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Cosmetic surgeries on knots in S^3

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Abstract. Two Dehn surgeries on a knot are called purely cosmetic, if they yield manifolds that are homeomorphic as oriented manifolds. Suppose there exist purely cosmetic surgeries on a knot in S^3, we show that the two surgery slopes must be the opposite of each other. One ingredient of our proof is a Dehn surgery formula for correction terms in Heegaard Floer homology.

1. Introduction

Given a knot K in a three-manifold Y, let α, β be two different slopes on K, and let $Y_\alpha(K), Y_\beta(K)$ be the manifolds obtained by α- and β-surgeries on K, respectively. If the two manifolds $Y_\alpha(K), Y_\beta(K)$ are homeomorphic, then we say the two surgeries are *cosmetic*; if $Y_\alpha(K) \cong Y_\beta(K)$ as oriented manifolds, then these two surgeries are *purely cosmetic*; if $Y_\alpha(K) \cong -Y_\beta(K)$ as oriented manifolds, then these two surgeries are *chirally cosmetic*.

Chirally cosmetic surgeries occur frequently for knots in S^3. For example, if K is amphicheiral, then $S^3_r(K) \cong -S^3_{-r}(K)$ for any slope r. If T is the right-hand trefoil knot, then

$$S^3_{(18k+9)/(3k+1)}(T) \cong -S^3_{(18k+9)/(3k+2)}(T)$$

for any nonnegative integer k (see [7]).

On the other hand, purely cosmetic surgeries are very rare. In fact, the following conjecture was proposed in Gordon's ICM talk [4, Conjecture 6.1] and Kirby's Problem List [6, Problem 1.81 A].

Conjecture 1.1 (Cosmetic surgery conjecture). *Suppose K is a knot in a closed oriented three-manifold Y such that $Y - K$ is irreducible and not homeomorphic to the solid torus. If two different Dehn surgeries on K are purely cosmetic, then there is a homeomorphism of $Y - K$ which takes one slope to the other.*

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This conjecture is highly nontrivial even when $Y = S^3$. In [5], Gordon and Luecke proved the famous knot complement theorem, which can be interpreted as that there are no cosmetic surgeries if one of the two slopes is ∞. In [2], Boyer and Lines proved the cosmetic surgery conjecture for any knot K with $\Delta''_K(1) \neq 0$. In recent years, Heegaard Floer homology [12] became a powerful tool to study this conjecture. In [14], Ozsváth and Szabó proved that if $S^3_{r_1}(K)$ is homeomorphic to $S^3_{r_2}(K)$, then either $S^3_{r_1}(K)$ is an L-space or r_1 and r_2 have opposite signs. Moreover, when $S^3_{r_1}(K)$ is homeomorphic to $S^3_{r_2}(K)$ as oriented manifolds, the second author [19] ruled out the case that $S^3_{r_1}(K)$ is an L-space, thus r_1 and r_2 must have opposite signs. In [18], Wang proved that genus 1 knots in S^3 do not admit purely cosmetic surgeries.

In this paper, we are going to put more restrictions on purely cosmetic surgeries for knots in S^3. Our main result is:

Theorem 1.2. *Suppose K is a nontrivial knot in S^3, and $r_1, r_2 \in \mathbb{Q} \cup \{\infty\}$ are two distinct slopes such that $S^3_{r_1}(K)$ is homeomorphic to $S^3_{r_2}(K)$ as oriented manifolds. Then the following assertions are true:*

(a) $r_1 = -r_2$.

(b) *If $r_1 = p/q$, where p, q are coprime integers, then* $q^2 \equiv -1 \pmod{p}$.

(c) $\tau(K) = 0$, *where τ is the concordance invariant defined by Ozsváth–Szabó* [9] *and Rasmussen* [15].

Remark 1.3. Suppose $K \subset S^3$ is a nontrivial knot, and f is a homeomorphism of $S^3 - K$. Then f must send the longitude to the longitude (up to orientation reversing) for homological reason, and the meridian to the meridian (up to orientation reversing) by the knot complement theorem [5]. Thus if f sends a slope to a different slope, then f extends to an orientation reversing homeomorphism of S^3, which means that K is amphicheiral. So Conjecture 1.1 for knots in S^3 can be reformulated as: if $S^3_{r_1}(K)$ is homeomorphic to $S^3_{r_2}(K)$ as oriented manifolds, then $r_1 = -r_2$ and K is amphicheiral. Our Theorem 1.2 proves the part of this conjecture concerning the slopes, and asserts that $\tau(K)$ is the same as the τ invariant of amphicheiral knots.

Remark 1.4. Ozsváth and Szabó [14] remarked that their method can be used to exclude cosmetic surgeries for certain numerators p. To illustrate, they proved that $p \neq 3$. Our Theorem 1.2 (b) implies a more precise restriction on p: -1 must be a quadratic residue modulo p.

Remark 1.5. Ozsváth and Szabó [14] gave the example of $K = 9_{44}$. This knot is a genus 2 knot with $\tau(K) = 0$ and

$$\Delta_K(T) = T^{-2} - 4T^{-1} + 7 - 4T + T^2.$$

Heegaard Floer homology does not obstruct K from admitting purely cosmetic surgeries. In fact, $S^3_1(K)$ and $S^3_{-1}(K)$ have the same Heegaard Floer homology. However, these two manifolds are not homeomorphic since they have different hyperbolic volumes.

An important ingredient of the proof of Theorem 1.2 is a surgery formula for Ozsváth and Szabó's correction terms. The statement of the formula is as follows, where the terms in the formula will be explained in Section 2.

Proposition 1.6. *Suppose $p, q > 0$, and fix $0 \leq i \leq p-1$. Then*

$$d(S^3_{p/q}(K), i) = d(L(p,q), i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i+p(-1)}{q} \rfloor}\}.$$

The above formula has independent interests, since the correction terms have been very useful in many applications of Heegaard Floer homology.

We also compute the rank of the reduced Heegaard Floer homology of surgeries on knots which admits purely cosmetic surgeries.

Proposition 1.7. *Suppose $K \subset S^3$ is a knot with $S^3_r(K) \cong S^3_{-r}(K)$ for some $r \in \mathbb{Q}\backslash\{0\}$. Then there exists a constant $C = C(K)$, such that*

$$\operatorname{rank} HF_{\mathrm{red}}(S^3_{p/q}(K)) = |q| \cdot C$$

for any coprime nonzero integers p, q. In fact, the constant $C(K)$ is the rank of $HF_{\mathrm{red}}(S^3_p(K))$ for $p > 0$.

This paper is organized as follows. In Section 2, we use Ozsváth and Szabó's rational surgery formula [14] to prove Proposition 1.6. This gives a bound of the correction terms by the correction terms of the corresponding lens spaces. A necessary and sufficient condition for the bound to be reached is found. In Section 3, we review the Casson–Walker and Casson–Gordon invariants. Combining these with the bound obtained in Section 2, we show that if there are purely cosmetic surgeries, then the correction terms are exactly the correction terms of the corresponding lens spaces. Our main theorem is then proved in Section 4. In Section 5, we will prove Proposition 1.7.

2. Rational surgeries and the correction terms

2.1. The rational surgery formula. In this subsection, we recall the rational surgery formula of Ozsváth and Szabó [14], and then compute the example of surgeries on the unknot.

Remark 2.1. For simplicity, throughout this paper we will use $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ coefficients for Heegaard Floer homology. Our proofs work for $\mathbb{Z}$ coefficients as well.

Given a knot K in an integer homology sphere Y. Let $C = CFK^\infty(Y, K)$ be the knot Floer chain complex [11, 15] of (Y, K). There are chain complexes

$$A^+_k = C\{i \geq 0 \text{ or } j \geq k\}, \quad k \in \mathbb{Z}$$

and $B^+ = C\{i \geq 0\} \cong CF^+(Y)$. As in [13], there are chain maps

$$v_k, h_k : A^+_k \to B^+.$$

Given $\frac{p}{q} \in \mathbb{Q} \setminus \{0\}$, let

$$\mathbb{A}_i^+ = \bigoplus_{s\in\mathbb{Z}} \big(s, A^+_{\lfloor \frac{i+ps}{q} \rfloor}(K)\big), \quad \mathbb{B}_i^+ = \bigoplus_{s\in\mathbb{Z}} (s, B^+).$$

Here, the first entry s in the parentheses is simply a decoration used to distinguish identical copies of A_k^+ or B^+. Define maps

$$v^+_{\lfloor \frac{i+ps}{q} \rfloor} : \big(s, A^+_{\lfloor \frac{i+ps}{q} \rfloor}(K)\big) \to (s, B^+), \quad h^+_{\lfloor \frac{i+ps}{q} \rfloor} : \big(s, A^+_{\lfloor \frac{i+ps}{q} \rfloor}(K)\big) \to (s+1, B^+).$$

Adding these up, we get a chain map

$$D^+_{i,p/q} : \mathbb{A}_i^+ \to \mathbb{B}_i^+,$$

with

$$D^+_{i,p/q}\{(s, a_s)\}_{s\in\mathbb{Z}} = \{(s, b_s)\}_{s\in\mathbb{Z}},$$

where

$$b_s = v^+_{\lfloor \frac{i+ps}{q} \rfloor}(a_s) + h^+_{\lfloor \frac{i+p(s-1)}{q} \rfloor}(a_{s-1}).$$

In [14], Ozsváth and Szabó gave an identification of $\mathrm{Spin}^c(Y_{p/q}(K))$ with $\mathbb{Z}/p\mathbb{Z}$. This identification can be made explicit by the procedure in [14, Sections 4, 7]. For our purpose in this paper, we only need to know that such an identification exists. We will use this identification throughout this paper.

Theorem 2.2 (Ozsváth–Szabó). *Let $\mathbb{X}^+_{i,p/q}$ be the mapping cone of $D^+_{i,p/q}$, then there is a relatively graded isomorphism of groups*

$$H_*(\mathbb{X}^+_{i,p/q}) \cong HF^+(Y_{p/q}(K), i).$$

The above isomorphism is actually U-equivariant, so the two groups are isomorphic as $\mathbb{F}_2[U]$-modules.

Remark 2.3. For $K \subset S^3$, the absolute grading on $\mathbb{X}^+_{i,p/q}$ is determined by an absolute grading on $\mathbb{B}^+$ which is independent of K. The absolute grading on $\mathbb{B}^+$ is chosen so that the grading of

$$\mathbf{1} \in H_*(\mathbb{X}^+_{i,p/q}(O)) \cong \mathcal{T}^+ := \mathbb{F}_2[U, U^{-1}]/U\mathbb{F}_2[U]$$

is $d(L(p,q), i)$, where O is the unknot. This absolute grading on $\mathbb{X}^+_{i,p/q}(K)$ then agrees with the absolute grading on $HF^+(S^3_{p/q}(K), i)$. See [14, Section 7.2] for a discussion of the absolute grading.

Let

$$\mathfrak{A}_k^+ = H_*(A_k^+), \quad \mathfrak{B}^+ = H_*(B^+),$$

let

$$\mathfrak{v}_k^+, \mathfrak{h}_k^+ : \mathfrak{A}_k^+ \to \mathfrak{B}^+$$

be the maps induced on homology, and let

$$\mathfrak{D}^+_{i,p/q} : H_*(\mathbb{A}_i^+) \to H_*(\mathbb{B}_i^+)$$

be the map induced by $D^+_{i,p/q}$ on homology. There is a natural short exact sequence of chain complexes:

$$0 \longrightarrow \mathbb{B}^+ \xrightarrow{\text{incl}} \mathbb{X}^+_{i,p/q} \xrightarrow{\text{proj}} \mathbb{A}^+_i \longrightarrow 0.$$

Then Theorem 2.2 implies that there is an exact triangle

$$\begin{tikzcd} H_*(\mathbb{A}^+_i) \arrow[r, "\mathfrak{D}^+_{i,p/q}"] & H_*(\mathbb{B}^+) \arrow[d, "\text{incl}_*"] \\ & HF^+(Y_{p/q}(K), i). \arrow[ul, "\text{proj}_*"] \end{tikzcd} \tag{1}$$

If $K = O$ is the unknot, then the $\frac{p}{q}$-surgery on K gives rise to the lens space $L(p,q)$. Then

$$\mathfrak{A}^+_k \cong A^+_k \cong \mathfrak{B}^+_k \cong B^+_k \cong \mathcal{T}^+.$$

We have

$$v^+_k = \begin{cases} U^{|k|} & \text{if } k \le 0, \\ 1 & \text{if } k \ge 0, \end{cases} \qquad h^+_k = \begin{cases} 1 & \text{if } k \le 0, \\ U^{|k|} & \text{if } k \ge 0. \end{cases}$$

Suppose $p, q > 0$. Let $0 \le i \le p-1$, then $\lfloor \frac{i+ps}{q} \rfloor \ge 0$ if and only if $s \ge 0$. We have $b_0 = a_0 + a_{-1}$. For $\xi \in \mathcal{T}^+$, define

$$\iota(\xi) = \{(s, \xi_s)\}_{s \in \mathbb{Z}} \in \mathbb{A}^+_i$$

by letting

$$\xi_s = \begin{cases} \xi, & \text{if } s \in \{-1, 0\}, \\ U^{\lfloor \frac{i+p(s-1)}{q} \rfloor} \xi_{s-1}, & \text{if } s > 0, \\ U^{-\lfloor \frac{i+p(s+1)}{q} \rfloor} \xi_{s+1}, & \text{if } s < -1. \end{cases}$$

Then ι maps $\mathcal{T}^+$ isomorphically to the kernel of $D^+_{i,p/q}$. It is clear that $D^+_{i,p/q}$ is surjective (see also Lemma 2.8 for a proof), hence $\ker D^+_{i,p/q} \cong \mathbb{X}^+_{i,p/q}$. In particular, $\iota(\mathbf{1})$ should have absolute grading $d(L(p,q), i)$. The absolute grading on $\mathbb{B}^+$ can be determined by the fact

$$v^+_{\lfloor \frac{i}{q} \rfloor}(0, \mathbf{1}) = h^+_{\lfloor \frac{i+p(-1)}{q} \rfloor}(-1, \mathbf{1}) = (0, \mathbf{1}) \in (0, B^+). \tag{2}$$

2.2. Bounding the correction terms. For a rational homology three-sphere Y equipped with a Spin^c structure $\mathfrak{s}$, $HF^+(Y, \mathfrak{s})$ can be decomposed as the direct sum of two groups: The first group is the image of $HF^\infty(Y, \mathfrak{s})$ in $HF^+(Y, \mathfrak{s})$, whose minimal absolute $\mathbb{Q}$ grading is an invariant of $(Y, \mathfrak{s})$ and is denoted by $d(Y, \mathfrak{s})$, the *correction term* [8]; the second group is the quotient modulo the above image and is denoted by $HF_{\text{red}}(Y, \mathfrak{s})$. Altogether, we have

$$HF^+(Y, \mathfrak{s}) = \mathcal{T}^+_{d(Y,\mathfrak{s})} \oplus HF_{\text{red}}(Y, \mathfrak{s}).$$

For a knot $K \subset S^3$, let $\mathfrak{A}^T_k = U^n \mathfrak{A}^+_k$ for $n \gg 0$, then $\mathfrak{A}^T_k \cong \mathcal{T}^+$. Let $\mathfrak{D}^T_{i,p/q}$ be the restriction of $\mathfrak{D}^+_{i,p/q}$ on

$$\mathcal{A}^T_i = \bigoplus_{s \in \mathbb{Z}} \big(s, \mathfrak{A}^T_{\lfloor \frac{i+ps}{q} \rfloor}(K)\big).$$

Since $\mathfrak{v}_k^+, \mathfrak{h}_k^+$ are graded isomorphisms at sufficiently high gradings and are U-equivariant, $\mathfrak{v}_k^+|\mathfrak{A}_k^T$ is modeled on multiplication by U^{V_k} and $\mathfrak{h}_k^+|\mathfrak{A}_k^T$ is modeled on multiplication by U^{H_k}, where $V_k, H_k \geq 0$. Note that the numbers V_k and H_k are invariants of the double-filtered chain homotopy equivalence type of $CFK^\infty(Y, K)$. Hence, they are invariants of the knot K.

Lemma 2.4. $V_k \geq V_{k+1}$, $H_k \leq H_{k+1}$.

Proof. The map v_k^+ factors through the map v_{k+1}^+ via the factorization

$$C\{i \geq 0 \text{ or } j \geq k\} \longrightarrow C\{i \geq 0 \text{ or } j \geq k+1\} \xrightarrow{v_{k+1}^+} C\{i \geq 0\}.$$

Hence it is easy to see that $V_k \geq V_{k+1}$. A similar argument works for H_k by considering the factorization of h_{k+1}^+:

$$C\{i \geq 0 \text{ or } j \geq k+1\} \xrightarrow{U} C\{i \geq -1 \text{ or } j \geq k\} \longrightarrow C\{i \geq 0 \text{ or } j \geq k\} \xrightarrow{h_k^+} C\{i \geq 0\}.$$

□

It is obvious that $V_k = 0$ when $k \geq g$ and $H_k = 0$ when $k \leq -g$, $V_k \to +\infty$ as $k \to -\infty$, $H_k \to +\infty$ as $k \to +\infty$.

Theorem 2.5. *Suppose $p, q > 0$ are coprime integers. Then*

$$d(S^3_{p/q}(K), i) \leq d(L(p,q), i)$$

for all $i \in \mathbb{Z}/p\mathbb{Z}$. The equality holds for all i if and only if $V_0 = H_0 = 0$.

Remark 2.6. The first part of Theorem 2.5 easily follows from [8, Theorem 9.6] and [10, Corollary 1.5]. We will present a different proof, which enables us to get the conclusion about V_0 and H_0.

Lemma 2.7. *For any knot $K \subset S^3$, we have $V_0 = H_0$. Hence $V_k \geq H_k$ if $k \leq 0$ and $V_k \leq H_k$ if $k \geq 0$.*

Proof. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ is a doubly pointed Heegaard diagram for (S^3, K), then $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$ is also a Heegaard diagram for (S^3, K). Hence the roles of i, j can be interchanged. It follows that v_0^+ is equivalent to h_0^+, hence $V_0 = H_0$. □

Lemma 2.8. *Suppose that $p, q > 0$. Then the map $\mathfrak{D}^T_{i,p/q}$ is surjective for each $0 \leq i \leq p-1$.*

Proof. Suppose

$$\boldsymbol{\eta} = \{(s, \eta_s)\}_{s \in \mathbb{Z}} \in H_*(\mathbb{B}_i^+).$$

Let

$$\xi_{-1} = U^{-H_{\lfloor \frac{i+p(-1)}{q} \rfloor}} \eta_0, \quad \xi_0 = 0.$$

Here $\xi_{-1} = U^{-H_{\lfloor \frac{i+p(-1)}{q} \rfloor}} \eta_0$ means that ξ_{-1} is an element with $U^{H_{\lfloor \frac{i+p(-1)}{q} \rfloor}} \xi_{-1} = \eta_0$. For other s, let

$$\xi_s = \begin{cases} U^{-V_{\lfloor \frac{i+ps}{q} \rfloor}} \big(\eta_s - U^{H_{\lfloor \frac{i+p(s-1)}{q} \rfloor}} \xi_{s-1}\big), & \text{if } s > 0, \\ U^{-H_{\lfloor \frac{i+ps}{q} \rfloor}} \big(\eta_{s+1} - U^{V_{\lfloor \frac{i+p(s+1)}{q} \rfloor}} \xi_{s+1}\big), & \text{if } s < -1. \end{cases}$$

By the definition of direct sum, $\eta_s = 0$ when $|s| \gg 0$. Using the facts that

$$\begin{aligned} H_{\lfloor \frac{i+p(s-1)}{q} \rfloor} - V_{\lfloor \frac{i+ps}{q} \rfloor} &\to +\infty, \quad \text{as } s \to +\infty, \\ V_{\lfloor \frac{i+p(s+1)}{q} \rfloor} - H_{\lfloor \frac{i+ps}{q} \rfloor} &\to +\infty, \quad \text{as } s \to -\infty, \end{aligned}$$

we see that $\xi_s = 0$ when $|s| \gg 0$. So $\boldsymbol{\xi} = \{(s, \xi_s)\}_{s \in \mathbb{Z}} \in \mathcal{A}_i^T$. Clearly

$$\mathfrak{D}^T_{i,p/q}(\boldsymbol{\xi}) = \boldsymbol{\eta}. \qquad \square$$

Our key idea is the following lemma.

Lemma 2.9. *Suppose $p/q > 0$. Under the identification*

$$H_*(\mathbb{X}^+_{i,p/q}) \cong HF^+(S^3_{p/q}(K), i),$$

$U^n HF^+(S^3_{p/q}(K), i)$ is identified with a subgroup of the homology of the mapping cone of $D^T_{i,p/q}$ when $n \gg 0$.

Proof. By Lemma 2.8, $\mathfrak{D}^T_{i,p/q}$ is surjective, hence $\mathfrak{D}^+_{i,p/q}$ is also surjective. By the exact triangle (1), we conclude that $HF^+(Y_{p/q}(K), i) \cong \ker \mathfrak{D}^+_{i,p/q}$. Suppose $\boldsymbol{\xi} \in U^n \ker \mathfrak{D}^+_{i,p/q}$ for $n \gg 0$. Then

$$\boldsymbol{\xi} \in U^n H_*(\mathbb{A}^+_{i,p/q}) = \mathcal{A}^T_{i,p/q}.$$

Hence $\boldsymbol{\xi}$, being an element in $\ker \mathfrak{D}^+_{i,p/q}$, is actually an element in $\ker \mathfrak{D}^T_{i,p/q}$. This proves that $U^n \ker \mathfrak{D}^+_{i,p/q}$ is a subgroup of the homology of the mapping cone of $D^T_{i,p/q}$. $\square$

Now we are ready to prove Proposition 1.6 stated in the introduction.

Proof of Proposition 1.6. Lemmas 2.4 and 2.7 imply that

$$\begin{cases} H_{\lfloor \frac{i+p(s-1)}{q} \rfloor} \geq H_0 = V_0 \geq V_{\lfloor \frac{i+ps}{q} \rfloor} & \text{if } s > 0, \\ H_{\lfloor \frac{i+p(s-1)}{q} \rfloor} \leq H_0 = V_0 \leq V_{\lfloor \frac{i+ps}{q} \rfloor} & \text{if } s < 0. \end{cases} \tag{3}$$

Given $\xi \in \mathcal{T}^+$, define

$$\rho(\xi) = \{(s, \xi_s)\}_{s \in \mathbb{Z}}$$

as follows. If

$$V_{\lfloor \frac{i}{q} \rfloor} \geq H_{\lfloor \frac{i+p(-1)}{q} \rfloor}, \tag{4}$$

let

$$\xi_{-1} = U^{V_{\lfloor \frac{i}{q} \rfloor} - H_{\lfloor \frac{i+p(-1)}{q} \rfloor}} \xi, \quad \xi_0 = \xi;$$

if

$$V_{\lfloor \frac{i}{q} \rfloor} < H_{\lfloor \frac{i+p(-1)}{q} \rfloor}, \tag{5}$$

let

$$\xi_{-1} = \xi, \quad \xi_0 = U^{H_{\lfloor \frac{i+p(-1)}{q} \rfloor} - V_{\lfloor \frac{i}{q} \rfloor}} \xi.$$

For other s, using (3), let

$$\xi_s = \begin{cases} U^{H_{\lfloor \frac{i+p(s-1)}{q} \rfloor} - V_{\lfloor \frac{i+ps}{q} \rfloor}} \xi_{s-1}, & \text{if } s > 0, \\ U^{V_{\lfloor \frac{i+p(s+1)}{q} \rfloor} - H_{\lfloor \frac{i+ps}{q} \rfloor}} \xi_{s+1}, & \text{if } s < -1. \end{cases}$$

As argued in the proof of Lemma 2.8, $\xi_s = 0$ when $|s| \gg 0$. Then ρ maps $\mathcal{T}^+$ into $\ker \mathfrak{D}^T_{i,p/q}$ and $U\rho(\mathbf{1}) = 0$. In light of Lemma 2.9, the grading of $\rho(\mathbf{1})$ is $d(S^3_{p/q}(K), i)$.

If (4) holds, the map

$$\mathfrak{v}^+_{\lfloor \frac{i}{q} \rfloor} : \left(0, \mathfrak{A}^T_{\lfloor \frac{i}{q} \rfloor}\right) \to (0, \mathfrak{B}^+)$$

is $U^{V_{\lfloor \frac{i}{q} \rfloor}}$. Using Remark 2.3 and comparing (2), the grading of $\rho(\mathbf{1})$ can be computed by

$$d(L(p,q), i) - 2V_{\lfloor \frac{i}{q} \rfloor}.$$

If (5) holds, the map

$$\mathfrak{h}^+_{\lfloor \frac{i+p(-1)}{q} \rfloor} : \left(-1, \mathfrak{A}^T_{\lfloor \frac{i+p(-1)}{q} \rfloor}\right) \to (0, \mathfrak{B}^+)$$

is $U^{H_{\lfloor \frac{i+p(-1)}{q} \rfloor}}$. The grading of $\rho(\mathbf{1})$ can be computed by

$$d(L(p,q), i) - 2H_{\lfloor \frac{i+p(-1)}{q} \rfloor}.$$

□

Remark 2.10. The argument in the proof of Lemma 2.7 implies that $V_k = H_{-k}$ for any $k \in \mathbb{Z}$. So Proposition 1.6 may be stated as

$$d(S^3_{p/q}(K), i) = d(L(p,q), i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, V_{\lfloor \frac{p+q-1-i}{q} \rfloor}\}.$$

Proof of Theorem 2.5. The first part of Theorem 2.5 immediately follows from Proposition 1.6.

If $d(S^3_{p/q}(K), i) = d(L(p,q), i)$ for all i, then

$$\max\{V_{\lfloor \frac{i}{q} \rfloor}, H_{\lfloor \frac{i+p(-1)}{q} \rfloor}\} = 0 \tag{6}$$

for all i. In particular, $V_0 = 0$. It follows from Lemma 2.7 that $H_0 = 0$.

If $V_0 = H_0 = 0$, then (6) holds for all i. So $d(S^3_{p/q}(K), i) = d(L(p,q), i)$. □

3. Casson–Walker, Casson–Gordon invariants and the correction term

3.1. Casson–Walker invariant. The Casson invariant is one of the many invariants of a closed three-manifold Y that can be obtained by studying representations of its fundamental group in a certain non-abelian group G. Roughly speaking, the Casson invariant of an integral homology sphere Y is obtained by counting representations of $\pi_1(Y)$ in $G = SU(2)$. The geometric structure used to obtain a topological invariant is a Heegaard splitting of Y. An alternative gauge-theoretical approach uses flat bundles together with a Riemannian metric on Y and leads to a refinement of the Casson invariant, the Floer homology.

By extending Casson's $SU(2)$ intersection theory to include reducible representations, Walker extended the Casson invariant to rational homology spheres. Most remarkably, Walker's invariant admits a purely combinatorial definition in terms of surgery presentations. The following proposition is the special case of a more general surgery formula, when K is a null-homologous knot in a rational homology sphere Y (see [2, Theorem 2.8]). Our convention here is that $\lambda(S^3_{+1}(T)) = 1$, where T is the right-hand trefoil.

Proposition 3.1. *Let K be a null-homologous knot in a rational homology three-sphere Y, and let $L(p,q)$ be the lens space obtained by (p/q)-surgery on the unknot in S^3. Then*

$$\lambda(Y_{p/q}(K)) = \lambda(Y) + \lambda(L(p,q)) + \frac{q}{2p}\Delta''_K(1). \tag{7}$$

Here, the Alexander polynomial Δ_K is normalized to be symmetric and satisfy $\Delta_K(1) = 1$.

Definition 3.2. Given two coprime numbers p and q, the *Dedekind sum* $s(q,p)$ is

$$s(q,p) := \operatorname{sign}(p) \cdot \sum_{k=1}^{|p|-1} \Big(\Big(\frac{k}{p}\Big)\Big)\Big(\Big(\frac{kq}{p}\Big)\Big),$$

where

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

The next proposition also follows from [2, Theorem 2.8].

Proposition 3.3. *For a lens space $L(p,q)$, one has $\lambda(L(p,q)) = -\frac{1}{2}s(q,p)$.*

When $p, q > 0$, write p/q as a continued fraction

$$\frac{p}{q} = [a_1, \dots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\ddots}}}.$$

We learn from Rasmussen [16, Lemma 4.3] that the Casson–Walker invariant of $L(p,q)$ can be calculated alternatively by the formula

$$s(q,p) = \frac{1}{12}\Big(\frac{q}{p} + \frac{q'}{p} + \sum_{i=1}^{n}(a_i - 3)\Big), \tag{8}$$

where $0 < q' < p$ is the unique integer such that $qq' \equiv 1 \pmod p$.

Lemma 3.4. *The Casson–Walker invariant of a lens space $\lambda(L(p,q))$ vanishes if and only if $q^2 \equiv -1 \pmod p$.*

Proof. If $\lambda(L(p,q)) = 0$, we must have $q + q' \equiv 0 \pmod p$ in view of formula (8). Together with the definition of q', we immediately see

$$q^2 \equiv -qq' \equiv -1 \pmod p.$$

On the other hand, it is well known from the classification result of lens spaces that $L(p,q)$ is orientation-preserving homeomorphic to $L(p,q')$. Hence,

$$\lambda(L(p,q)) = \lambda(L(p,q')).$$

If $q^2 \equiv -1 \pmod p$, then $q' \equiv -q \pmod p$; so we have

$$\lambda(L(p,q)) = \lambda(L(p,-q)) = -\lambda(L(p,q)).$$

This implies $\lambda(L(p,q)) = 0$. □

The Casson–Walker invariant is closely related to the correction terms and the Euler characteristic of HF_{red}. The following theorem is established as [17, Theorem 3.3], whose special case was also known in [8, Theorem 5.1].

Theorem 3.5. *For a rational homology sphere Y, we have*

$$|H_1(Y;\mathbb{Z})|\lambda(Y) = \sum_{\mathfrak{s}\in\text{Spin}^c(Y)} \Big(\chi(HF_{\text{red}}(Y,\mathfrak{s})) - \frac{1}{2}d(Y,\mathfrak{s})\Big).$$

3.2. Casson–Gordon invariant. Let us recall the following G-signature theorem for closed four-manifolds [1, Proposition 6.18].

Theorem 3.6 (G-signature theorem). *Suppose $\tilde{X} \xrightarrow{\pi} X$ is an m-fold cyclic cover of closed four-manifolds branched over a closed surface F in X. Then,*

$$\text{sig}(\tilde{X}) = m \cdot \text{sig}(X) - [F]^2 \cdot \frac{m^2-1}{3m}.$$

Here $[F]^2 = \langle \text{PD}^{-1}([F]) \smile \text{PD}^{-1}([F]), [X]\rangle$.

Consider a closed oriented three-manifold Y with $H_1(Y;\mathbb{Z}) = \mathbb{Z}_m$. It has a unique m-fold cyclic cover $\tilde{Y} \to Y$. Pick up an m-fold cyclic branched covering of four-manifold $\tilde{W} \to W$, branched over a properly embedded surface F in W, such that

$$\partial(\tilde{W} \to W) = (\tilde{Y} \to Y).$$

The existence of such (W, F) follows from [3, Lemma 2.2].

Definition 3.7. The total Casson–Gordon invariant of Y is given by

$$\tau(Y) = m \cdot \text{sig}(W) - \text{sig}(\tilde{W}) - [F]^2 \cdot \frac{m^2-1}{3m}.$$

It is a standard argument to see the independence of the definition on the choice of the four-manifolds cover $\tilde{W} \to W$. Suppose $\tilde{W'} \to W'$ is another cover that bounds $\tilde{Y} \to Y$, then we can construct a branched cover $-\tilde{W'} \cup_{\tilde{Y}} \tilde{W} \to -W' \cup_Y W$ of closed four-manifolds. It follows readily from Novikov additivity and the G-signature theorem that the invariant is well defined.

Definition 3.8. Let K be a knot in an integral homology sphere Y. The *generalized signature function* $\sigma_K(\xi)$ is the signature of the matrix $A(\xi) := (1-\bar{\xi})A + (1-\xi)A^T$ for a Seifert matrix A of K, where $|\xi| = 1$.

A surgery formula for the total Casson–Gordon invariant was established in [2, Lemma 2.22].

Proposition 3.9. *Let K be a knot in an integral homology sphere Y, then*

$$\tau(Y_{p/q}(K)) = \tau(L(p,q)) - \sigma(K,p), \tag{9}$$

where $\sigma(K,p) = \sum_{r=1}^{p-1} \sigma_K(e^{2i\pi r/p})$.

Quite amazingly, the total Casson–Gordon invariant of the lens space $L(p,q)$ is also related to the Dedekind sum.

Proposition 3.10. *For a lens space $L(p,q)$, one has $\tau(L(p,q)) = -4p \cdot s(q,p)$.*

3.3. Cosmetic surgeries with slopes of opposite signs. In this subsection, we derive an obstruction for purely cosmetic surgeries with slopes of opposite signs. Recall that both $\mathrm{Spin}^c(S^3_{p/q}(K))$ and $\mathrm{Spin}^c(L(p,q))$ are identified with $\mathbb{Z}/p\mathbb{Z}$. This leads to an explicit identification of $\mathrm{Spin}^c(S^3_{p/q}(K))$ with $\mathrm{Spin}^c(L(p,q))$ in the statement of the next proposition.

Proposition 3.11. *Given $p, q_1, q_2 > 0$ and a knot K in S^3. If*

$$Z = S^3_{p/q_1}(K) \cong S^3_{-p/q_2}(K)$$

as oriented manifolds, then

$$\Delta''_K(1) = 0, \qquad \sum_{\mathfrak{s} \in \mathrm{Spin}^c(Z)} \chi(HF_{\mathrm{red}}(Z,\mathfrak{s})) = 0,$$

and there exists a one-to-one correspondence

$$\sigma : \mathrm{Spin}^c(L(p,q_1)) \to \mathrm{Spin}^c(L(p,q_2))$$

such that

$$d(S^3_{p/q_1}(K),\mathfrak{s}) = d(L(p,q_1),\mathfrak{s}) = d(S^3_{-p/q_2}(K),\sigma(\mathfrak{s})) = -d(L(p,q_2),\sigma(\mathfrak{s}))$$

for every $\mathfrak{s}$.

Proof. Using the surgery formulae (7) and (9), we can compute the Casson–Walker and Casson–Gordon invariants of Z from its two surgery presentations and obtain

$$\lambda(Z) = \lambda(L(p,q_1)) + \frac{q_1}{2p}\Delta''_K(1) = \lambda(L(p,-q_2)) + \frac{-q_2}{2p}\Delta''_K(1),$$
$$\tau(Z) = \tau(L(p,q_1)) - \sigma(K,p) = \tau(L(p,-q_2)) - \sigma(K,p).$$

In light of Propositions 3.3 and 3.10, we must have $\Delta''_K(1) = 0$, hence

$$\lambda(Z) = \lambda(S^3_{p/q_1}(K)) = \lambda(L(p,q_1)).$$

This, according to Theorem 3.5, implies

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(Z)} \Big(\chi(HF_{\mathrm{red}}(Z,\mathfrak{s})) - \frac{1}{2}d(Z,\mathfrak{s})\Big) = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(L(p,q_1))} -\frac{1}{2}d(L(p,q_1),\mathfrak{s}).$$

It follows from Theorem 2.5 that

$$d(S^3_{p/q_1}(K),\mathfrak{s}) \le d(L(p,q_1),\mathfrak{s})$$

for any knot K and $p/q_1 > 0$. Therefore,

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(Z)} \chi(HF_{\mathrm{red}}(Z,\mathfrak{s})) \le 0.$$

On the other hand,

$$\lambda(Z) = \lambda(S^3_{-p/q_2}(K)) = \lambda(L(p,-q_2)).$$

Again, this implies

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(Z)} \Big(\chi(HF_{\mathrm{red}}(Z,\mathfrak{s})) - \frac{1}{2}d(Z,\mathfrak{s})\Big) = \sum_{\mathfrak{s}\in\mathrm{Spin}^c(L(p,-q_2))} -\frac{1}{2}d(L(p,-q_2),\mathfrak{s}).$$

With negative surgery coefficient $-p/q_2$, Theorem 2.5 implies that

$$d(S^3_{-p/q_2}(K),\mathfrak{s}) = -d(S^3_{p/q_2}(\overline{K}),\mathfrak{s}) \ge -d(L(p,q_2),\mathfrak{s}) = d(L(p,-q_2),\mathfrak{s}).$$

Therefore,

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(Z)} \chi(HF_{\mathrm{red}}(Z,\mathfrak{s})) \ge 0.$$

This implies

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(Z)} \chi(HF_{\mathrm{red}}(Z,\mathfrak{s})) = 0$$

and

$$d(S^3_{p/q_i}(K),\mathfrak{s}) = d(L(p,q_i),\mathfrak{s})$$

for $i = 1, 2$ and every $\mathfrak{s}$. □

It is a natural question to ask what three-manifolds may be obtained via purely cosmetic surgeries on knots in S^3. The above obstruction enables us to eliminate the following class of three-manifolds that includes all Seifert fibred rational homology spheres.

Corollary 3.12. *If Z is a plumbed three-manifold of a negative-definite graph with at most one bad point, then Z can not be obtained via purely cosmetic surgeries on knots in S^3.*

Proof. By [10, Corollary 1.4], all elements of $HF^+(Z)$ have even $\mathbb{Z}/2\mathbb{Z}$ grading. This implies that in the case $HF_{\mathrm{red}}(Z) \neq 0$, it must be that

$$\sum_{\mathfrak{s} \in \mathrm{Spin}^c(Z)} \chi(HF_{\mathrm{red}}(Z, \mathfrak{s})) = \operatorname{rank} HF_{\mathrm{red}}(Z) \neq 0,$$

hence we can apply Proposition 3.11. The other case where $HF_{\mathrm{red}}(Z) = 0$ follows from discussions in [19]. □

4. Proof of the main theorem

Let

$$\hat{A}_k = C\{\max\{i, j-k\} = 0\}, \quad \hat{B} = C\{i = 0\}$$

and

$$\nu(K) = \min\{k \in \mathbb{Z} \mid \hat{v}_k : \hat{A}_k \to \widehat{CF}(S^3) \text{ induces a non-trivial map in homology}\}$$

be the knot invariant defined by Ozsváth–Szabó [14].

Proposition 4.1. *Suppose that $K \subset S^3$ is a knot with $V_0 = H_0 = 0$. Then $\nu(K) \leq 0$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \hat{A}_k \arrow[r, "i_A"] \arrow[d, "\hat{v}_k"'] & A_k^+ \arrow[d, "v_k^+"] \\ \hat{B} \arrow[r, "i_B"] & B^+ \end{tikzcd}$$

and the induced commutative diagram of homology. Since $U\mathbf{1} = 0$, $\mathbf{1} \in \mathfrak{A}_k^+$ is in the image of $(i_A)_*$. Since $V_0 = 0$, we have $v_0^+(\mathbf{1}) = \mathbf{1}$. The above commutative diagram shows that the induced map $(\hat{v}_0)_*$ is nontrivial in homology. Thus $\nu(K) \leq 0$. □

Proof of Theorem 1.2. By the result in [19], we only need to consider the case that r_1, r_2 have opposite signs. Suppose $r_1 = p/q_1$ and $r_2 = -p/q_2$, where p, q_1, q_2 are positive integers, $\gcd(p, q_1) = \gcd(p, q_2) = 1$. By Proposition 3.11, $d(S^3_{p/q_1}(K), i) = d(L(p, q_1), i)$. Theorem 2.5 implies that $V_0 = H_0 = 0$. By Proposition 4.1, we have $\nu(K) \leq 0$. Since $\nu(K) = \tau(K)$ or $\tau(K) + 1$ (see [14, Lemma 9.2] and [9, 15]), $\tau(K) \leq 0$. The same argument can be applied to $\overline{K}$ to show that $\tau(\overline{K}) \leq 0$. Since $\tau(\overline{K}) = -\tau(K)$, we must have $\tau(K) = 0$.

Since $\nu(K) = \tau(K)$ or $\tau(K) + 1$ and $\nu(K) \leq 0$, we must have $\nu(K) = 0$. The same argument can be applied to $\overline{K}$ to show that $\nu(\overline{K}) = 0$. So we can apply [14, Proposition 9.9] to conclude that $r_1 = -r_2$.

Using Proposition 3.11 and (7), we conclude that

$$\lambda(L(p, q_1)) = \lambda(S^3_{p/q_1}(K)) = \lambda(S^3_{-p/q_1}(K)) = \lambda(L(p, -q_1)) = -\lambda(L(p, q_1)).$$

So $\lambda(L(p, q_1)) = 0$. The fact that $q_1^2 \equiv -1 \pmod{p}$ follows from Lemma 3.4. □

5. The computation of HF_{red}

In order to get more information about the knot K, we need to consider the reduced Heegaard Floer homology HF_{red} of the surgered manifolds. If K admits purely cosmetic surgeries, our computation (Proposition 1.7) shows that $HF_{\mathrm{red}}(S^3_{p/q}(K))$ looks like the Heegaard Floer homology of the surgery on an amphicheiral knot. Thus it provides more evidence to Conjecture 1.1 for knots in S^3.

Let

$$\mathfrak{A}_{k,\mathrm{red}} = \mathfrak{A}_k^+/\mathfrak{A}_k^T$$

and

$$\mathcal{A}_{i,\mathrm{red}} = \bigoplus_{s\in\mathbb{Z}} \big(s, \mathfrak{A}_{\lfloor \frac{i+ps}{q} \rfloor,\mathrm{red}}(K)\big),$$

we have:

Proposition 5.1. *Suppose $K \subset S^3$ is a knot with $V_0 = H_0 = 0$. If either*

$$p/q > 0$$

or

$$p/q < 0, \quad d(S^3_{p/q}(K), i) = d(L(p,q), i),$$

then

$$\mathcal{A}_{i,\mathrm{red}} \cong HF_{\mathrm{red}}(S^3_{p/q}(K), i)$$

and the isomorphism preserves the absolute grading.

Lemma 5.2. *Suppose $K \subset S^3$ is a knot with $V_0 = H_0 = 0$. If $p/q > 0$, then $\mathfrak{D}^T_{i,p/q}$ is surjective and its kernel is isomorphic to $\mathcal{T}^+$; if $p/q < 0$, then $\mathfrak{D}^T_{i,p/q}$ is injective and its cokernel is isomorphic to $\mathcal{T}^+$.*

Proof. We always suppose $p > 0$ and $0 \le i \le p-1$. First consider the case that $p/q > 0$. The surjectivity of $\mathfrak{D}^T_{i,p/q}$ is guaranteed by Lemma 2.8. We define a map

$$\sigma : \mathcal{T}^+ \to H_*(\mathbb{A}_i^+)$$

as follows. Given $\xi \in \mathcal{T}^+$, let $\sigma(\xi) = \{(s, \xi_s)\}_{s\in\mathbb{Z}}$, where

$$\xi_s = \begin{cases} \xi, & \text{if } s \in \{-1, 0\}, \\ U^{H_{\lfloor \frac{i+p(s-1)}{q} \rfloor} - V_{\lfloor \frac{i+ps}{q} \rfloor}} \xi_{s-1} = U^{H_{\lfloor \frac{i+p(s-1)}{q} \rfloor}} \xi_{s-1}, & \text{if } s > 0, \\ U^{V_{\lfloor \frac{i+p(s+1)}{q} \rfloor} - H_{\lfloor \frac{i+ps}{q} \rfloor}} \xi_{s+1} = U^{V_{\lfloor \frac{i+p(s+1)}{q} \rfloor}} \xi_{s+1}, & \text{if } s < -1. \end{cases}$$

We claim that there is a short exact sequence

$$0 \longrightarrow \mathcal{T}^+ \xrightarrow{\ \sigma\ } \mathcal{A}_i^T \xrightarrow{\ \mathfrak{D}^T_{i,p/q}\ } H_*(\mathbb{B}_i^+) \longrightarrow 0.$$

In fact, σ is clearly injective and the image of σ is in the kernel of $\mathfrak{D}^T_{i,p/q}$. Suppose $\{(s, \xi_s)\}_{s\in\mathbb{Z}}$

is in the kernel of $\mathfrak{D}^T_{i,p/q}$, we want to show that it is in the image of σ. Since

$$V_{\lfloor \frac{i}{q} \rfloor} = H_{\lfloor \frac{i+p(-1)}{q} \rfloor} = 0,$$

one must have $\xi_{-1} = \xi_0$. Let $\xi = \xi_0$, then we can check $\sigma(\xi) = \{(s, \xi_s)\}$. This finishes the proof of the case where $p/q > 0$.

Next consider the case where $p/q < 0$. We have

$$V_{\lfloor \frac{i+ps}{q} \rfloor} = 0 \text{ when } s < 0, \quad H_{\lfloor \frac{i+ps}{q} \rfloor} = 0 \text{ when } s \geq 0. \tag{10}$$

Suppose that $\{(s, \xi_s)\}_{s\in\mathbb{Z}}$ is in the kernel of $\mathfrak{D}^T_{i,p/q}$. Then

$$U^{H_{\lfloor \frac{i+p(s-1)}{q} \rfloor}} \xi_{s-1} + U^{V_{\lfloor \frac{i+ps}{q} \rfloor}} \xi_s = 0 \tag{11}$$

for any $s \in \mathbb{Z}$. By the definition of direct sum, $\xi_s = 0$ when $|s|$ is sufficiently large. Suppose $\xi_s = 0$ for some $s > 0$, then it follows from (10) and (11) that $\xi_{s-1} = 0$. So we have $\xi_s = 0$ for all $s \geq 0$. Similarly, $\xi_s = 0$ for all $s < 0$. This proves that $\mathfrak{D}^T_{i,p/q}$ is injective.

Given

$$\boldsymbol{\eta} = \{(s, \eta_s)\}_{s\in\mathbb{Z}} \in H_*(\mathbb{B}^+),$$

let

$$\phi(\boldsymbol{\eta}) = \eta_0 + \sum_{s>0} U^{\sum_{j=1}^{s} V_{\lfloor \frac{i+p(j-1)}{q} \rfloor}} \eta_s + \sum_{s<0} U^{\sum_{j=1}^{-s} H_{\lfloor \frac{i-pj}{q} \rfloor}} \eta_s.$$

We claim that the sequence

$$0 \longrightarrow \mathcal{A}^T_i \xrightarrow{\mathfrak{D}^T_{i,p/q}} H_*(\mathbb{B}^+_i) \xrightarrow{\phi} \mathcal{T}^+ \longrightarrow 0$$

is exact. It is routine to check $\phi \circ \mathfrak{D}^T_{i,p/q} = 0$. Moreover, suppose $\phi(\boldsymbol{\eta}) = 0$. Let $M > 0$ be an integer such that $\eta_s = 0$ whenever $|s| > M$. Define

$$\xi_s = \begin{cases} \eta_s, & \text{if } s \leq -M, \\ \eta_s - U^{H_{\lfloor \frac{i+p(s-1)}{q} \rfloor}} \xi_{s-1}, & \text{if } -M < s < 0, \\ \eta_{s+1}, & \text{if } s \geq M-1, \\ \eta_{s+1} - U^{V_{\lfloor \frac{i+p(s+1)}{q} \rfloor}} \xi_{s+1}, & \text{if } 0 \leq s < M-1. \end{cases}$$

We can check $\mathfrak{D}^T_{i,p/q}\{(s, \xi_s)\} = \boldsymbol{\eta}$, where at $(0, \eta_0)$ we use the fact that $\phi(\boldsymbol{\eta}) = 0$. This proves $\ker\phi = \operatorname{im}\mathfrak{D}^T_{i,p/q}$. The image of ϕ is clearly $\mathcal{T}^+$, so $\mathcal{T}^+$ is isomorphic to the cokernel of $\mathfrak{D}^T_{i,p/q}$. □

Proof of Proposition 5.1. When $p/q > 0$, we can identify $HF^+(S^3_{p/q}, i)$ with the kernel of $\mathfrak{D}^+_{i,p/q}$. Then there is a natural projection map

$$\pi : HF^+(S^3_{p/q}, i) \to \mathcal{A}_{i,\mathrm{red}}.$$

We claim that there is a short exact sequence

$$0 \longrightarrow \mathcal{T}^+ \xrightarrow{\sigma} HF^+(S^3_{p/q}, i) \xrightarrow{\pi} \mathcal{A}_{i,\mathrm{red}} \longrightarrow 0,$$

where σ is the map defined in Lemma 5.2.

From Lemma 5.2 we know that σ is injective, and $\operatorname{im}\sigma \subset \ker \pi$. If $\boldsymbol{\xi} \in \ker \mathfrak{D}^+_{i,p/q}$ is in the kernel of π, then $\boldsymbol{\xi}$ is contained in $\mathbb{A}^T_{i,p/q}$, so

$$\boldsymbol{\xi} \in \ker \mathfrak{D}^T_{i,p/q} = \operatorname{im}\sigma.$$

Next we show that π is surjective. Let $\pi' : \mathbb{A}^+_i \to \mathbb{A}_{i,\mathrm{red}}$ be the projection map. We need to show that for any $\boldsymbol{\zeta} \in \mathbb{A}_{i,\mathrm{red}}$, there exists a $\boldsymbol{\xi} \in \ker \mathfrak{D}^+_{i,p/q}$ with $\pi'(\boldsymbol{\xi}) = \boldsymbol{\zeta}$. In fact, let $\boldsymbol{\xi}_1$ be any element with $\pi'(\boldsymbol{\xi}_1) = \boldsymbol{\zeta}$. Since $\mathfrak{D}^T_{i,p/q}$ is surjective, there exists $\boldsymbol{\xi}_2 \in \mathbb{A}^T_i$ with

$$\mathfrak{D}^T_{i,p/q}(\boldsymbol{\xi}_2) = \mathfrak{D}^+_{i,p/q}(\boldsymbol{\xi}_1),$$

then $\boldsymbol{\xi} = \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$ is the element we want. This finishes the proof of the claim.

The claim immediately implies our conclusion when $p/q > 0$.

When $p/q < 0$, suppose $d(S^3_{p/q}(K), i) = d(L(p,q), i)$. We claim that

$$\operatorname{im} \mathfrak{D}^+_{i,p/q} = \operatorname{im} \mathfrak{D}^T_{i,p/q}.$$

Otherwise, $\operatorname{im} \mathfrak{D}^+_{i,p/q}$ is strictly larger than $\operatorname{im} \mathfrak{D}^T_{i,p/q}$. Then $\phi(\operatorname{im} \mathfrak{D}^+_{i,p/q})$ would contain a nonzero element, where ϕ is the map defined in Lemma 5.2. Hence $\mathbf{1} \in \phi(\operatorname{im} \mathfrak{D}^+_{i,p/q})$. By the exact triangle (1), $U^n HF^+(S^3_{p/q}(K), i)$ is contained in $\mathrm{incl}_*(\operatorname{coker} \mathfrak{D}^+_{i,p/q})$ when $n \gg 0$. It follows that the bottommost element in $U^n HF^+(S^3_{p/q}(K), i)$ for $n \gg 0$ has grading higher than the grading of $(0, \mathbf{1}) \in (0, B^+)$, which is $d(L(p,q), i)$. This gives a contradiction.

Now our conclusion easily follows from the claim and Lemma 5.2. $\square$

By the proof of Theorem 1.2, Proposition 1.7 is an easy corollary of the following proposition.

Proposition 5.3. *Suppose $K \subset S^3$ is a knot with $V_0 = H_0 = 0$. Then there exists a constant $C = C(K)$, such that*

$$\operatorname{rank} HF_{\mathrm{red}}(S^3_{p/q}(K)) = |q| \cdot C$$

for any coprime integers p, q with $p/q > 0$. Moreover, if $d(S^3_{p/q}(K), i) = d(L(p,q), i)$ for all i, then the above equality also holds for $p/q < 0$.

Proof. Let $C = \sum_{k\in\mathbb{Z}} \operatorname{rank} \mathfrak{A}_{k,\mathrm{red}}$. In

$$\bigoplus_{i\in\mathbb{Z}/p\mathbb{Z}} \mathbb{A}_{i,\mathrm{red}} = \bigoplus_{i=0}^{p-1} \bigoplus_{s\in\mathbb{Z}} \big(s, \mathfrak{A}_{\lfloor \frac{i+ps}{q} \rfloor,\mathrm{red}}(K)\big),$$

each $\mathfrak{A}_{k,\mathrm{red}}$ appears exactly $|q|$ times. It follows from Proposition 5.1 that

$$\operatorname{rank} HF_{\mathrm{red}}(S^3_{p/q}(K)) = |q| \cdot C,$$

whenever the conditions in the statement of the theorem are satisfied. Note that the constant $C(K)$ is indeed the same as the rank of $HF_{\mathrm{red}}(S^3_p(K))$ for $p > 0$. $\square$

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