Correction terms, $\mathbb{Z}_2$-Thurston norm, and triangulations

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We explicitly construct layered-triangulations for manifolds admitting a genus-one open book decomposition with connected binding. This construction gives an upper bound to the complexity of these manifolds. In a different direction, we show that the correction terms in Heegaard Floer homology give a lower bound to the genus of one-sided Heegaard splittings and the $\mathbb{Z}_2$-Thurston norm. Using a result of Jaco–Rubinstein–Tillmann, this gives a lower bound to the complexity of certain closed 3-manifolds. We illustrate our theorems with some examples.

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1. Introduction

Given a three-manifold Y, the *complexity* $C(Y)$ of Y is the minimal number of tetrahedra one needs to triangulate Y. This invariant is notoriously hard to compute. In order to estimate $C(Y)$, one needs to find both upper bounds and lower bounds to it. Upper bounds to $C(Y)$ can be obtained by constructing triangulations, and lower bounds to $C(Y)$ can be obtained by studying certain invariants of Y.

To get a good upper bound to $C(Y)$, we want to construct triangulations with as few as possible tetrahedra. A class of such triangulations, called layered-triangulations, was studied in [6]. In this paper, we consider the special case when the manifold Y admits a genus-one open book decomposition with connected binding. It is well-known that such a Y is the double branched cover of S^3 over the closure of a 3-braid σ. We will construct a layered-triangulation of Y explicitly in terms of σ.

Theorem 1.1. *Suppose that Y is the double branched cover of S^3 over the closure of a 3-braid σ, and the word length of σ is l with respect to the generator set $\{\sigma_2^{\pm 1}, (\sigma_2 \sigma_1)^{\pm 1}\}$. Then Y has a one-vertex triangulation with l tetrahedra.*

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To get a lower bound to the complexity, we will use results from Heegaard Floer homology [11]. One important feature of Heegaard Floer homology which makes it so useful is that it gives a lower bound to the genus of surfaces in a given homology class. In dimension 3, it determines the Thurston norm [15]. In dimension 4, the adjunction inequality [12] gives a lower bound to the genus of surfaces which is often sharp, and the concordance invariant [14] gives a lower bound to the slice genus of knots.

In [10], we studied a new type of genus bounds. Roughly speaking, given a torsion class $a \in H_1(Y; \mathbb{Z})$, where Y is a 3-manifold, one can consider the minimal rational genus of all knots representing a. This defines a function Θ on the torsion subgroup of $H_1(Y; \mathbb{Z})$, which was introduced by Turaev [19] as an analogue of the Thurston norm [18]. When the homology class a has order 2, this Θ is essentially the minimal genus of embedded non-orientable surfaces in a given $\mathbb{Z}_2$-homology class. To state the theorem, let us recall that the correction term of a rational homology sphere Y with a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ is a rational number $d(Y, \mathfrak{s})$. There is an affine action of $H^2(Y; \mathbb{Z})$ on $\mathrm{Spin}^c(Y)$ which is denoted by addition.

The following theorem is essentially [5, Theorem 3.3], which is a corollary of the main theorem in [10].

Theorem 1.2. *Let Y be a rational homology 3-sphere,*

$$\beta \colon H_2(Y; \mathbb{Z}_2) \to H_1(Y; \mathbb{Z})$$

be the Bockstein homomorphism, and

$$\mathrm{PD} \colon H_i(Y; \mathbb{Z}) \to H^{3-i}(Y; \mathbb{Z})$$

be the Poincaré duality map. Suppose that a nonzero class $A \in H_2(Y; \mathbb{Z}_2)$ is represented by a closed connected non-orientable surface of genus h, then

$$h \geq 2 \max_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \big\{ d(Y, \mathfrak{s} + \mathrm{PD} \circ \beta(A)) - d(Y, \mathfrak{s}) \big\}.$$

A stronger version of the above theorem is obtained by Levine–Ruberman–Strle [9], who proved a non-orientable genus bound in dimensional four.

Bredon and Wood [2] initiated the study of minimal genus non-orientable surfaces representing a given $\mathbb{Z}_2$-homology class. They have completely determined the minimal genus for lens spaces $L(2k, q)$. (Lens spaces with odd order do not contain embedded non-orientable surfaces, since such surfaces always represent nontrivial $\mathbb{Z}_2$-homology classes.) This problem is closely related to one-sided Heegaard splittings introduced by Rubinstein [17], since the Heegaard surfaces in this case are non-orientable. As a consequence, Theorem 1.2 gives a lower bound to the one-sided Heegaard genus of a rational homology sphere in terms of Heegaard Floer correction terms, while such kind of bounds are not known for the usual two-sided Heegaard genus.

Non-orientable genus is also closely related to the $\mathbb{Z}_2$-Thurston norm, which is discussed in Section 2. As a corollary of Theorem 1.2, we get the following lower bound to the $\mathbb{Z}_2$-Thurston norm for certain 3-manifolds.

Corollary 1.3. *If an irreducible rational homology 3-sphere Y satisfies the condition*

$$H_1(S; \mathbb{Z}_2) \to H_1(Y; \mathbb{Z}_2) \text{ is surjective for any orientable incompressible surface } S \subset Y, \qquad (1)$$

then the $\mathbb{Z}_2$-Thurston norm of any $A \in H_2(Y; \mathbb{Z}_2)$ is bounded below by

$$-2 + 2 \max_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \big\{ d(Y, \mathfrak{s} + \mathrm{PD} \circ \beta(A)) - d(Y, \mathfrak{s}) \big\}.$$

Jaco–Rubinstein–Tillmann [7] found a lower bound to $C(Y)$ for some 3-manifolds in terms of the $\mathbb{Z}_2$-Thurston norm. Using Corollary 1.3 and its variants, we can compute the $\mathbb{Z}_2$-Thurston norm for some 3-manifolds, thus give lower bounds to $C(Y)$.

We carry out the above computation/construction explicitly for two classes of manifolds.

Proposition 1.4. *Let L be the closure of the braid*

$$\sigma = \sigma_1 \sigma_2^{-2a_1} \sigma_1 \sigma_2^{-2a_2} \cdots \sigma_1 \sigma_2^{-2a_{2n-1}} \sigma_1 \sigma_2^{-2a_{2n}},$$

where $a_i, n > 0$, $\Sigma(L)$ be the double branched cover of S^3 branched over L. Then the complexity $C(\Sigma(L))$ of $\Sigma(L)$ is in the range

$$2n - 4 + 2\sum_{i=1}^{2n} a_i \le C(\Sigma(L)) \le 4n + 2\sum_{i=1}^{2n} a_i.$$

Proposition 1.5. *Let L be the closure of the braid $\sigma_1 \sigma_2^{-2a-1} \sigma_1 \sigma_2^{-2b-1} \sigma_1 \sigma_2^{-2c-1}$, where a, b, c are nonnegative integers. Then the complexity $C(\Sigma(L))$ of $\Sigma(L)$ is in the range*

$$2(a + b + c) + 2 \le C(\Sigma(L)) \le 2(a + b + c) + 9.$$

This paper is organized as follows: In Section 2, we review our earlier work on rational genus bounds. We derive Theorem 1.2 as a corollary of this work. We then show that this bound gives lower bounds to one-sided Heegaard genus and $\mathbb{Z}_2$-Thurston norm. In Section 3, we construct layered triangulations for manifolds admitting a genus one open book decomposition with connected binding. In Section 4, we review Ozsváth and Szabó's algorithm of computing the correction terms of the double branched cover of S^3 branched over alternating links. In Section 5, we carry out the computation for the double branched cover of S^3 branched over some alternating closed 3-braids, and determine their $\mathbb{Z}_2$-Thurston norm, hence give upper and lower bounds for the complexity of these manifolds.

2. Rational genus and non-orientable genus bounds

2.1. Rational genus bounds in Heegaard Floer homology

Heegaard Floer homology, introduced by Ozsváth and Szabó [11], is an invariant for closed oriented Spin^c 3-manifolds $(Y, \mathfrak{s})$, taking the form of a collection of related homology groups as $\widehat{HF}(Y, \mathfrak{s})$, $HF^{\pm}(Y, \mathfrak{s})$, and $HF^{\infty}(Y, \mathfrak{s})$. There is a U-action on Heegaard Floer homology groups. When $\mathfrak{s}$ is torsion, there is an absolute Maslov $\mathbb{Q}$-grading on the Heegaard Floer homology groups. The U-action decreases the grading by 2.

For a rational homology 3-sphere Y with a Spin^c structure $\mathfrak{s}$, $HF^+(Y, \mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $HF^{\infty}(Y, \mathfrak{s}) \cong \mathbb{Z}[U, U^{-1}]$ in $HF^+(Y, \mathfrak{s})$, which is isomorphic to $\mathcal{T}^+ = \mathbb{Z}[U, U^{-1}]/UZ[U]$, and its minimal absolute $\mathbb{Q}$-grading is an invariant of $(Y, \mathfrak{s})$, denoted by $d(Y, \mathfrak{s})$, the *correction term* [13]; the second group is the quotient modulo the above image and is denoted by $HF_{\mathrm{red}}(Y, \mathfrak{s})$. Altogether, we have

$$HF^+(Y, \mathfrak{s}) = \mathcal{T}^+ \oplus HF_{\mathrm{red}}(Y, \mathfrak{s}).$$

Suppose that Y is a closed oriented 3-manifold, there is a kind of "norm" function one can define on the torsion subgroup of $H_1(Y; \mathbb{Z})$. To define it, let us first recall the rational genus of a rationally null-homologous knot $K \subset Y$ defined by Calegari and Gordon [3].

Suppose that K is a rationally null-homologous oriented knot in Y, and $\nu(K)$ is a tubular neighborhood of K. A properly embedded oriented connected surface $F \subset Y \setminus \overset{\circ}{\nu}(K)$ is called a *rational Seifert surface* for K, if ∂F consists of coherently oriented parallel curves on $\partial \nu(K)$, and the orientation of ∂F is coherent with the orientation of K. The *rational genus* of K is defined to be

$$g_r(K) = \min_F \frac{\max\{0, -\chi(F)\}}{2|[\mu] \cdot [\partial F]|},$$

where F runs over all the rational Seifert surfaces for K, and $\mu \subset \partial \nu(K)$ is the meridian of K.

The rational genus is a natural generalization of the genus of null-homologous knots. Moreover, given a torsion class in $H_1(Y)$, one can consider the minimal rational genus for all knots in this torsion class. More precisely, given $a \in \text{Tors}\, H_1(Y)$, let

$$\Theta(a) = \min_{K \subset Y,\, [K] = a} 2g_r(K).$$

This Θ was introduced by Turaev [19] in a slightly different form. Turaev regarded Θ as an analogue of the Thurston norm [18], in the sense that it measures the minimal normalized Euler characteristic of a "folded surface" representing a given class in $H_2(Y; \mathbb{Q}/\mathbb{Z})$.

The main result in [10] gives a lower bound to Θ via Heegaard Floer correction terms.

Theorem 2.1 *(Ni–Wu). Suppose that Y is a rational homology 3-sphere, $K \subset Y$ is a knot, F is a rational Seifert surface for K. Then*

$$1 + \frac{-\chi(F)}{|[\partial F] \cdot [\mu]|} \geq \max_{\mathfrak{s} \in \text{Spin}^c(Y)} \left\{ d(Y, \mathfrak{s} + \text{PD}[K]) - d(Y, \mathfrak{s}) \right\}. \tag{2}$$

The right hand side of (2) only depends on the manifold Y and the homology class of K, so it gives a lower bound of $1 + \Theta(a)$ for the homology class $a = [K]$.

2.2. Non-orientable genus bound

Theorem 1.2 is a corollary of Theorem 2.1, as explained in [5, Theorem 3.3]. For self-containedness, we include a proof of Theorem 1.2 below.

Let Π_h be the closed connected non-orientable surface of genus h, then $H_1(\Pi_h) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}_2$. The following lemma about the unique torsion class in $H_1(\Pi_h)$ is elementary, and the proof is left to the reader.

Lemma 2.2. *The torsion class is the image of the $\mathbb{Z}_2$-fundamental class under the Bockstein homomorphism $\beta \colon H_2(\Pi_h; \mathbb{Z}_2) \to H_1(\Pi_h; \mathbb{Z})$. Moreover, the complement of a simple closed curve $c \subset \Pi_h$ is orientable if and only if c represents the torsion class.*

Definition 2.3. A simple closed curve $c \subset \Pi_h$ is called a *torsion curve* if it represents the torsion class.

Proof of Theorem 1.2. Suppose that $\Pi \subset Y$ is a closed non-orientable surface representing A. Let $K \subset \Pi$ be a torsion curve, then it follows from Lemma 2.2 that $\Pi - K$ is orientable and $[K] = \beta[\Pi]$. Since $\Pi - K$ is orientable and Π is non-orientable, either $\Pi \cap \nu(K)$ is a Möbius band or $\Pi \cap \nu(K)$ is an annulus and the induced orientations of the two components of $\partial(\Pi \cap \nu(K))$ are coherent on $\partial \nu K$. So $\Pi \cap (Y - \nu(K))$ is a rational Seifert surface for K with $|[\mu] \cdot [\partial(\Pi \cap \nu(K))]| = 2$, where $\mu \subset \partial \nu(K)$ is the meridian of K. If the genus of Π is h, then $\chi(\Pi \cap (Y - \nu(K))) = \chi(\Pi) = 2 - h$.

On the other hand, if a is in the image of $\beta \colon H_2(Y; \mathbb{Z}_2) \to H_1(Y; \mathbb{Z})$, then the order of a is 2. Let F be a rational Seifert surface for a knot K representing a. Without loss of generality, we may assume $|[\mu] \cdot [\partial F]| = 2$.

Depending on $|\partial F| = 1$ or 2, we can glue in a Möbius band or annulus to F to get a non-orientable surface $\widehat{F}$. Clearly, $\chi(F) = \chi(\widehat{F})$. Since $\beta \colon H_2(Y; \mathbb{Z}_2) \to H_1(Y; \mathbb{Z})$ is injective, $[\widehat{F}] \in H_2(Y; \mathbb{Z}_2)$ is determined by $[K] = \beta[\widehat{F}]$.

Our conclusion immediately follows from the above discussion and Theorem 2.1. □

2.3. One-sided Heegaard splittings

Theorem 1.2 can be used to study one-sided Heegaard splitting, introduced in [17].

Definition 2.4. Let Y be a closed orientable 3-manifold. A pair (Y, Π) is called a *one-sided Heegaard splitting* if $\Pi \subset Y$ is a closed non-orientable surface, such that its complement is an open handlebody. Moreover, $[\Pi] \in H_2(Y; \mathbb{Z}_2)$ is called the class *associated with* (Y, Π).

Rubinstein [17] proved that for any nonzero element $A \in H_2(Y; \mathbb{Z}_2)$, there exists a one-sided Heegaard splitting (Y, Π) such that $[\Pi] = A$. He also studied when there exists an incompressible one-sided Heegaard splitting.

Theorem 2.5 *(Rubinstein). If Y is irreducible and $b_1(Y) = 0$, then the condition* (1) *is equivalent to the condition that the complement of any non-orientable incompressible surface is an open handlebody. Under this condition, there is an incompressible one-sided splitting associated with any nonzero class.*

When Y is the lens space $L(2k, q)$, Rubinstein [17] proved that all incompressible surfaces in Y are isotopic, and the genus $N(2k, q)$ can be computed by [2]. Recently, Johnson [8] proved that any non-orientable surface in $L(2k, q)$ with a given genus is unique up to isotopy, thus $L(2k, q)$ has a unique genus g one-sided Heegaard splitting for any given genus $g \geq N(2k, q)$.

One can define the *one-sided Heegaard genus* of (Y, A) to be the minimal genus of Π such that (Y, Π) is a one-sided Heegaard splitting and $[\Pi] = A$. Theorem 1.2 clearly gives a lower bound to the one-sided Heegaard genus.

2.4. A lower bound to the $\mathbb{Z}_2$-Thurston norm

Theorem 1.2 may also be used to give a lower bound to the $\mathbb{Z}_2$-Thurston norm introduced in [7].

Definition 2.6. Given a closed, not necessarily orientable, surface S, let

$$\chi_-(S) = \sum_{S_i \subset S} \max\{-\chi(S_i), 0\},$$

where the sum is taken over all the components of S. For any $A \in H_2(Y; \mathbb{Z}_2)$, define the $\mathbb{Z}_2$-*Thurston norm* of A to be

$$||A|| := \min\{\chi_-(S) | S \subset Y, [S] = A\}.$$

A surface $S \subset Y$ is $\mathbb{Z}_2$-*taut*, if no component of S is a sphere or $\mathbb{R}P^2$ and $||[S]|| = -\chi(S)$.

Clearly, if $S \subset Y$ is $\mathbb{Z}_2$-taut, then each component of S is non-separating and incompressible.

Proof of Corollary 1.3. By Theorem 2.5, any non-orientable incompressible surface must be connected. Our conclusion then follows from Theorem 1.2. □

Remark 2.7. The condition (1) in Corollary 1.3 is used to ensure that any $\mathbb{Z}_2$-taut surface is connected. This condition may be removed if we can prove an analogue of Theorem 2.1 for links.

When the rank of $H_2(Y; \mathbb{Z}_2)$ is small, it is easy to remove the condition (1). For example, if $H_2(Y; \mathbb{Z}_2) \cong \mathbb{Z}_2$, any $\mathbb{Z}_2$-taut surface is necessarily connected. If $H_2(Y; \mathbb{Z}_2) \cong \mathbb{Z}_2^2$, we may use the following lemma.

Lemma 2.8. *Suppose that $H_2(Y; \mathbb{Z}_2) \cong \mathbb{Z}_2^2$, and the three nonzero elements in $H_2(Y; \mathbb{Z}_2)$ are α_1, α_2, α_3. Let h_i be a lower bound to the genus of the closed **connected** non-orientable surfaces representing α_i, $i = 1, 2, 3$. If h_i's satisfy the inequalities*

$$h_i + h_{i+1} \geq h_{i+2} + 2, \quad i = 1, 2, 3,$$

then $h_i - 2 \leq ||\alpha_i||$. Here the indices are understood modulo 3.

Proof. Let S be a $\mathbb{Z}_2$-taut surface representing α_1. If S is connected, our conclusion obviously holds. If S is disconnected, we may assume that no component of S is null-homologous in $H_2(Y; \mathbb{Z}_2)$, and any two components of S are not homologous in $H_2(Y; \mathbb{Z}_2)$. If follows that S has exactly two components, and they represent α_2 and α_3, respectively. So $\chi_-(S) \geq (h_2 - 2) + (h_3 - 2) \geq h_1 - 2$. Hence $||\alpha_1|| \geq h_1 - 2$. Similarly, $h_i - 2 \leq ||\alpha_i||$ for $i = 2, 3$. $\square$

Jaco, Rubinstein and Tillmann [7] showed that the $\mathbb{Z}_2$-Thurston norm gives a lower bound to the complexity of some manifolds.

Theorem 2.9 *(Jaco–Rubinstein–Tillmann). Let Y be a closed, orientable, irreducible, atoroidal, connected 3-manifold with triangulation $\mathcal{T}$. Let $H \subset H_2(Y; \mathbb{Z}_2)$ be a rank 2 subgroup, then*

$$|\mathcal{T}| \geq 2 + \sum_{A \in H} ||A||.$$

Corollary 2.10. *Suppose that Y is irreducible and atoroidal, $b_1(Y) = 0$, and Y satisfies (1). Let $H \subset H_2(Y; \mathbb{Z}_2)$ be a rank 2 subgroup, then the complexity of Y is bounded below by*

$$-4 + 2 \sum_{A \in H} \max_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \left\{ d(Y, \mathfrak{s} + \mathrm{PD} \circ \beta(A)) - d(Y, \mathfrak{s}) \right\}.$$

Proof. This follows from Corollary 1.3 and Theorem 2.9. $\square$

3. Genus-one open books and layered-triangulations

In this section, we will construct layered-triangulations for manifolds admitting a genus one open book decomposition with connected binding. Our construction gives a lower bound to the complexity of the 3-manifolds considered in the last section. The concept of layered-triangulations was introduced by Jaco and Rubinstein [6]. We refer the reader to their original paper for more details on layered-triangulations.

Suppose that M is a compact 3-manifold with boundary, and $\mathcal{T}$ is a triangulation of M. Then $\mathcal{T}|_{\partial M}$ is a (2-dimensional) triangulation of ∂M. If e is one edge of $\mathcal{T}|_{\partial M}$, then we can add a tetrahedron $\tilde{\Delta}$ to $\mathcal{T}$ along e. The new space M' we get is still homeomorphic to M, but there is a new triangulation $\mathcal{T}' = \mathcal{T} \cup \Delta$, where Δ is the image of $\tilde{\Delta}$, and two faces of Δ are identified with the two triangles adjacent to e in $\mathcal{T}|_{\partial M}$. This process is called "layering a tetrahedron along an edge". As in Fig. 1, $\mathcal{T}'|_{\partial M'}$ differs from $\mathcal{T}|_{\partial M}$ by "flipping the diagonal e".
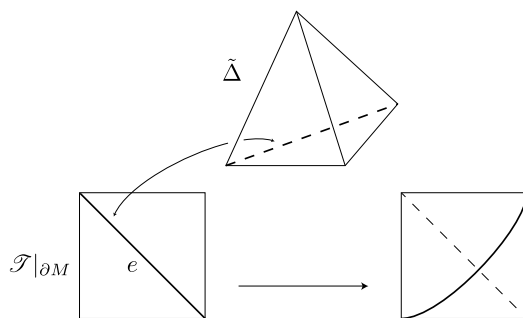
Fig. 1. Layering a tetrahedron along an edge results in a diagonal flip in the triangulation of the boundary.

The following lemma is obvious.

Lemma 3.1. *Let $\mathbb{T}_1$ be the genus 1 compact oriented surface with one boundary component. Then $\mathbb{T}_1$ has exactly one one-vertex triangulation $\mathscr{R}$ up to homeomorphism.*

The triangulation $\mathscr{R}$ has five edges, labeled as a_1, a_2, b_1, b_2, c, as in Fig. 2. Here c is the unique edge on $\partial\mathbb{T}_1$, and a_1, a_2, b_1, b_2 are called the *interior edges*.

Let $\mathbb{S} = \mathbb{T}_1 \times [0, 1]/ \sim$, where the equivalence relation $\sim$ is given by $(x, t_1) \sim (x, t_2)$ for any $x \in \partial\mathbb{T}_1$ and $t_1, t_2 \in [0, 1]$. Then $\mathbb{S}$ is homeomorphic to a genus 2 handlebody, and is homotopy equivalent to $\mathbb{T}_1$.

From now on, we only consider triangulations of $\mathbb{S}$ with the following properties: the triangulation has only one vertex, and one edge of the triangulation is on $\partial\mathbb{T}_1 \times \{0\}$.

We start with the triangulation $\mathscr{R}$ of $\mathbb{T}_1$, regarded as a degenerate triangulation of $\mathbb{S}$. Suppose $\mathscr{T}$ is a triangulation of $\mathbb{S}$ we have obtained, then we can layer a tetrahedron along an interior edge of $\mathscr{T}|_{\mathbb{T}_1 \times \{1\}}$ to get a new triangulation. The triangulations obtained in this way are called *layered-triangulations*.

Suppose that $\mathscr{T}$ is a layered-triangulation of $\mathbb{S}$. By Lemma 3.1, $\mathscr{T}|_{\mathbb{T}_1 \times \{0\}}$ is homeomorphic to $\mathscr{T}|_{\mathbb{T}_1 \times \{1\}}$, so we can glue them together by a homeomorphism

$$f_{\mathscr{T}} \colon \mathbb{T}_1 \times \{1\} \to \mathbb{T}_1 \times \{0\}$$

induced by a simplicial isomorphism. The resulting manifold $M(\mathscr{T})$ has a genus 1 open book decomposition with connected binding, and the resulting triangulation is also called a *layered-triangulation*. A well-known result of Birman–Hilden [1] implies that $M(\mathscr{T})$ is the double branched cover of S^3 over a closed 3-braid, and the map $f_{\mathscr{T}}$ doubly covers the map on the disk with 3 marked points corresponding to the 3-braid. We want to find out what the closed 3-braid is in terms of $\mathscr{T}$.

Lemma 3.2. *Let a_1', b_1' be simple closed curves in the interior of $\mathbb{T}_1$ which are freely isotopic to a_1, b_1, respectively. Let τ_a, τ_b be the left Dehn twists about a_1', b_1' on $\mathbb{T}_1$, respectively. Let $f = f_{\mathscr{T}}$. After layering a tetrahedron along an interior edge e of $\mathscr{T}|_{\mathbb{T}_1 \times \{1\}}$, we get a new triangulation $\mathscr{T}'$. Then the isotopy class of $f_{\mathscr{T}'}$ is given in the following table:*

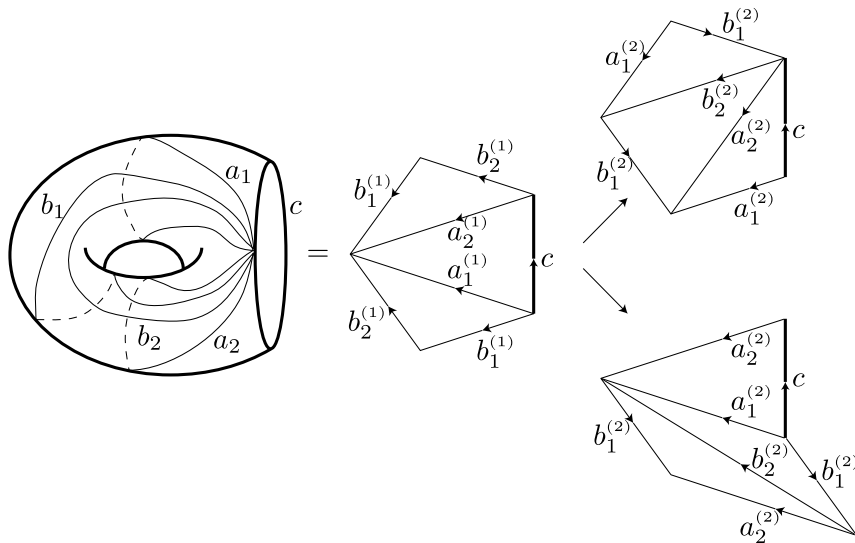| flipped edge e	$f_{\mathscr{T}'}$
a_1	$f\tau_b^{-1}\tau_a^{-1}$
a_2	$f\tau_a\tau_b$
b_1	$f\tau_a^{-1}$
b_2	$f\tau_a$

Fig. 2. The triangulation $\mathscr{R}$ and two diagonal flips of a_1 and b_1.

Proof. For our convenience, we orient the edges a_i, b_i as in Fig. 2. Homotopically, the Dehn twist τ_a fixes a_1, a_2, sends b_1 to $\bar{b}_2$, and sends b_2 to a loop freely homotopic to $b_2 * a_1$; the Dehn twist τ_b fixes b_1, and sends x to $x * b_1$ where $x \in \{a_1, a_2, b_2\}$.

Let $a_i^{(k)}$, $b_i^{(k)}$ be the interior edges of $\mathscr{T}|_{\mathbb{T}_1 \times \{k\}}$, $i = 1, 2$, $k = 0, 1$. After layering a new tetrahedron, the interior edges of $\mathscr{T}'|_{\mathbb{T}_1 \times \{1\}}$ are denoted $a_i^{(2)}$, $b_i^{(2)}$.

We first consider layering a tetrahedra along b_1. The central picture in Fig. 2 shows the labeling of $a_i^{(1)}$, $b_i^{(1)}$, and the bottom right picture shows the labeling of $a_i^{(2)}$, $b_i^{(2)}$. For simplicity, we suppress the superscripts in the figure. By definition, $f_{\mathscr{T}}$ maps $a_i^{(1)}$, $b_i^{(1)}$ to $a_i^{(0)}$, $b_i^{(0)}$, and $f_{\mathscr{T}'}$ maps $a_i^{(2)}$, $b_i^{(2)}$ to $a_i^{(0)}$, $b_i^{(0)}$. From Fig. 2, we see that $a_i^{(2)}$ is isotopic to $a_i^{(1)} = \tau_a(a_i^{(1)})$, $b_1^{(2)}$ is isotopic to $\bar{b}_2^{(1)} = \tau_a(b_1^{(1)})$, and $b_2^{(2)}$ is isotopic to $b_2^{(1)} * a_1^{(1)} = \tau_a(b_2^{(1)})$. Hence

$$f_{\mathscr{T}'} = f_{\mathscr{T}} \tau_a^{-1}$$

up to isotopy.

If we flip b_1, the new diagonal we create is b_2. Hence if we flip b_2, we will create b_1. Suppose we start with a triangulation $\mathscr{T}$, layer a tetrahedra along b_1, then layer a tetrahedra along b_2. The two new tetrahedra have exactly two common faces. As observed in [6], we can crush the two tetrahedra to get a triangulation with two less tetrahedra, which is exactly the initial triangulation $\mathscr{T}$. In this sense, layering a tetrahedra along b_2 is the inverse operation of layering a tetrahedra along b_1. As a result,

$$f_{\mathscr{T}'} = f_{\mathscr{T}} \tau_a$$

if $\mathscr{T}'$ is obtained by layering a tetrahedra along b_2.

Now we consider layering a tetrahedra along a_1, in which case the top right picture in Fig. 2 shows the labeling of $a_i^{(2)}$, $b_i^{(2)}$. We can see that

$$a_1^{(2)} \sim b_1^{(1)} = \tau_a \tau_b(a_1^{(1)}), \quad a_2^{(2)} \sim a_2^{(1)} * \bar{b}_2^{(1)} = \tau_a \tau_b(a_2^{(1)}),$$
$$b_1^{(2)} \sim \bar{b}_2^{(1)} = \tau_a \tau_b(b_1^{(1)}), \quad b_2^{(2)} \sim a_2^{(1)} = \tau_a \tau_b(b_2^{(1)}).$$
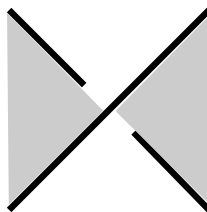
Fig. 3. The coloring convention.

So we can conclude that

$$f_{\mathscr{T}'} = f_{\mathscr{T}}(\tau_a \tau_b)^{-1}.$$

As before, layering a tetrahedra along a_2 is the inverse operation of layering a tetrahedra along a_1. So

$$f_{\mathscr{T}'} = f_{\mathscr{T}} \tau_a \tau_b$$

if $\mathscr{T}'$ is obtained by layering a tetrahedra along a_2. □

Proof of Theorem 1.1. By Birman–Hilden [1], under the double branched cover from an annulus to a disk with two branched points, a Dehn twist doubly covers a half Dehn twist which corresponds to a standard generator of the braid group. Consider the double branched cover from $\mathbb{T}_1$ to the disk with three branched points, we can get a one-to-one correspondence between the mapping classes of $\mathbb{T}_1$ and the 3-braids. (This correspondence is illustrated in [4, Fig. 9.15].) After labeling the three branched points appropriately, we may arrange so that τ_b corresponds to σ_1 and τ_a corresponds to σ_2. Our desired result then follows from Lemma 3.2. □

4. The double branched cover of alternating links

For any link $L \subset S^3$, let $\Sigma(L)$ be the double branched cover of S^3 branched along L. In this section, we will review an algorithm of Ozsváth and Szabó [16], which computes the correction terms of $\Sigma(L)$ when L is a non-split alternating link.

Let L be a link with a connected diagram. The diagram separates the plane into several regions. We can color the regions by black and white, such that any two adjacent regions have different colors. When the diagram is alternating, we can choose the coloring convention as in Fig. 3.

We now define the black graph $\mathcal{B}(L)$ of the diagram as follows: the vertices of $\mathcal{B}(L)$ are in one-to-one correspondence with the black regions, and the edges of $\mathcal{B}(L)$ are in one-to-one correspondence with the crossings. For each crossing between two regions R_1, R_2, there is an edge connecting the corresponding vertices v_1, v_2.

Now we choose a maximal subtree T of $\mathcal{B}(L)$, and let $\{e_i\}_{i=1}^b$ be the edges in $Z_T = \mathcal{B}(L) - T$. We orient these e_i in any way we like. Since T is a maximal subtree, for any $e_i \in Z_T$ there is a unique circuit C_i in $T \cup \{e_i\}$, with the orientation coherent with e_i.

Let $V = \mathbb{Z}^b$ be generated by $e_1, e_2, \ldots, e_b$. We define a quadratic form $Q: V \otimes V \to \mathbb{Z}$ as follows. For any e_i, e_j,

$$Q(e_i \otimes e_j) = \pm E(C_i \cap C_j).$$

Here $E(C_i \cap C_j)$ is the number of edges in $C_i \cap C_j$, the sign is positive if the orientations of C_i and C_j are different on $C_i \cap C_j$, and negative if the orientations of C_i and C_j are the same on $C_i \cap C_j$. In particular, $Q(e_i \otimes e_i) = -E(C_i)$.

Let $V^* = \mathrm{Hom}(V, \mathbb{Z})$ be the dual group of V. For any $\alpha \in V^*$, define

$$|\alpha|^2 = \max_{v \in V \otimes \mathbb{R} - \{0\}} \frac{(\alpha(v))^2}{Q(v, v)}. \tag{3}$$

We define a homomorphism $q \colon V \to V^*$ by letting

$$(q(a))(v) = Q(a, v).$$

Notation 4.1. Under the basis $\{e_1, e_2, \ldots, e_b\}$, elements in V can be represented by column vectors. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_b\}$ be the dual basis of V^*, namely, $\alpha_i(e_j) = \delta_{ij}$, then elements in V^* will be represented by row vectors.

Lemma 4.2. *For* $\alpha = \sum a_i \alpha_i$, *let* $a = (a_1, a_2, \ldots, a_b)$. *Suppose that* Q *is represented by the symmetric matrix* Q, *then*

$$|\alpha|^2 = aQ^{-1}a^T. \tag{4}$$

Proof. For any $X_0 \in V \otimes \mathbb{R}$, the tangent hyperplane of the ellipsoid

$$X^T Q X = X_0^T Q X_0$$

at X_0 is

$$X_0^T Q X = X_0^T Q X_0.$$

On the other hand, the hyperplane defined by α is

$$aX = aX_0.$$

These two hyperplanes coincide when

$$a = X_0^T Q.$$

So

$$\begin{aligned}
|\alpha|^2 &= \left. \frac{|\alpha(X_0)|^2}{Q(X_0, X_0)} \right|_{X_0 = (aQ^{-1})^T} \\
&= \frac{|a(aQ^{-1})^T|^2}{(aQ^{-1})Q(aQ^{-1})^T} \\
&= aQ^{-1}a^T. \qquad \square
\end{aligned}$$

An element $\kappa \in V^*$ is *characteristic* if

$$\kappa(v) \equiv Q(v, v) \pmod{2}, \qquad \text{for any } v \in V.$$

Let $\mathcal{C} \subset V^*$ be the set of all characteristic elements. Two characteristic elements κ_1, κ_2 are *equivalent* if

$$\kappa_1 - \kappa_2 = 2q(v), \qquad \text{for some } v \in V.$$

Let $C_1(V) = \mathcal{C}/(2q(V))$ be the set of all equivalent classes of characteristic elements, which is an affine space over the group $H = V^*/(q(V))$. More precisely, given $c \in C_1(V)$ and $[\alpha] \in H$, $c + 2[\alpha]$ is a well-defined element in $C_1(V)$, and $c + 2[\alpha] = c$ if and only if $[\alpha] = 0$.

For any $c \in C_1(V)$, define

$$d(V, c) = \max_{\kappa \in c} \frac{|\kappa|^2 + b}{4}. \tag{5}$$

Theorem 4.3 *(Ozsváth–Szabó). Suppose that $Y = \Sigma(L)$, where L is an alternating link, let $Q \colon V \otimes V \to \mathbb{Z}$ be a quadratic form constructed as before, then $C_1(V)$ is isomorphic to $\mathrm{Spin}^c(Y)$ as affine spaces, and there exists an affine isomorphism between them such that $d(V, c)$ is the corresponding correction term of Y.*

5. Computation

In this section, we use Theorem 1.2 to determine the $\mathbb{Z}_2$-Thurston norm of the double branched cover of some alternating closed 3-braids. As a result, we get bounds on the complexity of these manifolds.

Let σ_1, σ_2 be the two standard generators of B_3. We summarize our results below.

Proposition 5.1. *Let L be the closure of the braid*

$$\sigma = \sigma_1 \sigma_2^{-2a_1} \sigma_1 \sigma_2^{-2a_2} \cdots \sigma_1 \sigma_2^{-2a_{2n-1}} \sigma_1 \sigma_2^{-2a_{2n}},$$

where $a_i, n > 0$, then the $\mathbb{Z}_2$-Thurston norms of the three nonzero homology classes in $H_2(\Sigma(L); \mathbb{Z}_2)$ are

$$\sum_{i\ odd} a_i + n - 2, \quad \sum_{i\ even} a_i + n - 2, \quad \sum_{i=1}^{2n} a_i - 2.$$

Proposition 5.2. *Let L be the closure of the braid $\sigma_1 \sigma_2^{-2a-1} \sigma_1 \sigma_2^{-2b-1} \sigma_1 \sigma_2^{-2c-1}$, where a, b, c are nonnegative integers. Then the $\mathbb{Z}_2$-Thurston norms of the three nonzero homology classes in $H_2(\Sigma(L); \mathbb{Z}_2)$ are $a + b$, $b + c$ and $c + a$.*

We point out that each of the family considered above consists of pure braids only. Hence, $H_2(\Sigma(L); \mathbb{Z}_2) \cong \mathbb{Z}_2^2$ and there are exactly three nonzero $\mathbb{Z}_2$-homology classes.

As a consequence of these two propositions, we can prove Propositions 1.4 and 1.5.

Proof of Proposition 1.4. The lower bound follows from Theorem 2.9 and Proposition 5.1.

For the upper bound, we notice that $\sigma_2 \sigma \sigma_2^{-1}$ is equal to

$$(\sigma_2 \sigma_1) \sigma_2^{-2a_1-1} (\sigma_2 \sigma_1) \sigma_2^{-2a_2-1} \cdots (\sigma_2 \sigma_1) \sigma_2^{-2a_{2n-1}-1} (\sigma_2 \sigma_1) \sigma_2^{-2a_{2n}-1}$$

and apply Theorem 1.1. $\square$

Similarly, using Proposition 5.2, we can prove Proposition 1.5.

5.1. Proof of Proposition 5.1

Case 1. $n = 1$, namely, $\sigma = \sigma_1 \sigma_2^{-2a} \sigma_1 \sigma_2^{-2b}$, $a, b > 0$.

Using the notation from the previous section, we construct the black graph $\mathcal{B}(L)$ in Fig. 4. Choose a maximal subtree T, then Z_T has two edges e_1, e_2. So $V = \langle e_1, e_2 \rangle$. The quadratic form is represented by the symmetric matrix
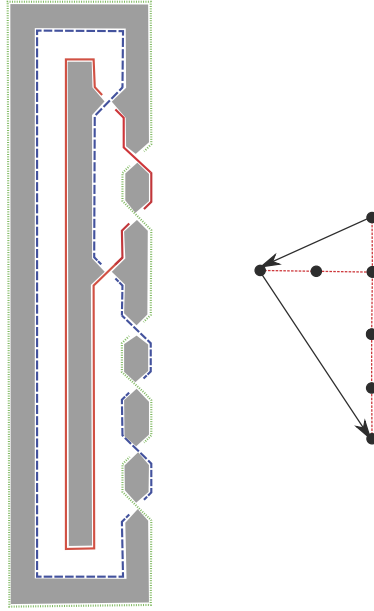
Fig. 4. The closure of the pure braid $\sigma = \sigma_1 \sigma_2^{-2a} \sigma_1 \sigma_2^{-2b}$ ($a = 1$, $b = 2$ here) and its black graph on the right. The dashed edges indicate a maximal subtree T, and the remaining two edges are oriented.

$$Q = \begin{pmatrix} -2a - 2 & 2 \\ 2 & -2b - 2 \end{pmatrix}$$

The set of characteristic elements is

$$\mathcal{C} = \{ a_1 \alpha_1 + a_2 \alpha_2 \,|\, a_1 \equiv a_2 \equiv 0 \pmod 2 \}.$$

Let

$$\kappa_0 = (0,0), \quad \kappa_1 = q(e_1) = (-2a - 2, 2),$$
$$\kappa_2 = q(e_2) = (2, -2b - 2), \quad \kappa_3 = q(e_1 + e_2) = (-2a, -2b)$$

be characteristic elements in the affine space $\mathcal{C}_1(V)$. Straightforward computation yields

$$|\kappa_0|^2 = \kappa_0 Q^{-1} \kappa_0^T = 0,$$
$$|\kappa_1|^2 = \kappa_1 Q^{-1} \kappa_1^T = -2(1 + a),$$
$$|\kappa_2|^2 = \kappa_2 Q^{-1} \kappa_2^T = -2(1 + b),$$
$$|\kappa_3|^2 = \kappa_3 Q^{-1} \kappa_3^T = -2(a + b).$$

Lemma 5.3. *Each of the characteristic elements κ_i maximizes the above quadratic form within its equivalent class of characteristic elements, i.e.,*

$$|\kappa_i + 2q(v)|^2 \leq |\kappa_i|^2$$

for any $v \in V$, $i = 0, 1, 2, 3$.

Proof. The characteristic elements equivalent to κ_i have the form

$$\kappa = q(me_1 + ne_2), \quad m, n \in \mathbb{Z}$$

where

- m, n even, when $i = 0$
- m odd n even, when $i = 1$
- m even n odd, when $i = 2$
- m, n odd, when $i = 3$

and

$$
\begin{aligned}
|\kappa|^2 &= \kappa Q^{-1} \kappa^T \\
&= (m, n) Q \begin{pmatrix} m \\ n \end{pmatrix} \\
&= -(2a + 2)m^2 - (2b + 2)n^2 + 4mn \\
&= -2((m - n)^2 + am^2 + bn^2).
\end{aligned}
$$

Since $a, b > 0$, it is clear that each $|\kappa_i|^2$ is a maximizer for the given parity on m and n. □

Hence, we can compute the correction terms via the formula (5) and apply Theorem 1.2 and Lemma 2.8 to get a lower bound to the $\mathbb{Z}_2$-Thurston norm of $\Sigma(L)$. Note that $\kappa_1 - \kappa_0$, $\kappa_2 - \kappa_0$, $\kappa_3 - \kappa_0$ represent the three different homology classes of order 2 in $H^2(Y; \mathbb{Z})$. For any $\alpha \in H_2(Y; \mathbb{Z}_2)$, let $h(\alpha)$ be the minimal genus of closed connected non-orientable surfaces representing α. We obtain:

$$
h(\beta^{-1}\mathrm{PD}^{-1}(\kappa_1 - \kappa_0)) \geq 2(d(Y, [\kappa_0]) - d(Y, [\kappa_1])) = a + 1.
$$
$$
h(\beta^{-1}\mathrm{PD}^{-1}(\kappa_2 - \kappa_0)) \geq 2(d(Y, [\kappa_0]) - d(Y, [\kappa_2])) = b + 1.
$$
$$
h(\beta^{-1}\mathrm{PD}^{-1}(\kappa_3 - \kappa_0)) \geq 2(d(Y, [\kappa_0]) - d(Y, [\kappa_3])) = a + b.
$$

Thus by Lemma 2.8, the corresponding $\mathbb{Z}_2$-Thurston norms are bounded below by

$$
a - 1, b - 1, a + b - 2.
$$

Finally, we show that the lower bounds here are sharp. Indeed, the lift of the disk bounded by each component of the link in $\Sigma(L)$ has Euler characteristic exactly $1 - a$, $1 - b$ and $2 - a - b$, respectively. (For example, the disk bounded by the dotted curve in Fig. 4 intersects with the solid curve and the dashed curve at $a + b$ points in total. By the Riemann–Hurwitz formula, the Euler characteristic of its double branched cover is $2\chi(D^2) - (a + b) = 2 - a - b$.)

Case 2. $n > 1$.

This case (Fig. 5) is very similar to the previous case. The quadratic form Q is represented by the symmetric matrix

$$
Q = \begin{pmatrix}
-2a_1 - 2 & 1 & 0 & \cdots & 0 & 1 \\
1 & -2a_2 - 2 & 1 & \cdots & 0 & 0 \\
0 & 1 & -2a_3 - 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -2a_{2n-1} - 2 & 1 \\
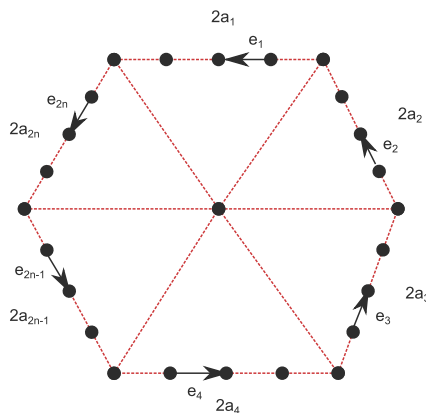1 & 0 & 0 & \cdots & 1 & -2a_{2n} - 2
\end{pmatrix}.
$$

Fig. 5. The black graph of the closure of σ in this case. The graph is supported in a $2n$-gon P. The ith side of P consists of $2a_i$ edges, and there is an edge connecting the center of P with each corner vertex of P. The maximal tree T is obtained by removing an edge from each side of P.

Given $X = \sum_{i=1} x_i e_i$, we have

$$Q(X, X) = X^T Q X = -\sum_{i=1}^{2n} \left(2a_i x_i^2 + (x_i - x_{i+1})^2 \right),$$

where the subscripts are understood modulo $2n$.

The set of characteristic elements is

$$\mathcal{C} = \left\{ \sum_{i=1}^{2n} a_i \alpha_i \,\middle|\, a_i \equiv 0 \pmod{2} \right\},$$

and $q(e_i) = (Q e_i)^T$ is the ith row of Q.

Let

$$\kappa_0 = 0 \in V^*,$$

$$\kappa_1 = q\left(\sum_{i \text{ odd}} e_i \right) = (-2a_1 - 2, 2, -2a_3 - 2, 2, \cdots, -2a_{2n-1} - 2, 2),$$

$$\kappa_2 = q\left(\sum_{i \text{ even}} e_i \right) = (2, -2a_2 - 2, 2, -2a_4 - 2, \cdots, 2, -2a_{2n} - 2),$$

$$\kappa_3 = \kappa_1 + \kappa_2 = (-2a_1, -2a_2, -2a_3, -2a_4, \cdots, -2a_{2n-1}, -2a_{2n})$$

be characteristic elements in the affine space $\mathcal{C}_1(V)$. Straightforward computation yields

$$|\kappa_0|^2 = \kappa_0 Q^{-1} \kappa_0^T = 0,$$

$$|\kappa_1|^2 = \kappa_1 Q^{-1} \kappa_1^T = \kappa_1 \sum_{i \text{ odd}} e_i = -2(n + \sum_{i \text{ odd}} a_i),$$

$$|\kappa_2|^2 = \kappa_2 Q^{-1} \kappa_2^T = \kappa_2 \sum_{i \text{ even}} e_i = -2(n + \sum_{i \text{ even}} a_i),$$

$$|\kappa_3|^2 = \kappa_3 Q^{-1} \kappa_3^T = \kappa_3 (\sum_{i=1}^{2n} e_i) = -2 \sum_{i=1}^{2n} a_i.$$

Lemma 5.4. *Each of the characteristic element κ_j maximizes the above quadratic form within its equivalent class of characteristic elements, i.e.,*

$$|\kappa_j + 2q(v)|^2 \leq |\kappa_j|^2$$

for any $v \in V$, $j = 0, 1, 2, 3$.

Proof. The characteristic elements equivalent to k_j have the form

$$\kappa = q(\sum_{i=1}^{2n} x_i e_i), \quad x_i \in \mathbb{Z}$$

where

- x_i even for any i, when $j = 0$
- $x_i \equiv i \pmod 2$, when $j = 1$
- $x_i \equiv i + 1 \pmod 2$, when $j = 2$
- x_i odd for any i, when $j = 3$

and

$$\begin{aligned}
|\kappa|^2 &= (QX)^T Q^{-1}(QX) \\
&= X^T Q X \\
&= -\sum_{i=1}^{2n} (2a_i x_i^2 + (x_i - x_{i+1})^2).
\end{aligned}$$

Since $a_i > 0$, it is clear that each $|\kappa_i|^2$ is a maximizer for the given parity on x_i's. $\quad\square$

Similar to the previous case, $\kappa_1 - \kappa_0$, $\kappa_2 - \kappa_0$, $\kappa_3 - \kappa_0$ represent the three different homology classes of order 2 in $H^2(Y; \mathbb{Z})$, and

$$h(\beta^{-1}\mathrm{PD}^{-1}(\kappa_1 - \kappa_0)) \geq 2(d(Y, [\kappa_0]) - d(Y, [\kappa_1])) = \sum_{i \text{ odd}} a_i + n.$$

$$h(\beta^{-1}\mathrm{PD}^{-1}(\kappa_2 - \kappa_0)) \geq 2(d(Y, [\kappa_0]) - d(Y, [\kappa_2])) = \sum_{i \text{ even}} a_i + n.$$

$$h(\beta^{-1}\mathrm{PD}^{-1}(\kappa_3 - \kappa_0)) \geq 2(d(Y, [\kappa_0]) - d(Y, [\kappa_1])) = \sum_{i=1}^{2n} a_i.$$

By Lemma 2.8, we get lower bounds to the $\mathbb{Z}_2$-Thurston norm. The lower bounds here are also sharp – the lift of the disks bounded by each component of the link in $\Sigma(L)$ has negative Euler characteristic

$$\sum_{i \text{ odd}} a_i + n - 2, \quad \sum_{i \text{ even}} a_i + n - 2, \quad \sum_{i=1}^{2n} a_i - 2,$$

respectively.

5.2. Proof of Proposition 5.2

We construct a black graph and its associated maximal subtree T (Fig. 6) for the closure of the braid $\sigma = \sigma_1 \sigma_2^{-2a-1} \sigma_1 \sigma_2^{-2b-1} \sigma_1 \sigma_2^{-2c-1}$, analogous to Fig. 4 before.
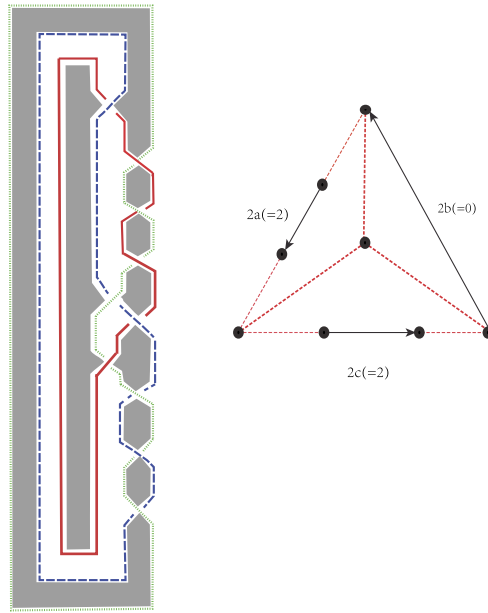
Fig. 6. The closure of the pure braid $\sigma = \sigma_1 \sigma_2^{-2a-1} \sigma_1 \sigma_2^{-2b-1} \sigma_1 \sigma_2^{-2c-1}$ ($a = 1$, $b = 0$, $c = 1$ here) and its black graph on the right. The graph is supported in a triangle P with $2a$, $2b$ and $2c$ interior dots on each side, and there is an edge connecting the center of P with each corner vertex of P. The maximal tree T is obtained by removing an edge from each side of P.

The quadratic form Q is represented by the symmetric matrix

$$Q = \begin{pmatrix} -2a-3 & 1 & 1 \\ 1 & -2b-3 & 1 \\ 1 & 1 & -2c-3 \end{pmatrix}$$

The set of characteristic elements

$$\mathcal{C} = \{a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 \,|\, a_1 \equiv a_2 \equiv a_3 \equiv 1 \pmod{2}\},$$

and

$$q(e_1) = (-2a-3, 1, 1), \ q(e_2) = (1, -2b-3, 1), \ q(e_3) = (1, 1, -2c-3).$$

Let

$$\kappa_0 = (1, -1, 1), \ \kappa_1 = \kappa_0 + q(e_1 - e_2) = (-2a-3, 2b+3, 1)$$

be characteristic elements in the affine space $\mathcal{C}_1(V)$. We claim:

Lemma 5.5. *Each of the characteristic elements κ_i maximizes the quadratic form within its equivalent class of characteristic elements, i.e.,*

$$|\kappa_i + 2q(v)|^2 \le |\kappa_i|^2$$

for any $v \in V$, $i = 0, 1$.

Proof. The characteristic elements equivalent to κ_i have the form

$$\kappa = \kappa_0 + q(x_1 e_1 + x_2 e_2 + x_3 e_3), \ x_i \in \mathbb{Z}$$

where

- x_1, x_2, x_3 even, when $i = 0$
- x_1, x_2 odd, x_3 even, when $i = 1$

and

$$\begin{aligned}
|\kappa|^2 &= |\kappa_0|^2 - (2a+3)x_1^2 - (2b+3)x_2^2 - (2c+3)x_3^2 \\
&\quad + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + 2x_1 - 2x_2 + 2x_3 \\
&= |\kappa_0|^2 - \Big(2ax_1^2 + 2bx_2^2 + 2cx_3^2 + \sum_i (x_i - x_{i+1})^2 \\
&\quad + (x_1 - 1)^2 + (x_2 + 1)^2 + (x_3 - 1)^2 - 3\Big).
\end{aligned}$$

Since $a, b, c \geq 0$, it is clear that each $|\kappa_j|^2$ is a maximizer for the given parity on x_i's. $\quad\square$

Note that $\kappa_1 - \kappa_0$ represents one of the homology classes of order 2 in $H^2(Y; \mathbb{Z})$. Hence,

$$h(\beta^{-1}\text{PD}^{-1}(\kappa_1 - \kappa_0)) \geq 2(d(Y, [\kappa_0]) - d(Y, [\kappa_1])) = a + b + 2.$$

Similar arguments apply to the other two homology classes of order 2, we get lower bounds to h given by $b + c + 2$ and $c + a + 2$. By Lemma 2.8, we get lower bounds to the $\mathbb{Z}_2$-Thurston norms given by

$$a + b, \quad b + c, \quad c + a.$$

Meanwhile, the lower bounds are sharp – the lift of one of the disks bounded by a component of the link in $\Sigma(L)$ has the desired Thurston norm.

Acknowledgements

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