Invariant Metrics with Nonnegative Curvature on Compact Lie Groups

Nathan Brown, Rachel Finck, Matthew Spencer, Kristopher Tapp and Zhongtao Wu

Abstract. We classify the left-invariant metrics with nonnegative sectional curvature on SO(3) and $U(2)$.

1 Introduction

Every compact Lie group admits a bi-invariant metric, which has nonnegative sectional curvature. This observation is the starting point of almost all known constructions of nonnegatively and positively curved manifolds. The only exceptions are found in [1, 6, 12], and many of these exceptions admit a different metric which does come from a bi-invariant metric on a Lie group; see [10].

In order to generalize this cornerstone starting point, we consider the following problem: for each compact Lie group G, classify the left-invariant metrics on G which have nonnegative sectional curvature. The problem is well motivated because each such metric, g_l, provides a new starting point for several known constructions.

For example, suppose $H \subset G$ is a closed subgroup. The left-action of H on (G, g_l) is by isometries. The quotient space $H\backslash G$ inherits a metric of nonnegative curvature, which is generally inhomogeneous. Geroch proved [5] that this quotient metric can only have positive curvature if $H\backslash G$ admits a normal homogeneous metric with positive curvature, so new examples of positive curvature cannot be found among such metrics. However, it seems worthwhile to search for new examples with quasi- or almost-positive curvature (meaning positive curvature on an open set, or on an open and dense set).

A second type of example occurs when $H \subset G$ is a closed subgroup, and F is a compact Riemannian manifold of nonnegative curvature on which H acts isometrically. Then H acts diagonally on $(G, g_l) \times F$. The quotient, $M = H\backslash(G \times F)$, inherits a metric with nonnegative curvature. M is the total space of a homogeneous F-bundle over $H\backslash G$. Examples of this type with quasi-positive curvature are given in [9]. It is not known whether new positively curved examples of this type could exist.

In this paper, we explore the problem for small-dimensional Lie groups. In Section 3, we classify the left-invariant metrics on SO(3) with nonnegative curvature. This case is very special because the bi-invariant metric has positive curvature, and therefore so does any nearby metric. The solution is straightforward and known to

some, but does not appear in the literature. In Section 4, we classify the left-invariant metrics on $U(2)$ with nonnegative curvature. This classification does not quite reduce to the previous, since not all such metrics lift to product metrics on $U(2)$'s double-cover $S^1 \times \mathrm{SU}(2)$.

The only examples in the literature of left-invariant metrics with nonnegative curvature on compact Lie groups are obtained from a bi-invariant metric by shrinking along a subalgebra (or a sequence of nested subalgebras). We review this construction in Section 5. Do all examples arise in this way? In Section 6, we outline a few ways to make this question precise. Even for $U(2)$, the answer is essentially no. Also, since each of our metrics on $U(2)$ induces via Cheeger's method a left-invariant metric with nonnegative curvature on any compact Lie group which contains a subgroup isomorphic to $U(2)$, our classification for small-dimensional groups informs the question for large-dimensional groups.

2   Background

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let g_0 be a bi-invariant metric on G. Let g_l be a left-invariant metric on G. The values of g_0 and g_l at the identity are inner products on $\mathfrak{g}$ which we denote as $\langle \cdot , \cdot \rangle_0$ and $\langle \cdot , \cdot \rangle_l$. We can always express the latter in terms of the former:

$$\langle A, B \rangle_l = \langle \phi(A), B \rangle_0$$

for some positive-definite self-adjoint $\phi \colon \mathfrak{g} \to \mathfrak{g}$. The eigenvalues and eigenvectors of ϕ are called eigenvalues and eigenvectors vectors of the metric g_l.

There are several existing formulas for the curvature tensor of g_l at the identity, which we denote as $R \colon \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Püttmann's formula [8] says that for all $x, y, z, w \in \mathfrak{g}$,

$$(2.1) \quad \langle R(x, y)z, w \rangle_l = -\frac{1}{4} \Big(\langle [\phi x, y], [z, w] \rangle_0 + \langle [x, \phi y], [z, w] \rangle_0$$

$$+ \langle [x, y], [\phi z, w] \rangle_0 + \langle [x, y], [z, \phi w] \rangle_0 \Big)$$

$$- \frac{1}{4} \Big(\langle [x, w], [y, z] \rangle_l - \langle [x, z], [y, w] \rangle_l - 2 \langle [x, y], [z, w] \rangle_l \Big)$$

$$- \Big(\langle B(x, w), \phi^{-1} B(y, z) \rangle_0 - \langle B(x, z), \phi^{-1} B(y, w) \rangle_0 \Big) ,$$

where B is defined by $B(x, y) = \frac{1}{2}([x, \phi y] + [y, \phi x])$.

Sectional curvatures can be calculated from this formula. An alternative formula by Milnor for the sectional curvatures of g_l is found in [7]. If $\{e_1, \ldots, e_n\} \subset \mathfrak{g}$ is a g_l-orthonormal basis, then the sectional curvature of the plane spanned by e_1 and e_2 is:

$$(2.2) \quad \kappa(e_1, e_2) = \sum_{k=1}^{n} \Big(\frac{1}{2} \alpha_{12k} \left(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12} \right)$$

$$- \frac{1}{4} \left(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12} \right) \left(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12} \right) - \alpha_{k11} \alpha_{k22} \Big) ,$$

where $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle_l$ are the *structure constants*.

3   **Metrics on** SO(3)

Let $G = \mathrm{SO}(3)$, which has Lie algebra $\mathfrak{g} = so(3)$. We use the following bi-invariant metric:

$$\langle A, B \rangle_0 = (1/2) \operatorname{trace}(AB^T) \text{ for } A, B \in \mathfrak{g}.$$

A left-invariant metric g_l on G has three eigenvalues, $\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$. The metric can be re-scaled so that $\lambda_3 = 1$.

Proposition 3.1   *A left-invariant metric on* SO(3) *with eigenvalues* $\{\lambda_1^2, \lambda_2^2, 1\}$ *has nonnegative sectional curvature if and only if the following three inequalities hold:*

(3.1)
$$2\lambda_1^2 + 2\lambda_2^2 - 3 + \lambda_1^4 - 2\lambda_1^2\lambda_2^2 + \lambda_2^4 \geq 0,$$
$$2\lambda_1^2 - 3\lambda_1^4 + 2\lambda_1^2\lambda_2^2 - 2\lambda_2^2 + 1 + \lambda_2^4 \geq 0,$$
$$2\lambda_2^2 - 3\lambda_2^4 + 2\lambda_1^2\lambda_2^2 - 2\lambda_1^2 + 1 + \lambda_1^4 \geq 0.$$

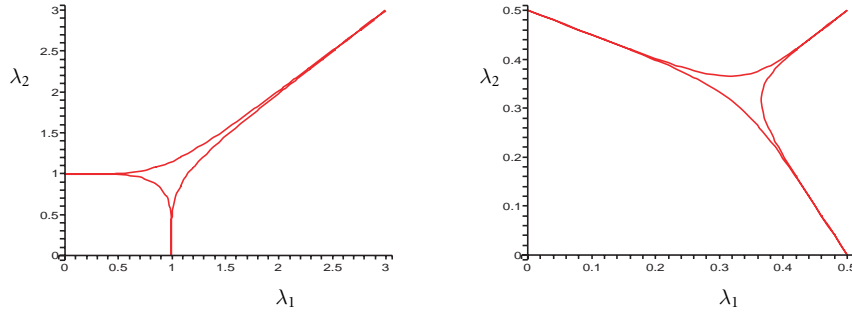Figure 1: The values of λ_1 and λ_2 for which SO(3) has nonnegative sectional curvature, when $\lambda_3 = 1$ (left) and when $\sum \lambda_i = 1$ (right)

The intersection of the graph in Figure 1 (left) with the identity function $\lambda_1 = \lambda_2$ is interesting. The eigenvalues $\{\lambda^2, \lambda^2, 1\}$ yield nonnegative curvature if and only if $\lambda^2 \geq 3/4$. Re-scaled, this means that the eigenvalues $\{1, 1, \mu^2\}$ yield nonnegative curvature if and only if $\mu^2 \leq 4/3$. Metrics with two equal eigenvalues on SO(3) (or on its double-cover S^3) are commonly called *Berger metrics*. They are obtained by scaling the Hopf fibers by a factor of μ^2. The nonnegative curvature cut-off $\mu^2 \leq 4/3$ is well known.

Proof Let g_l be a left-invariant metric on SO(3). Let $\{u_1, u_2, u_3\} \subset \mathfrak{g}$ denote an oriented g_0-orthonormal basis of eigenvectors of g_l, with eigenvalues $\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$. Then,

$$\left\{ e_1 = \frac{u_1}{\lambda_1}, e_2 = \frac{u_2}{\lambda_2}, e_3 = \frac{u_3}{\lambda_3} \right\}$$

is a g_l-orthonormal basis of $\mathfrak{g}$. The Lie bracket structure of $\mathfrak{g}$ is given by:

(3.2) $$[u_1, u_2] = u_3, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = u_2.$$

The structure constants, $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle_l$, are:

$$\alpha_{123} = -\alpha_{213} = \frac{\lambda_3}{\lambda_1 \lambda_2}, \quad \alpha_{231} = -\alpha_{321} = \frac{\lambda_1}{\lambda_2 \lambda_3}, \quad \alpha_{312} = -\alpha_{132} = \frac{\lambda_2}{\lambda_1 \lambda_3},$$

with all others equal to zero. Equation (2.2) gives:

$$\kappa(e_1, e_2) = \frac{1}{2} \frac{\lambda_3}{\lambda_1 \lambda_2} \left(-\frac{\lambda_3}{\lambda_1 \lambda_2} + \frac{\lambda_1}{\lambda_2 \lambda_3} + \frac{\lambda_2}{\lambda_1 \lambda_3} \right)$$

$$- \frac{1}{4} \left(\frac{\lambda_3}{\lambda_1 \lambda_2} - \frac{\lambda_1}{\lambda_2 \lambda_3} + \frac{\lambda_2}{\lambda_1 \lambda_3} \right) \left(\frac{\lambda_3}{\lambda_1 \lambda_2} + \frac{\lambda_1}{\lambda_2 \lambda_3} - \frac{\lambda_2}{\lambda_1 \lambda_3} \right).$$

When $\lambda_3 = 1$, the inequality $\kappa(e_1, e_2) \geq 0$ is equivalent to the first inequality of (3.1). The second and third inequalities are equivalent to $\kappa(e_2, e_3) \geq 0$ and $\kappa(e_1, e_3) \geq 0$, respectively.

It remains to prove the following: if the planes spanned by pairs of $\{e_1, e_2, e_3\}$ are nonnegatively curved, then all planes are nonnegatively curved. It is immediate from Püttmann's formula (2.1) that $R(e_i, e_j)e_k = 0$ whenever i, j, k are distinct. This implies that the curvature operator is diagonal in the basis $\{e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1\}$. The result follows. ∎

To classify up to isometry the left-invariant metrics of nonnegative sectional curvature on SO(3) it remains to observe the following proposition:

Proposition 3.2 *Two left-invariant metrics on* SO(3) *are isometric if and only if they have the same eigenvalues.*

Proof Suppose first that $f\colon (\text{SO}(3), g_1) \to (\text{SO}(3), g_2)$ is an isometry, where g_1 and g_2 are left-invariant metrics. We lose no generality in assuming that $f(I) = I$. The linear map $\wedge^2(\mathfrak{g}) \to \wedge^2(\mathfrak{g})$ induced by $df_I\colon \mathfrak{g} \to \mathfrak{g}$ must send eigenvectors of the curvature operator of g_1 to eigenvectors of the curvature operator of g_2. Since these eigenvectors are wedge products of eigenvectors of the metrics, $df_I\colon \mathfrak{g} \to \mathfrak{g}$ must send the eigenvectors of g_1 to the eigenvectors of g_2. Since df_I preserves lengths, the two metrics must have the same eigenvalues.

Conversely, suppose that g_1 and g_2 are left-invariant metrics on SO(3) with eigenvectors $E = \{e_1, e_2, e_3\}$ and $F = \{f_1, f_2, f_3\}$, respectively. Assume that the eigenvalues of the two metrics agree. The bases E and F can be chosen to be g_0-orthonormal and to have the same orientation. There exists $x \in \text{SO}(3)$ such that $\text{Ad}_x\colon \mathfrak{g} \to \mathfrak{g}$ sends E to F. It is straightforward to verify that $C_x\colon (\text{SO}(3), g_1) \to (\text{SO}(3), g_2)$ (conjugation by x) is an isometry. ∎

4 Metrics on $U(2)$

Let $G = U(1) \times \mathrm{SU}(2)$. In this section, we classify the left-invariant metrics on G with nonnegative curvature. This is equivalent to solving the problem for $U(2)$, which has the same Lie algebra.

Let g_0 denote a bi-invariant metric on G. Suppose that g_l is a left-invariant metric on G. The Lie algebra of G is $\mathfrak{g} = u(1) \oplus su(2)$. The factors $u(1)$ and $su(2)$ are orthogonal with respect to g_0. If they are orthogonal with respect to g_l, then g_l is a product metric, so the problem reduces to the one from the previous section.

We will see, however, that nonnegatively curved metrics need not be product metrics. This might be surprising, since when g_l is nonnegatively curved, the splitting theorem [2, Ch. 4] implies that the pull-back of g_l to the universal cover $\mathbb{R} \times \mathrm{SU}(2)$ is isometric to a product metric. The subtlety is that the metric product structure and the group product structure need not agree. Further, the metric g_l on G need only be locally isometric to a product metric, not globally.

Let $\{u_1, u_2, u_3\}$ denote an orthonormal basis of eigenvectors of the restriction of g_l to the $su(2)$ factor. Let $\{\lambda_1^2, \lambda_2^2, \lambda_3^2\}$ denote the corresponding eigenvalues, which we call "restricted eigenvalues" of g_l. Notice that

$$\left\{ e_1 = \frac{u_1}{\lambda_1}, e_2 = \frac{u_2}{\lambda_2}, e_3 = \frac{u_3}{\lambda_3} \right\}$$

is g_l-orthonormal. Let $e_0 \in \mathfrak{g}$ span the $u(1)$ factor. Let E_0 span the g_l-orthogonal compliment of the $su(2)$ factor, and be g_l-unit length.

Proposition 4.1 (G, g_l) *has nonnegative curvature if and only if* $\{\lambda_1, \lambda_2, \lambda_3\}$ *satisfy the restrictions of Proposition* 3.1*, and one of the following conditions holds:*

(i) E_0 *is parallel to* e_0 *(in this case,* g_l *is a product metric);*
(ii) $\lambda_1 = \lambda_2 = \lambda_3$ *(in this case,* E_0 *is arbitrary);*
(iii) $\lambda_1 = \lambda_2$ *and* $E_0 \in \mathrm{span}\{e_0, e_3\}$ *(or analogously if a different pair of* λ's agree*).*

We prove the proposition with a sequence of lemmas.

Lemma 4.2   *The map* $\mathrm{ad}_{E_0} \colon \mathfrak{g} \to \mathfrak{g}$ *is skew-adjoint with respect to* g_l *if and only if one of the three conditions of Proposition* 4.1 *holds.*

Proof   Write $E_0 = ae_0 + be_1 + ce_2 + de_3$. In the g_l-orthonormal basis $\{E_0, e_1, e_2, e_3\}$ of $\mathfrak{g}$, the structure constants $\alpha_{0jk} = \langle [E_0, e_j], e_k \rangle_l$ are as follows:

$$\alpha_{012} = d\frac{\lambda_2}{\lambda_1}, \quad \alpha_{021} = -d\frac{\lambda_1}{\lambda_2}, \quad \alpha_{023} = b\frac{\lambda_3}{\lambda_2},$$

$$\alpha_{032} = -b\frac{\lambda_2}{\lambda_3}, \quad \alpha_{031} = c\frac{\lambda_1}{\lambda_3}, \quad \alpha_{013} = -c\frac{\lambda_3}{\lambda_1}.$$

Here we assume for convenience that g_0 is scaled such that the Lie bracket structure of the SU(2) factor is given by equation (3.2). The lemma follows by inspection, since ad_{E_0} is skew-adjoint if and only if $\alpha_{0ij} = -\alpha_{0ji}$ for all i, j.   ∎

The next lemma applies more generally to any left-invariant metric on any Lie group.

Lemma 4.3   *The left-invariant vector field $x \in \mathfrak{g}$ is parallel if and only if ad_x is skew-adjoint and $\langle x, [\mathfrak{g}, \mathfrak{g}] \rangle_l = 0$.*

Proof   For all $y, z \in \mathfrak{g}$,

$$(4.1) \qquad 2 \langle y, \nabla_z x \rangle_l = \langle z, [y, x] \rangle_l + \langle x, [y, z] \rangle_l - \langle y, [x, z] \rangle_l.$$

If ad_x is skew-adjoint, then the first and last terms of (4.1) sum to 0. If additionally $\langle x, [\mathfrak{g}, \mathfrak{g}] \rangle_l = 0$, then the middle term is also 0. So $\nabla_z x = 0$ for all $z \in \mathfrak{g}$, which means that x is parallel.

Conversely, assume that x is parallel, so the left side of (4.1) equals 0 for all $y, z \in \mathfrak{g}$. When $y = z$, this yields $2 \langle y, [y, x] \rangle_l = 0$ for all $y \in \mathfrak{g}$, which implies that ad_x is skew-adjoint. This property makes the first and third terms of (4.1) sum to zero, so $\langle x, [y, z] \rangle_l = 0$ for all $y, z \in \mathfrak{g}$. In other words, $\langle x, [\mathfrak{g}, \mathfrak{g}] \rangle_l = 0$. ∎

The next lemma [7] also applies more generally to any left-invariant metric on any Lie group. We use r to denote the Ricci curvature of g_l.

Lemma 4.4 (Milnor)   *If $x \in \mathfrak{g}$ is g_l-orthogonal to the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$, then $r(x) \leq 0$, with equality if and only if ad_x is skew-adjoint with respect to g_l.*

Proof of Proposition 4.1   Suppose that g_l has nonnegative curvature. Then ad_{E_0} is skew-adjoint by Lemma 4.4. Next, Lemma 4.2 implies that one of the three conditions of the proposition hold. It remains to prove that $\{\lambda_1, \lambda_2, \lambda_3\}$ satisfy the constraints for eigenvalues of a nonnegatively curved metric on SO(3) (or equivalently on SU(2)). By Lemma 4.3, E_0 is parallel. Using (4.1), this implies that the SU(2) factor of G is totally geodesic, so its induced metric has nonnegative curvature, which gives the constraints on the $\lambda's$.

Conversely, suppose $\{\lambda_1, \lambda_2, \lambda_3\}$ satisfies the constraints for eigenvalues of a nonnegatively curved metric on SU(2), and that one of the three conditions of the proposition holds. By Lemma 4.2, ad_{E_0} is skew-adjoint, so by Lemma 4.3, E_0 is parallel. This implies that the SU(2) factor of G is totally geodesic. It has nonnegative curvature because of the constraints on the λ's. Consider the curvature operator of g_l expressed in the following basis of $\wedge^2(\mathfrak{g})$:

$$\{E_0 \wedge e_1, E_0 \wedge e_2, E_0 \wedge e_3, e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1\}.$$

Since E_0 is parallel, $R(E_0 \wedge e_i) = 0$. Furthermore, $R(e_i \wedge e_j)$ is calculated in the totally geodesic SU(2). It follows that the curvature operator is nonnegative. ∎

We refer to metrics of types (i) and (ii) in Proposition 4.1 as "twisted metrics." We know from the splitting theorem that nonnegatively curved twisted metrics are

locally isometric to untwisted (product) metrics. We end this section by explicitly exhibiting the local isometry between a twisted metric and a product metric.

Suppose that g_l is a twisted metric. Recall that $\{E_0, e_1, e_2, e_3\}$ is a g_l-orthonormal basis of $\mathfrak{g}$. Let $\tilde{g}_l$ denote the pull-back of g_l to the universal cover, $\mathbb{R} \times SU(2)$, of G. Let g' denote the product metric on $\mathbb{R} \times SU(2)$ for which $\{e_0, e_1, e_2, e_3\}$ forms a g'-orthonormal basis of $\mathfrak{g}$. Define $f \colon (\mathbb{R} \times SU(2), g') \to (\mathbb{R} \times SU(2), \tilde{g}_l)$ as follows:

$$f(t, g) = \text{flow from } (0, g) \text{ for time } t \text{ along } E_0 = R_{e^{tE_0}}(0, g) = (at, ge^{t\hat{E}_0}),$$

where $E_0 = ae_0 + be_1 + ce_2 + de_3$ and $\hat{E}_0 = be_1 + ce_2 + de_3$.

Proposition 4.5   *If g_l has nonnegative curvature, then f is an isometry.*

Proof   First,

$$(4.2) \qquad df_{(t,g)}(e_0) = \frac{d}{ds}\Big|_{s=0} (a(t+s), ge^{(t+s)\hat{E}_0}) = (a, (ge^{t\hat{E}_0})\hat{E}_0),$$

which is the value at $f(t, g)$ of the left-invariant vector field E_0. So df sends the left-invariant field e_0 to the left-invariant field E_0. Next, for $i \in \{1, 2, 3\}$,

$$(4.3) \qquad df_{(t,g)}(e_i g) = \frac{d}{ds}\Big|_{s=0} f(t, e^{se_i}g) = \frac{d}{ds}\Big|_{s=0} (at, e^{se_i}ge^{t\hat{E}_0}) = (0, e_i(ge^{t\hat{E}_0})),$$

which is the value at $f(g, t)$ of the right-invariant vector field determined by e_i. In other words, df sends the right-invariant vector field determined by e_i to itself.

In summary, (4.2) implies that df sends the g'-unit-length field e_0 to the $\tilde{g}_l$-unit-length field E_0. Further, (4.3) implies that df sends the g'-orthogonal compliment e_0 to the $\tilde{g}_l$-orthogonal compliment of E_0 (both of which equal span$\{e_1, e_2, e_3\}$). It remains to verify that the restriction $df \colon \text{span}\{e_1, e_2, e_3\} \to \text{span}\{e_1, e_2, e_3\}$ is an isometry.

Under condition (ii) of Proposition 4.1, g' restricts to a bi-invariant metric on the $SU(2)$ factor. Under condition (iii), g' restricts to a left-invariant and Ad_H-invariant metric, where $H = \exp(\text{span}(\hat{E}_0))$. In either case, the metric has exactly enough right-invariance to give the desired result from (4.3).   ∎

Propositions 3.2 and 4.5 together imply

Corollary 4.6   *Two left-invariant metrics with nonnegative curvature on $U(1) \times SU(2)$ are locally isometric if and only if their restricted eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ are the same.*

These local isometries are really isometries between the universal covers. In general, they are not group isomorphisms. They do not generally descend to global isometries between twisted and product metrics on $U(1) \times SU(2)$.

5 Review of Cheeger's Method

In the literature, the only examples of left-invariant metrics with nonnegative curvature on compact Lie groups come from a construction which, in its greatest generality, is due to Cheeger [1]. In this section, we review Cheeger's method.

Let $H \subset G$ be compact Lie groups with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Let g_0 be a left-invariant and Ad_H-invariant metric on G with nonnegative curvature. Let g_H be a right-invariant metric on H with nonnegative curvature. The right diagonal action of H on $(G, g_0) \times (H, g_H)$ is by isometries. The quotient is diffeomorphic to G and inherits a submersion metric, g_1, which is left-invariant and has nonnegative curvature: $(G, g_1) = ((G, g_0) \times (H, g_H))/H$. If g_H is bi-invariant, then g_1 is Ad_H-invariant.

The metrics g_0 and g_1 agree orthogonal to $\mathfrak{h}$. To describe g_1 on $\mathfrak{h}$, let $\{e_1, \ldots, e_k\}$ denote a g_H-orthonormal basis of $\mathfrak{h}$. Let A and $\tilde{A}$ denote the k-by-k matrices whose entries are $a_{ij} = g_0(e_i, e_j)$ and $\tilde{a}_{ij} = g_1(e_i, e_j)$, respectively. The restrictions of g_1 and g_0 to $\mathfrak{h}$ are shown in [1] to be related by the equation:

$$(5.1) \qquad \tilde{A} = A(I + A)^{-1}.$$

A common special case occurs when g_0 is bi-invariant, and g_H is a multiple, λ, of the restriction of g_0 to H. In this case, which was first studied in [4], g_1 restricted to $\mathfrak{h}$ is obtained from g_0 by scaling by the factor $t = \lambda/(1 + \lambda)$. In this case, g_1 is Ad_H-invariant, and is therefore Ad_K-invariant for any closed subgroup $K \subset H$. So Cheeger's method can be applied again:

$$(G, g_2) = ((G, g_1) \times (K, g_K))/K,$$

and so on. Iterating Cheeger's method in this way is studied in [3].

In summary, whenever $H_0 \subset H_1 \subset \cdots \subset H_{l-1} \subset H_l = G$ is a chain of closed subgroups of G with Lie algebras $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \cdots \subset \mathfrak{h}_{l-1} \subset \mathfrak{h}_l = \mathfrak{g}$, one can apply Cheeger's method l-times. One chooses a starting bi-invariant metric g_0 on G and l constants $\{\lambda_0, \ldots, \lambda_{l-1}\}$. The result is a new left-invariant metric with nonnegative curvature on G. The sub-algebras $\mathfrak{h}_i$ are separately scaled by factors determined by the λ's, with $\mathfrak{h}_i$ scaled a greater amount than $\mathfrak{h}_{i+1}$. More precisely, the eigenvalues of the metric are a strictly increasing sequence $t_0 < t_1 < \cdots < t_{l-1} < t_l = 1$. The $\lambda's$ can be chosen to give any such strictly increasing sequence. The eigenspace of t_0 equals $\mathfrak{h}_0$. For $i > 1$, the eigenspace of t_i equals the g_0-orthogonal compliment of $\mathfrak{h}_{i-1}$ in $\mathfrak{h}_i$.

6 Do All Metrics Come From Cheeger's Method?

Let G be a compact Lie group. Does every left-invariant metric on G with nonnegative curvature come from Cheeger's method? In this section, we outline several ways to precisely formulate this question, and address the cases $G = SO(3)$ and $G = U(2)$.

First, one might ask whether all examples arise by starting with a bi-invariant metric g_0 on G and applying Cheeger's method to a chain of subgroups $H_0 \subset \cdots \subset H_{l-1} \subset G$, each time choosing the metric on H_i to be a multiple, λ_i, of the restriction of g_0 to H_i. The answer is clearly no, since not all metrics on $SO(3)$ arise in this

fashion. But SO(3) is a very special case, since its bi-invariant metric has positive curvature. There are several ways to modify the question to reflect the guess that SO(3) is the only exception.

What if, in each application of Cheeger's method, one allows more general right-invariant metrics on the H_i's? For example, if $H_i \cong$ SO(3) or SU(2), we allow any right-invariant metric with nonnegative curvature. With this added generality, the answer is still no, since metrics on SO(3) which are nonnegatively but not positively curved do not arise in this fashion:

Proposition 6.1   *If g_0 is a bi-invariant metric on* SO(3) *and g_R is a right-invariant metric with nonnegative curvature on* SO(3), *then the following has strictly positive curvature:*

$$(\mathrm{SO}(3), g_l) := ((\mathrm{SO}(3), g_0) \times (\mathrm{SO}(3), g_R))/ \mathrm{SO}(3).$$

Further, every positively-curved left-invariant metric g_l can be described in this way for some bi-invariant metric g_0 and some positively curved right-invariant metric g_R.

Proof   Let $\pi \colon (\mathrm{SO}(3), g_0) \times (\mathrm{SO}(3), g_R) \to (\mathrm{SO}(3), g_l)$ denote the projection, which is a Riemannian submersion. Let $\{e_1, e_2, e_3\} \subset so(3)$ be a g_0-orthonormal basis of eigenvectors of the metric g_R, with eigenvalues denoted $\{\lambda_1, \lambda_2, \lambda_3\}$. The horizontal space of π at the identity is the following subspace of $so(3) \times so(3)$:

$$(6.1) \qquad \mathcal{H}_{(e,e)} = \{(V, W) \mid \langle (V, W), (e_i, e_i) \rangle = 0 \text{ for all } i = 1, 2, 3\}$$

$$= \{(V, W) \mid \langle V, e_i \rangle_0 + \lambda_i \langle W, e_i \rangle_0 = 0 \text{ for all } i = 1, 2, 3\}.$$

Now let $X_1 = (V_1, W_1), X_2 = (V_2, W_2) \in \mathcal{H}_{(e,e)}$ be linearly independent vectors. From (6.1), we see that V_1, V_2 are linearly independent as well. Since $(\mathrm{SO}(3), g_0)$ has positive curvature, $\kappa(V_1, V_2) > 0$, so $\kappa(X_1, X_2) > 0$. Finally, O'Neill's formula implies that $(\mathrm{SO}(3), g_l)$ has strictly positive curvature.

To prove the second statement of the proposition, notice that by (5.1), the eigenvalues $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\}$ of the metric g_l are determined from the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ of the metric g_R as follows:

$$(6.2) \qquad\qquad\qquad \tilde{\lambda}_i = \frac{\lambda_i}{1 + \lambda_i} \text{ for each } i = 1, 2, 3.$$

Suppose that $(\tilde{\lambda}_1, \tilde{\lambda}_2)$ is an arbitrary pair such that the triplet $\{\tilde{\lambda}_1, \tilde{\lambda}_2, 1\}$ strictly satisfies the inequalities of Proposition 3.1. Define

$$\lambda_1 := \frac{a\tilde{\lambda}_1}{1 + a(1 - \tilde{\lambda}_1)}, \quad \lambda_2 := \frac{a\tilde{\lambda}_2}{1 + a(1 - \tilde{\lambda}_2)}, \quad \lambda_3 := a.$$

For small enough a, the triplet $\{\lambda_1, \lambda_2, \lambda_3\}$ strictly satisfies the inequalities of Proposition 3.1. This is because, when scaled such that the third equals 1, the triplet ap-

proaches $\{\tilde{\lambda}_1, \tilde{\lambda}_2, 1\}$ as $a \to 0$. If g_R has eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$, then by equation 6.2, g_l has the following eigenvalues:

$$\left\{ \left(\frac{a}{1+a} \right) \tilde{\lambda}_1, \left(\frac{a}{1+a} \right) \tilde{\lambda}_2, \left(\frac{a}{1+a} \right) \right\}.$$

Further, g_R has positive curvature because the inner products which extend to right-invariant metrics with positive curvature are the same ones that extend to left-invariant metrics of positive curvature. This is because $a \mapsto a^{-1}$ is an isometry between left- and right-invariant metrics determined by the same inner product.

Thus, g_R can be chosen such that g_l is a multiple of any prescribed positively curved metric. Then, by scaling the bi-invariant metric g_0, this multiple can be made to be 1. ∎

Beginning with a bi-invariant metric g_0 on $G = \text{SU}(2) \times U(1)$ and applying Cheeger's method, one cannot obtain product metrics for which the SU(2)-factor has nonnegative but not positive curvature, nor can one obtain twisted metrics of type (2). Notice that the only chains of increasing-dimension subgroups are $U(1) \subset G$ (possibly embedded diagonally), $T^2 \subset G$ (any maximal torus), $U(1) \subset T^2 \subset G$, and $\text{SU}(2) \subset G$.

To obtain more metrics, in addition to allowing general right-invariant metrics on the H_i's as above, one could allow a more general starting metric g_0. For example, when $G = \text{SU}(2) \times U(1)$, one could allow g_0 to be any left-invariant product metric with nonnegative curvature. Even with this added generality, it is straightforward to see that one cannot obtain twisted metrics of type (iii) for which the totally geodesic SU(2) has nonnegative but not positive curvature.

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Department of Mathematics   *Department of Mathematics*
Stanford University   *Brown University*
Stanford, CA 94305-2125   *Providence, RI 02912*
U.S.A.   *e-mail: mpsprs@gmail.com*
e-mail: nbrown@math.stanford.edu
e-mail: rfinck@alum.mit.edu

Department of Mathematics   *88 College Road*
Williams College   *Princeton, MA 08544*
Williamstown, MA 01267   *U.S.A.*
U.S.A.   *e-mail: ztwu@alum.mit.edu*
e-mail: ktapp@williams.edu