)

\( \Rightarrow \) Suppose \( V \) is generated by \( v \) and \( \dim(V) = n. \) Then

\[ U(v) = a_0 v + a_1 T(v) + \cdots + a_n T^n(v). \]

Let

\[ g(t) = a_0 v + a_1 t + \cdots + a_n t^n, \]

then

\[ g(T)(v) = U(v). \]
Now for $\forall k \in \mathbb{N}$, we have
\[
g(T)(T^K(v)) = (a_0 + a_1T + \cdots + a_n T^n)(T^k(v)) = a_0 T^k(v) + a_1 T^{k+1}(v) + \cdots + a_n T^{k+n}(v).
\]
On the other hand
\[
U(T^K(v)) = a_0 T^k(v) + a_1 T^{k+1}(v) + \cdots + a_n T^{k+n}(v) = g(T)(T^K(v)).
\]
Notice that $V = \text{span}\{v, T(v), \cdots, T^k(v), \cdots\}$, then $U = g(T)$.

**Ex 5.4.25**

(a) Let $v \in E_{\lambda_i}^T$, then $TU(v) = UT(v) = U(\lambda_i v) = \lambda_i(U(v))$, i.e. $U(v) \in E_{\lambda_i}^T$. Hence $E_{\lambda_i}^T$ is $U$-invariant. On the other hand, $U$ is diagonalizable, hence $U$ is diagonalizable on $E_{\lambda_i}^T$ too by Exercise 24. Now choose a basis $\beta_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ for $E_{\lambda_i}^T$, s.t. $U(v_{ij}) = \lambda_{ij} v_{ij}, \ 1 \leq j \leq n_i$. Let $\beta = \beta_1 \cup \beta_2 \cdots \cup \beta_k$ where $k$ is the number of distinct eigenvalues of $T$, we have
\[
[T]_\beta = \text{diag}(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2, \cdots, \lambda_k, \cdots, \lambda_k),
\]
\[
[U]_\beta = \text{diag}(\lambda_{11}, \cdots, \lambda_{1n_1}, \lambda_{21}, \cdots, \lambda_{2n_2}, \cdots, \lambda_{kn_k}.
\]
Hence $T$ and $U$ are simultaneously diagonalizable.

(b) Matrix version:
If $A$ and $B$ are two diagonalizable matrices in $M_{n\times n}(F)$ such that $AB = BA$, then $A$ and $B$ are simultaneously diagonalizable.

Proof: By exercise 5.2.17, we know $A$ and $B$ are simultaneously diagonalizable iff $L_A$ and $L_B$ are simultaneously diagonalizable on $F^n$. Now since $AB = BA$, we have $L_A L_B = L_B L_A$. Moreover, $A$ and $B$ are diagonalizable matrices, so $L_A$ and $L_B$ are diagonalizable operators. From (a), we have $L_A$, $L_B$ are simultaneously diagonalizable, so as $A$ and $B$.  

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