Proceedings of 21st Gökova
Geometry-Topology Conference
pp. 250 – 263

On minimal genus problem

Zhongtao Wu

Abstract. We give a brief survey on minimal genus problems in low dimensional topology and explain the joint work with Ni which we discussed at the 2014 Gökova Geometry/Topology Conference.

1. Introduction

In low-dimensional topology, minimal genus problems refer to a type of questions that asks for the least genus needed to represent a given homology class. We list three interesting instances below:

(i) Finding the minimal genus of a connected embedded surface which represents a given 2-dimensional homology class $a \in H_2(X^4, \mathbb{Z})$ in a 4-manifold X^4.
(ii) Finding the minimal genus of an embedded surface which represents a given 2-dimensional homology class $a \in H_2(Y^3, \mathbb{Z})$ in a 3-manifold Y^3.
(iii) Finding the minimal genus of a knot which represents a given 1-dimensional homology class $a \in H_1(Y^3, \mathbb{Z})$ in a 3-manifold Y^3.

1.1. Thom conjecture

The first question on our list, known as the minimal genus problem in 4-manifold, has a long history which involved the development of many of the important techniques used in 4-dimensional topology. In the special case of the complex projective plane, there is the long-standing Thom conjecture on genera of surfaces in $\mathbb{CP}^2$.

Conjecture 1.1 (Thom Conjecture). *The minimal genus of a surface representing a fixed homology class in $\mathbb{CP}^2$ is always realized by a smooth algebraic curve (with either orientation).*

Remember that $H_2(\mathbb{CP}^2) \cong \mathbb{Z}$ is generated by the hyperplane $\mathbb{CP}^1 \subset \mathbb{CP}^2$; a smooth algebraic curve in the complex projective plane of degree d represents the class $d[\mathbb{CP}^1]$ and always has genus $(d-1)(d-2)/2$. We can thus rephrase the conjecture: if Σ is any smooth connected surface representing $d[\mathbb{CP}^1]$, then the genus of the surface Σ satisfies

$$g(\Sigma) \geq (d-1)(d-2)/2.$$

Key words and phrases. rational genus, Turaev norm, Thurston norm.
The author was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. 24300714).

Thom conjecture was first proved by Kronheimer-Mrowka [6] and Morgan-Szabó-Taubes [8] using the Seiberg-Witten invariants. Later, Ozsváth-Szabó obtained a generalization of the conjecture [18]. Known as the symplectic Thom conjecture, it states that: *a symplectic surface of a symplectic* 4*-manifold is genus minimizing within its homology class.*

The general minimal genus problem for 4-manifolds is far from settled. While the *adjunction inequality* from Seiberg-Witten theory gives a lower bound of the genus, it is unclear when it is sharp – are there surfaces that actually realize the equality?

1.2. Thurston norm

We now proceed to the Thurston norm - the second problem on our list. Suppose $S = \sqcup_{i=1}^{n} S_i$ is a surface with connected components S_i. Define

$$\chi_-(S) = \sum_{i=1}^{n} \max\{0, -\chi(S_i)\}. \tag{1}$$

Here, χ is the Euler characteristic, which equals $2 - 2g$ for a closed orientable surface of genus g. Roughly speaking, $\chi_-(S)$ measures the sum of the total genus over all components S_i that are not spheres. This quantity is also called the *complexity* of S, and it can be viewed as an extension of the genus to surfaces that are possibly disconnected or with boundary.

Back to the minimal genus problem, the Thurston norm for every homology class $a \in H_2(Y^3, \mathbb{Z})$ is defined as the minimal complexity needed for representing $[a]$:

$$\|a\| := \min_{[S]=a} \chi_-(S) \tag{2}$$

where S is an embedded orientable surface. Unlike in dimension 4, the embedded surface S representing a given class $a \in H_2(Y^3, \mathbb{Z})$ often needs to be disconnected[1]. Indeed, that is the reason why we minimize complexity instead of genus here.

Thurston norm is also defined for 3-manifolds with boundary, in which case one minimizes complexity over all surfaces with boundary that represent a *relative* homology class $[a] \in H_2(Y, \partial Y, \mathbb{Z})$. Relevant to knot theory is the special case when $Y = S^3 - N(K)$ is the complement of a tubular neighborhood $N(K)$ of a given knot K in S^3. Let $a \in H_2(Y, \partial Y, \mathbb{Z}) \cong \mathbb{Z}$ be a generator, then the surface S that represents a is a Seifert surface of K. So the Thurston norm $\|a\| = 2g(K) - 1$, where $g(K)$ denotes the *Seifert genus* (or *knot genus*) of the knot K.

Unlike dimension 4, we have a pretty good understanding of Thurston norm in dimension 3. There are a number of algorithms that determine the Thurston norm.

---

[1] As an example of the nonexistence of connected surface representative, take $Y^3 = S^2 \times S^1$ and $a = 2 \cdot [S^2] \in H_2(Y, \mathbb{Z})$. In fact, any non-primitive class $a \in H_2(Y^3, \mathbb{Z})$ cannot be represented by a connected surface S.

1.3. Rational genus and Turaev norm

Having seen Thom conjecture and the Thurston norm, we are motivated to discuss the third question on our list: Finding the minimal genus of a knot which represents a given 1-dimensional homology class $a \in H_1(Y, \mathbb{Z})$ in a 3-manifold Y. Foremost we need to define the genus of a knot. Remember that for a knot in S^3, its knot genus is the minimal genus of all Seifert surfaces of K; for a knot in a general 3-manifold, the definition does not directly apply. A necessary and sufficient condition for K to bound an embedded surface in its complement is that $[K] = 1 \in H_1(Y, \mathbb{Z})$. However, if $[K]$ has finite order, one can (only) find an embedded surface in the complement of K whose boundary wraps some number of times around K.

Following this idea, Calegari-Gordon [2] generalized the notion of genus to a rationally null-homologous $K \subset Y$, that is, $[K] = 0 \in H_1(Y, \mathbb{Q})$. Denote $N(K)$ and X_K the tubular neighborhood and the knot exterior of K, respectively. An embedded oriented surface F is called a *rational Seifert surface* of K if $\partial F = l \subset \partial X_K$ and $[l] = p[K] \in H_1(N(K), \mathbb{Z})$ for certain positive integer p. Define the *rational genus*

$$\|K\| := \inf_{\partial F = p[K]} \frac{\chi_-(F)}{2p}, \tag{3}$$

where we take infimum over all p-Seifert surface of K and all positive integers p. Similar to Thurston norm, the reason to use χ_- instead of genus in (3) is that it is a good measure of complexity for surfaces with possible *multi-boundary* components. The coefficient of $2p$ in the denominator accounts for the fact that F wraps p time around K and that the Euler characteristic is about -2 times genus.

With the notion of genus specified, one can make sense of the minimal genus problem. Given $a \in \operatorname{Tors} H_1(Y, \mathbb{Z})$, define the *Turaev norm*

$$\Theta(a) = \min_{K \subset Y,\, [K] = a} 2\|K\|. \tag{4}$$

Up to the factor 2, $\Theta(a)$ is the minimal rational genus of all knots in the torsion class a.

The rest of the paper is devoted to the study of the Turaev norm. In Section 2, we give a quick introduction to Heegaard Floer theory, explaining some of the most relevant aspects of the theory. Then, we state our main result (*Theorem 3.1*) that gives a lower bound of $\Theta(a)$ in terms of invariants from Heegaard Floer homology in Section 3. We also sketch the main idea and steps in the proof, with the aim of providing an intuitive guide to the more formally-written proof presented in [11]. Finally, in Section 4, we describe a few applications of this genus bound in other classical questions of low-dimensional topology. In particular, two more minimal genus problems will be raised and discussed.

2. Heegaard Floer theory

We start with knot Floer homology, which is an invariant originally defined for null-homologous knots in 3-manifolds by Ozsváth-Szabó [14] and independently by

Rasmussen [19]. For the purposes of this article, we need to consider rationally null-homologous knots. Luckily, the theory admits a rather straightforward extension to this case.

Assume that K is a knot in a rational homology sphere Y (so K is automatically a rationally null-homologous knot). There are several different variants of the knot Floer homology of K. The simplest is the hat version, which takes the form of a bi-graded, finitely generated abelian group

$$\widehat{HFK}(Y,K) = \bigoplus_{m,A\in\mathbb{Q}} \widehat{HFK}_m(Y,K,A).$$

Here, m is called the *Maslov* (or *homological*) grading, and A is called the *Alexander* grading[2]. Knot Floer homology satisfies a symmetric relation

$$\widehat{HFK}_m(Y,K,A) \cong \widehat{HFK}_{m-2A}(Y,K,-A), \tag{5}$$

as well as detect the rational genus of a knot, similar to null-homologous knots.

Theorem 2.1 ([9, Theorem 1.1], [14])**.** *Suppose K is a knot in a rational homology sphere Y and $\|K\| > 0$. Then*

$$1 + 2\|K\| = 2 \cdot \max_{\widehat{HFK}(Y,K,A)\neq 0} A.$$

In particular, the theorem in the case of $K \subset S^3$ reduces to the more familiar statement:

$$g(K) = \max\{A \mid \widehat{HFK}(K,A) \neq 0\}.$$

2.1. Dual Heegaard diagram and symmetry

We like to sketch the proof of the isomorphism (5) as it contains one of the key ideas to the establishment of our main result below (Theorem 3.1). Recall that in the holomorphic curves definition of knot Floer homology, a *doubly-pointed Heegaard diagram* $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ consisting of the data (see Figure 1):

- A surface Σ_g of genus g.
- A collection $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_g\}$ of g pairwise disjoint, simple closed curves on Σ_g and a curve collection $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_g\}$ with similar properties.
- Two points z, w disjoint from each other and from the α and β curves.

Given this Heegaard diagram, one can recover Y from the data $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and recover the knot K from the two points z, w; conversely, every knot can be represented by such a Heegaard diagram. In general, one can construct a Heegaard diagram for a knot from a suitable Morse function on the knot complement.

The next step is to define the knot Floer complex $\widehat{CFK}(Y,K)$ from the Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$. Define:

- Generators: correspond to intersection points $\boldsymbol{x} = (x_1, \cdots x_g)$ with $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation σ.

[2] Unlike the case for knots in S^3, the gradings m and A here are generally not integers.

- Differential ∂: counting holomorphic curves
$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ n_z(\phi)=n_w(\phi)=0}} \#\widehat{\mathcal{M}}(\phi) \cdot \mathbf{y}$$
- Gradings: Each generator $\boldsymbol{x}$ can be endowed with an Alexander grading A and a Maslov grading m.

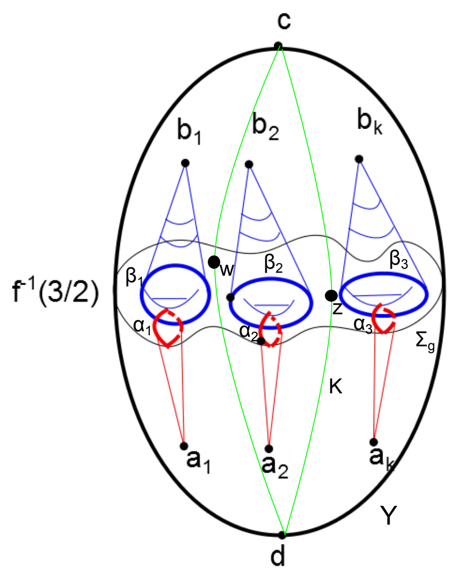

FIGURE 1. A Morse function interpretation of the Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ associated to $K \subset Y$. The closed curve passing through c and d represents the knot K.

The idea behind the proof of (5) can be summarized in one line, that is, *turn the Morse function upside-down*. Equivalently, this has the effect of modifying the Heegaard diagram for (Y, K):
$$(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w) \longrightarrow (-\Sigma_g, \boldsymbol{\beta}, \boldsymbol{\alpha}, w, z),$$
where we simultaneously change the orientation of the surface Σ_g, switch the collection of $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$ curves, and switch the base points z, w. The resulting diagram is called its *dual Heegaard diagram*. Observe that the generators and the differential ∂ remain the same in the knot Floer complex $\widehat{CFK}(Y, K)$ associated to the dual Heegaard diagram; only the gradings of the generators may change. With more work [11, Section 3], one can show that the induced isomorphism on homology shifts grading as desired (5).

2.2. Spin^c-structure and correction term

Next, we outline relevant background on Heegaard Floer homology of 3-manifolds, whose definition is very similar to that of knot Floer homology; in fact, one may think of knot Floer homology as a relative version of Heegaard Floer homology, associated to a pair consisting of a 3-manifold and a knot in it. Starting with a Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ with the same properties as before except that we do not have the other base point z, the Heegaard Floer complex $\widehat{CF}(Y)$ is generated by intersection points $\boldsymbol{x} = (x_1, \cdots x_g)$ with the differential ∂ counting holomorphic curves

$$\partial \mathbf{x} = \sum_{\mathbf{y}} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ n_w(\phi)=0}} \#\widehat{\mathcal{M}}(\phi) \cdot \mathbf{y}$$

The analogue of the decomposition of $\widehat{HFK}$ into Alexander gradings is a decomposition of $\widehat{HF}(Y)$ according to Spin^c structures:

$$\widehat{HF}(Y) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s}).$$

Without going to the detailed definition of Spin^c structures, let us note that the set of all Spin^c structures is an affine space that can be identified with the second cohomology $\mathrm{Spin}^c(Y) \cong H^2(Y, \mathbb{Z})$.

There is the same Maslov grading on $\widehat{CF}(Y)$ when Y is a rational homology sphere, which induces an absolute $\mathbb{Q}$-grading on the Heegaard Floer homology group $\widehat{HF}(Y)$. The *correction term* $d(Y, \mathfrak{s}) \in \mathbb{Q}$ is an invariant of Y that associates to each Spin^c structure a distinguished grading on the group $\widehat{HF}(Y, \mathfrak{s})$ [13].

3. Main results

Call a rational homology sphere Y an *L-space* if $\operatorname{rank} \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$. Including all lens spaces, they are the simplest 3-manifolds from the perspective of Heegaard Floer homology. A rationally null-homologous knot K in a 3-manifold is called *Floer simple* if $\operatorname{rank} \widehat{HFK}(Y, K) = \operatorname{rank} \widehat{HF}(Y)$. Examples of Floer simple knots include simple knots in lens spaces, which we will define presently.

This section is devoted to explaining the following main result from [11].

Theorem 3.1 ([11, Theorem 1.1, Proposition 5.1])**.** *(a) Let K be a Floer simple knot in an L-space Y. If the rational genus $\|K\| > 0$, then*

$$1 + 2||K|| = \max_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \big\{ d(Y, \mathfrak{s}) - d(Y, J\mathfrak{s} + \mathrm{PD}[K]) \big\}. \tag{6}$$

More generally,

(b) Let K be a knot in a rational homology sphere Y. Then

$$1 + 2||K|| \geq \max_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \big\{ d(Y, \mathfrak{s}) - d(Y, J\mathfrak{s} + \mathrm{PD}[K]) \big\}. \tag{7}$$

Note that the right hand side of (7) depends only on the manifold Y and the homology class of K. Using the identification $\mathrm{Spin}^c(Y) \cong H^2(Y, \mathbb{Z})$, we obtain a lower bound of $1 + \Theta(a)$ for the homology class $a = [K]$ (recall the definition (4)). Part (a) of the theorem says basically that this bound is sharp for a Floer simple knot in an L-space. The technical assumption $\|K\| > 0$ comes from Theorem 2.1.

3.1. Simple knots in lens spaces

As a special example, consider simple knots in lens spaces. Remember that a lens space $L(p, q)$ is an L-space. The notion of *simple knots* in lens space is describe as follows. In Figure 2, we draw the standard Heegaard diagram of a lens space $L(p, q)$. Here the opposite side of the rectangle is identified to give a torus, and there are one α and one β curve on the torus, intersecting at p points and dividing the torus into p regions. We then put two base points z, w and connect them in a proper way on the torus. Such a simple closed curve colored in green is called a simple knot. There is an alternative way of describing simple knots without referring to the Heegaard diagram: Take a genus 1 Heegaard splitting $U_0 \cup U_1$ of the lens space $L(p, q)$. Let D_0, D_1 be meridian disks in U_0, U_1 such that $\partial D_0 \cap \partial D_1$ consists of exactly p points. A simple knot in $L(p, q)$ is either the unknot or the union of two arcs $a_0 \subset D_0$ and $a_1 \subset D_1$.

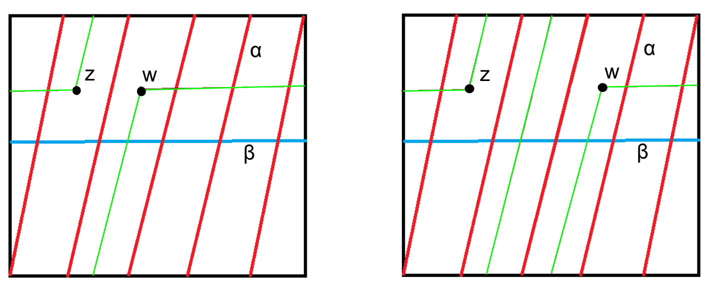

FIGURE 2. Heegaard diagrams of simple knots in $L(5, 1)$. The red α curve and the blue β curve intersect at 5 points. The simple closed green curves are simple knots representing two different homology classes in $H_1(L(5, 1); \mathbb{Z}) \cong \mathbb{Z}/5\mathbb{Z}$.

We claim that simple knots are Floer simple. This follows from the observation that the knot Floer complex $\widehat{CFK}(L(p, q), K)$ is generated by exactly the p intersection points of α and β curves. Moreover, there is exactly one simple knot in each homology class in $H_1(L(p, q))$ - this corresponds to the different relative positions of z and w in Figure 2. As a consequence of Theorem 3.1, we conclude that simple knots are indeed "simple", according to the measure of complexity in genus.

Corollary 3.2. *Simple knots in lens spaces are genus minimizers in their homology classes.*

The above statement can be viewed as a 1-lower-dimensional analog of the symplectic Thom conjecture described in Section 1.1. In view of the fact that the unknot is the only genus 0 knot in S^3, one may wonder whether simple knots are the *unique* genus minimizers in their homology classes. This question is answered negatively by Greene-Ni [3].

3.2. Floer simple knots in L-spaces

We give below a complete proof of Theorem 3.1(a).

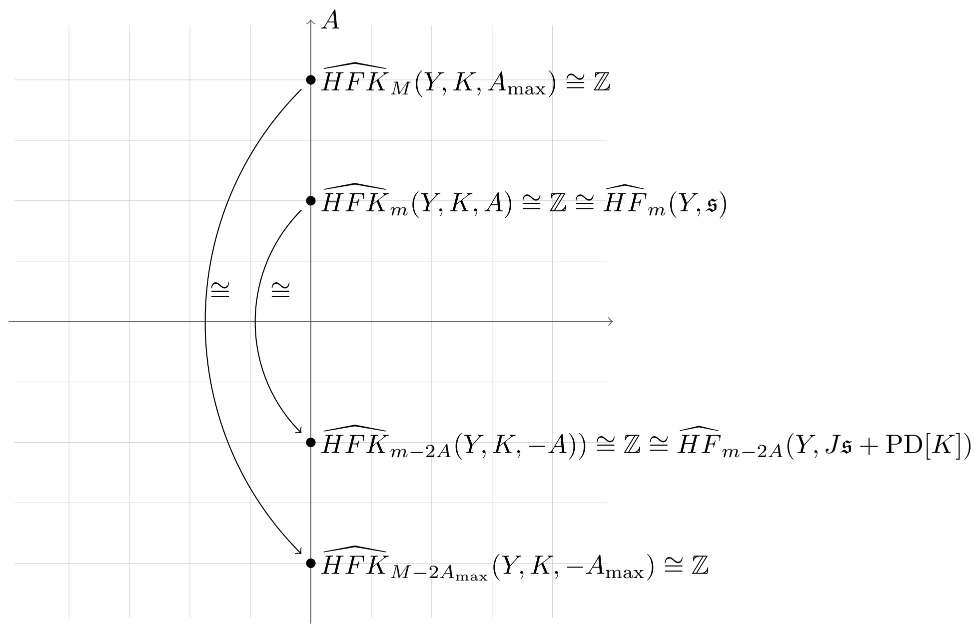

FIGURE 3. $\widehat{HFK}(Y,K)$ of a Floer simple knot in an L-space. Following (5), there are isomorphisms $\widehat{HFK}_m(Y,K,A) \cong \widehat{HFK}_{m-2A}(Y,K,-A)$

Proof of Theorem 3.1(a). Since $K \subset Y$ is a Floer simple knot in an L-space, the non-vanishing knot Floer homology $\widehat{HFK}(Y,K,A)$ for each Alexander grading A is isomorphic to the Heegaard Floer homology $\widehat{HF}(Y,\mathfrak{s})$ for a corresponding Spin^c structure $\mathfrak{s}$, both of which are isomorphic to $\mathbb{Z}$. Furthermore, there is a symmetry between $\widehat{HFK}_m(Y,K,A)$ and $\widehat{HFK}_{m-2A}(Y,K,-A)$. Remember that such an isometry is induced from the pair of a

Heegaard diagram and its dual $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w) \longrightarrow (-\Sigma_g, \boldsymbol{\beta}, \boldsymbol{\alpha}, w, z)$. If we restrict to the single-pointed Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, w) \longrightarrow (-\Sigma_g, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$, the intersection points of Spin^c structure $\mathfrak{s}$ in one diagram will correspond to the Spin^c structure $J\mathfrak{s} + \mathrm{PD}[K]$ in the other [12, Section 2.6]. See Figure 3.

On the other hand, the correction term $d(Y, \mathfrak{s})$ of an L-space is the same as the grading of the unique element in $\widehat{HF}(Y, \mathfrak{s})$. Hence

$$d(Y, \mathfrak{s}) - d(Y, J\mathfrak{s} + \mathrm{PD}[K]) = m - (m - 2A) = 2A.$$

Taking maximum of the left hand side over all Spin^c structures $\mathfrak{s}$, we have

$$\max_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \big\{ d(Y, \mathfrak{s}) - d(Y, J\mathfrak{s} + \mathrm{PD}[K]) \big\} = 2A_{\max}.$$

By Theorem 2.1, $1 + 2\|K\| = 2A_{\max}$. This proves the identity (6). $\square$

3.3. General case

For general knots in rational homology spheres, the knot Floer homology $\widehat{HFK}(Y, K)$ can no longer be identified with the group $\widehat{HF}(Y)$; neither can the correction term $d(Y, \mathfrak{s})$ be directly read off from $\widehat{HF}(Y, \mathfrak{s})$. A complete proof of Theorem 3.1(b) is indeed quite technical and it involves intensive diagram chasing of the knot Floer chain complex. Instead of presenting it in full detail here, we highlight the three main steps below.

(a) Express the correction term $d(Y, \mathfrak{s})$ as a min-max function of the Maslov grading of the generators of the chain complex $CFK^\infty(Y, K, \mathfrak{s})$ [11, Proposition 4.2].

(b) Use the associated dual diagram to express $d(Y, J\mathfrak{s} + \mathrm{PD}[K])$ as a min-max function of the Maslov grading of the generators of the chain complex $CFK^\infty(Y, K, \mathfrak{s})$ [11, Equation (11)].

(c) Relate Maslov and Alexander gradings using the isomorphism

$$\widehat{HFK}_m(Y, K, A) \cong \widehat{HFK}_{m-2A}(Y, K, -A).$$

Then apply Theorem 2.1 and a min-max inequality [11, Lemma 4.4] to obtain the desired genus bound.

3.4. An Example

In order to obtain interesting rational genus bounds from Theorem 3.1, we need effective algorithms to compute the Heegaard Floer correction terms. We consider below an example of surgeries on knots. Another example concerning double branched cover of knots will be presented in Section 4.1.

Suppose $K \subset S^3$ is a knot with genus g. Let $S^3_p(K)$ be the manifold obtained by p-surgery on K, and $K' \subset S^3_p(K)$ be the dual knot of the surgery. Then a Seifert surface of K in S^3, say F, is also a rational Seifert surface of K' in $S^3_p(K)$; indeed, the rational genus $\|K'\| = \frac{2g-1}{2p}$. On the other hand, one can show that

$$d(S^3_p(K), g) - d(S^3_p(K), g-1) = 1 + \frac{2g-1}{p}$$

when $p \geq 2g$ and K' was not genus minimizing[3]. It then follows from (7) that $\|L\| \geq \frac{2g-1}{2p}$ for any knots $[L] = [K']$. Hence, K' has to be a genus minimizer in its homology class with

$$\Theta([K']) = 2\|K'\| = \frac{2g-1}{p}$$

for large p-surgery.

4. Further applications

The rational genus bound is related to yet another minimal genus problem in low-dimensional topology, resembling the non-orientable version of the Thurston norm:

(ii') Finding the minimal genus of an embedded surface which represents a given 2-dimensional homology class $A \in H_2(Y^3, \mathbb{Z}_2)$ in a 3-manifold Y^3.

4.1. $\mathbb{Z}_2$-Thurston norm

Inspired by the definition of the Thurston norm (2), we denote

$$\|A\|_{\mathbb{Z}_2} := \min_{[S]=A} \chi_-(S)$$

for the $\mathbb{Z}_2$-*Thurston norm* of a homology class $A \in H_2(Y, \mathbb{Z}_2)$, where S is an embedded, *possibly non-orientable*, surface. Assume Y to be a rational homology 3-sphere and $\beta : H_2(Y, \mathbb{Z}_2) \to H_1(Y, \mathbb{Z})$ the Bockstein homomorphism. We claim that the $\mathbb{Z}_2$-Thurston norm of A is the same as the Turaev norm of $a = \beta(A) \in H_1(Y, \mathbb{Z})$, up to a factor of 2:

$$\|A\|_{\mathbb{Z}_2} = 2\Theta(a) \tag{8}$$

Equation (8) is proved by explicit constructions; refer to Figure 4. Suppose F is a minimal genus rational Seifert surface of the knot K representing $a \in H_1(Y, \mathbb{Z})$. Since a is in the image of the Bockstein homomorphism, it has order 2 in $H_1(Y, \mathbb{Z})$. Consequently, the boundary of F is either a connected closed curve or a pair of coherently oriented closed curves. We can thus close off F by gluing on a Möbius band or an annulus, respectively. The resulting closed surface $\hat{F}$ represents the homology class A. Conversely, suppose $\hat{F}$ is a minimal complexity surface representing $A \in H_2(Y, \mathbb{Z}_2)$. Since $\hat{F}$ is a non-orientable surface, it contains a torsion curve with the property that the complement of this curve is an open orientable surface. Let this torsion curve be our knot K; then $[K] = a \in H_1(Y, \mathbb{Z})$. A rational Seifert surface F can be obtained by deleting the tubular neighborhood of K from $\hat{F}$. Finally, note that

$$\|A\|_{\mathbb{Z}_2} = \chi_-(\hat{F}) = \chi_-(F) = 4\|K\| = 2\Theta(a).$$

This completes the proof.

---

[3]This follows from the surgery formula for correction terms $d(L(p,1), i) = \frac{(2i-p)^2-p}{4p}$ and $d(S^3_p(K), i)$ is given by $d(L(p,1), i) - 2\max\{V_i, H_{i-p}\}$. See [11] for details.

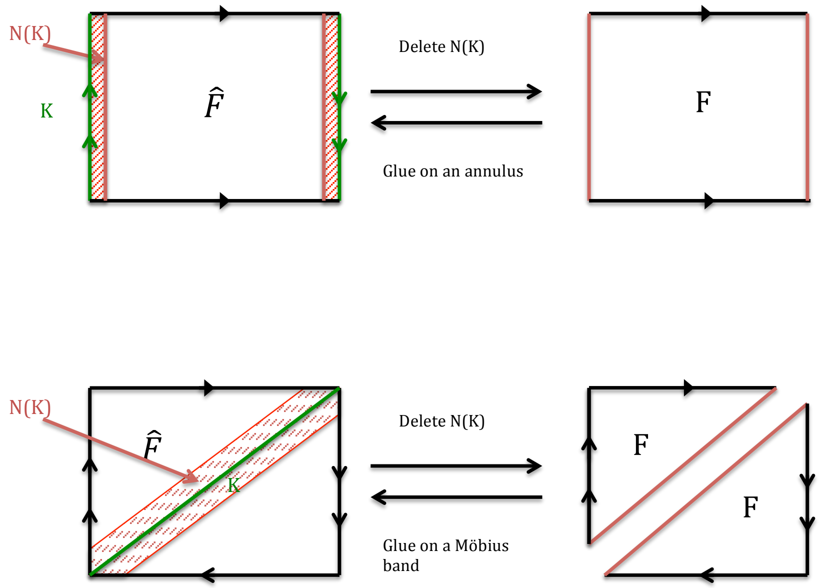

FIGURE 4. These are explicit constructions for the Klein bottle and the real projective plane, respectively. Removing a tubular neighborhood of a torsion curve K results in an open orientable surface, homeomorphic to a cylinder and a disk respectively. Conversely, gluing on an annulus or a Möbius band recovers the original non-orientable surface $\hat{F}$. All other higher-genus non-orientable surfaces are connected sums of orientable surfaces with Klein bottle or real projective plane.

As an application of (8), one can use the correction terms from Theorem 3.1 to bound $\mathbb{Z}_2$-Thurston norm. In particular, this reproves Bredon-Wood's [1] sharp minimal genus bound of non-orientable surfaces embedded in arbitrary lens spaces $L(2k, q)$.

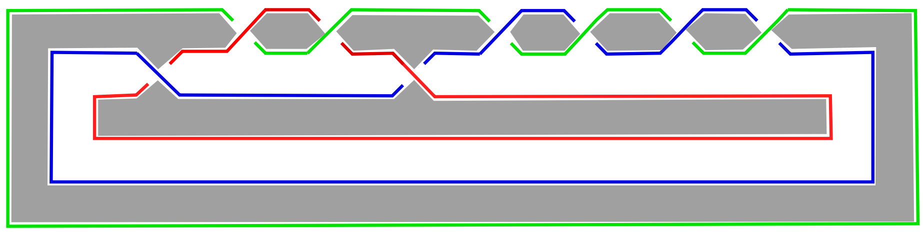

FIGURE 5. The closure of the pure braid $\sigma = \sigma_1\sigma_2^{2a}\sigma_1\sigma_2^{2b}$ ($a = -1$, $b = -2$ here)

As another example, consider the double branched cover of a closed 3-braids. Let σ_1, σ_2 be the two standard generators of B_3.

Example 4.1. $Y = \Sigma(L)$, where L is the closure of the braid $\sigma_1\sigma_2^{2a}\sigma_1\sigma_2^{2b}$, $ab > 0$.

Case 1: When both a, b are negative (Figure 5). This case is carefully studied in [10]. As L is a pure braid, we have $H_2(\Sigma(L), \mathbb{Z}_2) \cong \mathbb{Z}_2^2$ and there are three nonzero $\mathbb{Z}_2$-homology classes. Moreover, its correction terms can be determined from a general algorithm for double branched cover of an alternating link [17, Theorem 3.4]. In particular, there are four distinguished Spin^c structures $\mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ whose correction terms are equal to $\frac{1}{2}$, $\frac{a}{2}$, $\frac{b}{2}$, $\frac{a+b+1}{2}$, respectively, so that $\mathfrak{s}_1 - \mathfrak{s}_0, \mathfrak{s}_2 - \mathfrak{s}_0, \mathfrak{s}_3 - \mathfrak{s}_0$ represent the three different homology classes of order 2 in $H_1(Y, \mathbb{Z})$. From (8) we get the $\mathbb{Z}_2$-Thurston norm bound $-a-1$, $-b-1$ and $-a-b-2$. These bounds are in fact sharp: the lift of the disk bounded by each component of the link in $\Sigma(L)$ has the desired complexity[4].

Case 2: When both a, b are positive. Although not alternating, we observe that L is a pretzel link $P(-2, 2a, 2b)$ whose double branched cover is known to be the Seifert fibered space $Y = \{0; (-2, 1), (2a, 1), (2b, 1)\}$. Its orientation reversing $-Y$ has a negative-definite plumbing graph (Figure 6) when $a, b > 2$. There is a general algorithm for computing correction terms of negative-definite plumbed 3-manifolds by maximizing a certain associated quadratic form to the plumbed graph [16, Corollary 1.5]. The resulting $\mathbb{Z}_2$-Thurston norm bounds are $a-3$, $b-3$, $a+b-4$, respectively. They are also sharp bounds each of which can be realized by the lift of the disk bounded by a component of the link in $\Sigma(L)$.

[4] For example, the disk bounded by the green curve in Figure 5 intersects with the red and blue curves at $-a-b$ points. By Riemann-Hurwitz formula, the Euler characteristic of its double branched cover is $2\chi(D^2) + (a+b) = 2 + a + b$.

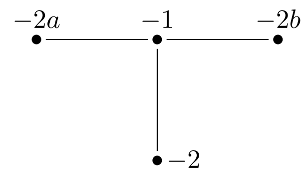

FIGURE 6. plumbing diagram for $-Y$

At this point, some readers may wonder why one would care about those strange non-orientable surfaces. It turns out that they have an unexpected connection to the complexity of a closed three-manifold, denoted by $C(Y)$. Defined as the minimal size of a triangulation, complexity is a very difficult invariant to determine. In [4], Jaco, Rubinstein and Tillmann give a lower bound of the complexity for a closed, orientable, irreducible, atoroidal, connected 3-manifold in terms of its $\mathbb{Z}_2$-Thurston norm:

$$C(Y) \geq 2 + \sum_{A \in H} \|A\|_{\mathbb{Z}_2},$$

where $H \subset H_2(Y;\mathbb{Z}_2)$ is a rank 2 subgroup. In joint work with Yi Ni, we used this relation to estimate the complexity of two families of 3-manifolds [10].

4.2. Four-ball rational genus

We conclude this brief article by posting yet another related question on minimal genus.

(iii') Finding the minimal four-ball genus of a knot which represents a given 1-dimensional homology class $a \in H_1(Y^3, \mathbb{Z})$ in a 3-manifold Y^3.

If K is a rationally null-homologous knot $K \subset Y \times \{1\} \subset Y \times [0,1]$, the rational four-ball genus $\|K\|_4$ is defined as the infimum of $\chi_-(F)/2p$ over all embedded surfaces $F \subset Y \times [0,1]$ whose boundary wraps multiple times around K. Clearly, $\|K\|_4 \leq \|K\|$. The question is whether Theorem 3.1 remain true when we replace genus by four-ball genus. Certain special cases are affirmatively proved by Levine-Ruberman-Strle [7].

Acknowledgements: The author likes to thank the organizers of the 2014 Gökova Geometry/Topology Conference for the invitation and their hospitality during this terrific event.

References

[1] G. E. Bredon, J. W. Wood, *Non-orientable surfaces in orientable 3-manifolds*, Invent. Math. 7 (1969), 83–110.

[2] D. Calegari, C. Gordon, *Knots with small rational genus*, Comment. Math. Helv. 88 (2013), no. 1, 85–130.

[3] J. Greene, Y. Ni, *Non-simple genus minimizers in lens spaces*, submitted, available at arXiv:1305.0517.

[4] W. Jaco, J. H. Rubinstein, S. Tillmann, *$\mathbb{Z}_2$–Thurston norm and complexity of 3-manifolds*, Math. Ann. 356 (2013), no. 1, 1–22.

[5] P. Kronheimer, T. Mrowka, *Gauge theory for embedded surfaces, I*, Topology 32 (1993), no. 4, 773–826.

[6] P. Kronheimer, T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. 1 (1994), no. 6, 797–808.

[7] A. Levine, D. Ruberman, S. Strle, *Non-orientable surfaces in homology cobordisms*, available at arXiv:1310.8516, to appear in Geom. Topol.

[8] J. Morgan, Z. Szabó, C. Taubes, *A product formula for the Seiberg–Witten invariants and the generalized Thom conjecture*, J. Differential Geom. 44 (1996), no. 4, 706–788.

[9] Y. Ni, *Link Floer homology detects the Thurston norm*, Geom. Topol. 13 (2009), no. 5, 2991–3019.

[10] Y. Ni, Z. Wu, *Correction terms, $\mathbb{Z}_2$-Thurston norm, and triangulations*, available at arXiv:1410.5342

[11] Y. Ni, Z. Wu, *Heegaard Floer correction terms and rational genus bounds*, Adv. Math. 267 (2014), 360–380.

[12] P. Ozsváth, Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2), 159 (2004), no. 3, 1027–1158.

[13] P. Ozsváth, Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003), no. 2, 179–261.

[14] P. Ozsváth, Z. Szabó, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004), no. 1, 58–116.

[15] P Ozsváth, Z Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004), 311–334.

[16] P Ozsváth, Z Szabó, *On knot Floer homology of plumbed three-manifolds*, Geor, Topol. 7 (2003) 185-224

[17] P. Ozsváth, Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. 194 (2005), no. 1, 1–33.

[18] P. Ozsváth, Z. Szabó, *The symplectic Thom conjecture*, Ann. of Math. 151 (1): 93-124.

[19] J Rasmussen, *Floer homology and knot complements*, PhD Thesis, Harvard University (2003), available at arXiv:math.GT/0306378

[20] W. Thurston, *A norm for the homology of* 3*–manifolds*, Mem. Amer. Math. Soc. 59 (1986), no. 339, i–vi and 99–130.

[21] V. Turaev, *A function on the homology of* 3*–manifolds*, Algebr. Geom. Topol. 7 (2007), 135–156.

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: ztwu@math.cuhk.edu.hk