PRESCRIBING Q-CURVATURE PROBLEM ON $\mathbf{S^n}$

JUNCHENG WEI AND XINGWANG XU

ABSTRACT. Let $\mathbf{P_n}$ be the $n-$th order Paneitz operator on $\mathbf{S}^n, n \geq 3$. We consider the following prescribing $Q-$curvature problem on $\mathbf{S^n}$:

$$\mathbf{P}_n u + (n-1)! = Q(x)e^{nu} \quad \text{on} \quad \mathbf{S^n},$$

where Q is a smooth positive function on $\mathbf{S^n}$ satisfying the following non-degeneracy condition:

$$(\Delta Q)^2 + |\nabla Q|^2 \neq 0.$$

Let $G^* : \mathbf{S}^n \to \mathbf{R}^{n+1}$ be defined by

$$G^*(x) = (-\Delta Q(x), \nabla Q(x)).$$

We show that if $Q > 0$ is non-degenerate and $deg(\frac{G^*}{|G^*|}, \mathbf{S}^n) \neq 0$, then the above equation has a solution. When n is even, this has been established in our earlier work [29]. When n is odd, P_n becomes a *pseudo-differential operator*. Here we develop a unified approach to treat both even and odd cases. The key idea is to write it as an integral equation and use Liapunov-Schmidt reduction method.

1. INTRODUCTION

On a general Riemannian manifold M with metric g, a metrically defined operator A_g is said to be *conformally invariant* if under the conformal change in metric $g_u = e^{2u}g$, the pair of corresponding operators A_{g_u} and A_g are related by

$$A_{g_u}(\varphi) = e^{-bu}A_g(e^{au}\varphi) \tag{1.1}$$

for all $\varphi \in C^\infty(M)$ and some constants a and b.

One such well known second order conformally covariant operator is the conformal Laplacian which is closely related to the Yamabe problem and more generally, to the problem of prescribing scalar curvature: *Given a smooth positive function K defined on a compact Riemannian*

1991 *Mathematics Subject Classification.* Primary 53C21; Secondary 58j60.
Key words and phrases. Q-curvature, existence, pseudo-differential operator.

manifold (M, g_0) *of dimension* $n \geq 2$*, does there exist a metric* g *conformal to* g_0 *for which* K *is the scalar curvature of the new metric* g*?* If $g = e^{2u} g_0$ for $n = 2$ or $g = u^{\frac{4}{n-2}} g_0$ for $n \geq 3$, our problem is reduced to finding solutions to the following nonlinear elliptic equations:

$$\Delta_{g_0} u + K e^{2u} = k_0 \tag{1.2}$$

for $n = 2$, or

$$\frac{4(n-1)}{n-2} \Delta_{g_0} u + K u^{\frac{n+2}{n-2}} = k_0 u, \; u > 0 \text{ on } M \tag{1.3}$$

for $n \geq 3$. (Here Δ_{g_0} denotes the Laplace-Beltrami operator of (M, g_0), k_0 is the Gaussian curvature of g_0 when $n = 2$ and the scalar curvature of g_0 when $n \geq 3$.) The problem of determining which K admits a solution to (1.2) (or (1.3)) has been studied extensively. See [5], [6], [9], [12], [13], [14], [21], [22], [28], [32] and the references therein.

In search for a higher order conformally invariant operator, Paneitz [25] discovered an interesting fourth order operator on a compact 4-manifold

$$\mathbf{P}_4 \varphi = \Delta^2 \varphi + \delta\left(\frac{2}{3} R I - 2 Ric\right) d\varphi$$

where δ denotes the divergence, d the differential and Ric the Ricci curvature of the metric g. Under the conformal change $g_u = e^{2u} g$, $\mathbf{P_4}$ undergoes the transformation $(\mathbf{P}_4)_u = e^{-4u} \mathbf{P_4}$ (i.e., $a = 0, b = 4$ in (1.1)). See [1], [4], [7], [10] and [11] for a discussion of general properties of Paneitz operators.

On a general compact manifold of dimension n, the existence of such an operator $\mathbf{P_n}$ with $(\mathbf{P}_n)_u = e^{-nu} \mathbf{P_n}$ for even dimension is established in [19]. However $\mathbf{P_n}$'s explicit form is known only for Euclidean space R^n with standard metric ($\mathbf{P_n} = (-\Delta)^{\frac{n}{2}}$) and hence only for the sphere $\mathbf{S^n}$ with standard metric g_0. The explicit formula for $\mathbf{P_n}$ on $\mathbf{S^n}$ which appears in [1] and [2] is

$$\mathbf{P_n} = \begin{cases} \Pi_{k=1}^{\frac{n-2}{2}} (-\Delta + k(n-k-1)), & \text{for } n \text{ even }, \\ (-\Delta + (\frac{n-1}{2})^2)^{\frac{1}{2}} \Pi_{k=0}^{\frac{n-3}{2}} (-\Delta + k(n-k-1)), & \text{for } n \text{ odd }. \end{cases}$$

In analogy to the second order case there exists some naturally defined curvature invariant Q_n of order n which, under the conformal change of metric $g_u = e^{2u} g_0$, is related to $\mathbf{P}_n u$ through the following differential equation

$$\mathbf{P}_n u + (Q_n)_0 = (Q_n)_u e^{nu} \qquad \text{on } M. \tag{1.4}$$

Motivated by the problem of the prescribing Gaussian curvature on S^2, we pose the following prescribing Q_n-curvature problem on $\mathbf{S^n}$: *Given a smooth function Q on $\mathbf{S^n}$, find a conformal metric $g_u = e^{2u}g_0$ for which $(Q_n)_u = Q$.*

We remark that there is a similar problem for general compact Riemannian manifolds. But since, in this case, the explicit expression for the operator $\mathbf{P_n}$ is unknown, we will not address the general prescribing Q_n curvature problem. When $n = 4$, there have been many works recently. See [3], [7], [15], [16] and the references therein.

Clearly the above question is equivalent to finding a solution of the differential equation

$$\mathbf{P}_n u + (n-1)! = Qe^{nu} \qquad \text{on } \mathbf{S^n}. \tag{1.5}$$

In our previous paper [29], we have treated the case $n = 2m$, i.e. n is **even**. (A different approach, based on curvature flows, was given recently in [24].) In that case, the operator $\mathbf{P_n}$ is a point-wise operator and by stereographic projection to $\mathbf{R^n}$, it simply becomes $(-\Delta)^m$. In this paper, we shall consider both **even** and **odd** cases. Note that when n is odd, the operator $\mathbf{P_n}$ involves $(-\Delta)^{\frac{n}{2}}$ which is a pseudo-differential operator.

Our basic idea is to transform (1.5) into an integral equation:

$$u(x) = \frac{1}{\beta_n}\int_{\mathbf{R^n}} \log[\frac{|y|}{|x-y|}]Q(y)e^{nu(y)}dy + C_0. \tag{1.6}$$

This approach was first taken in [8] and then later in [31]. As in [5] and [6], there are three main steps in the proofs: first a priori estimates, then a perturbation result, and finally a continuation argument. For a priori estimates, we work directly with the integral equation (1.6). (We note that a similar idea has been used in a recent paper [17].) For perturbation result, we use a direct Liapunov-Schmidt reduction method. The continuation argument is the same as before. The novelty of our approach is that we don't use any type of Moser-Trudinger inequalities. This also gives a new proof of the results in [5] and [6]. It is interesting to compare our approach here with the original approach of Chang-Yang in [5] and [6] and the flow-approach of Struwe [28].

To state the main results of this paper, we use the function introduced in [22]. For any smooth positive function Q on $\mathbf{S^n}$, Q is called *non-degenerate* if it satisfies the non-degeneracy condition:

$$(\Delta Q)^2 + |\nabla Q|^2 \neq 0 \tag{1.7}$$

on $\mathbf{S^n}$. For a non-degenerate function Q on $\mathbf{S^n}$, we can define the mapping $G^* : \mathbf{S^n} \longrightarrow \mathbf{R^{n+1}}$ by

$$G^*(x) = (-\Delta Q(x), \nabla Q(x)). \tag{1.8}$$

This mapping is well defined and it is never zero if the condition (1.7) is satisfied. Thus $\frac{G^*}{|G^*|}$ from $\mathbf{S^n}$ into $\mathbf{S^n}$ will be well defined. If we assume Q is C^3 function, then $\frac{G^*}{|G^*|}$ is C^1 on $\mathbf{S^n}$, hence its degree $deg(\frac{G^*}{|G^*|}, \mathbf{S^n})$ is well defined. Now we can state our main result as the following.

Theorem 1.1. *Suppose that $Q > 0$ on $\mathbf{S}^n$ $(n \geq 3)$ is non-degenerate and $deg(\frac{G^*}{|G^*|}, \mathbf{S^n}) \neq \mathbf{0}$, then the equation (1.5) has a solution.*

For example, if Q satisfies (1.7) and

$$\Delta Q(x)\Delta Q(-x) - \nabla Q(x) \cdot \nabla Q(-x) \geq 0, \ \forall x \in \mathbf{S^n}$$

then the equation (1.5) has a solution. The proof is similar to that of Corollary 1.1 of [22]. More examples of Q can also be found in [22].

Acknowledgment. The research of the first author is supported by an Earmarked Grant from RGC of Hong Kong. This research was initiated when the first author visited the Institute for Mathematical Sciences. He thanks the Institute of Mathematical Science at National University of Singapore for its hospitality. The research of the second author is partially supported by NUS research grant R-146-000-077-112. The part of this work was done when second author visited Mathematical Institute at Nanjing University, China. He would like to thank them for hospitality and providing excellent work condition. Both authors wish to thank the anonymous referees for careful reading and useful comments.

2. Preliminary results

The main aim in this section is to collect some materials we are going to use in this paper. The first result is the following classification of the solutions of an integral equation:

Lemma 2.1. *Let $Q(x) \equiv (n-1)!$ be a constant and let C_0 be a real constant. Assume $u \in C^2(\mathbf{R^n})$ is a solution to the integral equation*

$$u(x) = \frac{1}{\beta_n} \int_{\mathbf{R^n}} \log[\frac{|y|}{|x-y|}] Q(y) e^{nu(y)} dy + C_0, \tag{2.1}$$

for some constant C_0 such that $e^{nu} \in L^1(\mathbf{R^n})$. Then u is given by

$$u(x) = \log[\frac{2\lambda}{\lambda^2 + |x-x_0|^2}], \quad \lambda \in \mathbf{R}^+, \ \ x_0 \in \mathbf{R^n}. \tag{2.2}$$

Proof: See [8] or [31]. We just remark that the constant λ is defined by $\lambda = 2e^{-C_0} > 0$. □

As a starting step for estimates, we show the following lemma.

Lemma 2.2. *Suppose w is a C^2 function on $\mathbf{R^n}$ $(n \geq 3)$ such that*

(a) *Qe^{nw} is in $L^1(\mathbf{R^n})$ with $0 < m \leq Q \leq M$ for some constants m, M;*

(b) *in the sense of weak derivative, w satisfies the following equation:*

$$\Delta w + (n-2) \int_{\mathbf{R^n}} \frac{Q(y) e^{nw(y)}}{|x-y|^2} dy = 0. \tag{2.3}$$

Then there is a constant $C(w) > 0$ depending on m, n, M and w such that $|\Delta w|(x) \leq C(w)$ on $\mathbf{R^n}$.

Remark: In general, the constant $C(w)$ depends on the function w itself. Later in Theorem 3.3 of Section 3, we shall prove a priori estimates under some conditions on the function $Q(x)$.

Proof: Set $\alpha = \int_{\mathbf{R^n}} Q(y) e^{nw(y)} dy$. Then assumption (a) implies that $0 < \alpha < \infty$. This also implies that $\int_{\mathbf{R^n}} e^{nw(y)} dy$ is finite with upper bound depending only on m and α. Therefore there exists a large constant $R > 0$ such that

$$\int_{\mathbf{R}^n \setminus B_R(0)} Q(y) e^{nw(y)} dy \leq \frac{1}{8}. \tag{2.4}$$

Now notice that (2.3) holds almost everywhere. For any $x_0 \in \mathbf{R}^n \setminus B_{R+8}(0)$ such that (2.3) holds at x_0, we consider the solution h of the equation

$$\begin{cases} [(-\Delta)h](x) = (n-2) \int_{B_4(x_0)} \frac{Q(y)e^{nw(y)}}{|x-y|^2} dy & \text{in } B_4(x_0) \\ h = 0 & \text{on } \partial B_4(x_0). \end{cases} \tag{2.5}$$

Let

$$v_1(x) = \int_{B_4(x_0)} \left[\log \left(\frac{16}{|x-y|} \right) \right] Q(y) e^{nw(y)} dy \tag{2.6}$$

for all $x \in B_4(x_0)$. Since for all $x, y \in B_4(x_0)$, we have

$$|x-y| \leq |x-x_0| + |y-x_0| \leq 4 + 4 = 8.$$

Hence we conclude that

$$v_1(x) \geq 0 \quad \text{in} \quad B_4(x_0).$$

It is a routine calculation that

$$[(-\Delta)v_1](x) = (n-2) \int_{B_4(x_0)} \frac{Q(y)e^{nw(y)}}{|x-y|^2} dy. \tag{2.7}$$

Combine (2.5) and (2.7) to obtain:

$$\begin{cases} (-\Delta)[\pm h - v_1] \leq 0 & \text{in } \ B_4(x_0); \\ \pm h - v_1 \leq 0 & \text{on } \ \partial B_4(x_0), \end{cases} \tag{2.8}$$

in weak sense. The maximum principle ([20], Theorem 8.16) allows us to conclude that

$$|h(x)| \leq v_1(x) \ \ x \in B_4(x_0). \tag{2.9}$$

Now let us denote the measure $Q(y)e^{nw(y)}dy/\int_{B_4(x_0)} Q(y)e^{nw(y)}dy$ by $d\mu$. Therefore Jensen's inequality, together with (2.4), implies that

$$\begin{aligned}
&\int_{B_4(x_0)} \exp\left[4n|h(x)|\right] dx \\
\leq\ & \int_{B_4(x_0)} \exp\left[\frac{nv_1(x)}{2\int_{B_4(x_0)} Q(y)e^{nw(y)}dy}\right] dx \\
=\ & \int_{B_4(x_0)} \exp\left[\frac{n}{2}\int_{B_4(x_0)} \left\{\log\left(\frac{16}{|x-y|}\right)\right\} d\mu\right] dx \\
\leq\ & \int_{B_4(x_0)} \left\{\int_{B_4(x_0)} \left(\frac{16}{|x-y|}\right)^{n/2} d\mu\right\} dx \\
=\ & \int_{B_4(x_0)} \left\{\int_{B_4(x_0)} \left(\frac{16}{|x-y|}\right)^{n/2} dx\right\} d\mu \\
\leq\ & C,
\end{aligned} \tag{2.10}$$

where C is just a dimensional constant.

Now we consider the function $q(x) = w(x) - h(x)$ in the smaller ball $B_3(x_0)$. First we observe that, in weak sense,

$$\begin{aligned}
(\Delta q)(x) &= (\Delta w)(x) - (\Delta h)(x) \\
&= -(n-2)\left[\int_{\mathbf{R^n}} \frac{Q(y)e^{nw(y)}}{|x-y|^2}dy - \int_{B_4(x_0)} \frac{Q(y)e^{nw(y)}}{|x-y|^2}dy\right] \\
&= -(n-2)\int_{\mathbf{R}^n\setminus B_4(x_0)} \frac{Q(y)e^{nw(y)}}{|x-y|^2}dy.
\end{aligned}$$

If $x \in B_3(x_0)$ and $y \in \mathbf{R}^n\setminus B_4(x_0)$, then $|x-y| \geq |y-x_0|-|x-x_0| \geq 1$. Therefore we have

$$0 \leq (-\Delta q)(x) \leq (n-2)\alpha, \tag{2.11}$$

in weak sense.

Hence it follows from weak Harnack principle ([20], Theorem 8.17) that

$$\sup_{B_2(x_0)} q(x) \leq C[\|q^+\|_{L^2(B_3(x_0))} + \|\Delta q\|_{L^\infty(B_3(x_0))}]. \tag{2.12}$$

As we have seen in (2.11) the second term on the right is bounded independent of x_0. To estimate the previous one, we note that $q^+(x) =$

$(w-h)^+(x) \le w^+(x) + |h(x)|$ and also we have for all $t \ge 0$, $2e^t \ge t^2$. Thus we have

$$\begin{aligned}
&\int_{B_3(x_0)} [q^+(x)]^2 dx \\
\le\ & 2\int_{B_3(x_0)} e^{q^+(x)} dx \\
\le\ & 2\int_{B_3(x_0)} e^{w^+(x)} e^{|h(x)|} dx \\
\le\ & 2\int_{B_3(x_0)} \left\{1 + e^{w(x)}\right\} e^{|h(x)|} dx \\
\le\ & 2\left[\int_{B_3(x_0)} e^{nw(x)} dx\right]^{1/n} \left[\int_{B_3(x_0)} e^{n|h(x)|/(n-1)} dx\right]^{(n-1)/n} \\
& +2\int_{B_3(x_0)} e^{|h(x)|} dx \\
\le\ & C, \qquad (2.13)
\end{aligned}$$

where C is independent of x_0 by using (2.10), Hölder inequality as well as assumption (a).

Therefore, it follows that $w(y) = q(y) + h(y) \le C + |h(y)|$ in the even smaller ball $B_2(x_0)$. Therefore we reach the estimate:

$$\int_{B_2(x_0)} e^{4nw(y)} dy \le e^{4nC} \int_{B_2(x_0)} e^{4n|h(y)|} dy \le C_1, \qquad (2.14)$$

where we have used (2.10).

Next by the equation (2.3), we have, for any $|x_0|$ sufficiently large,

$$\begin{aligned}
|\Delta w|(x_0) &= (n-2)\int_{\mathbf{R^n}} \frac{Q(y)e^{nw(y)}}{|x_0-y|^2} dy \\
&\le M(n-2)\{\int_{\mathbf{R}^n\setminus B_2(x_0)} \frac{e^{nw(y)}}{|x_0-y|^2} dy + \int_{B_2(x_0)} \frac{e^{nw(y)}}{|x_0-y|^2} dy\} \\
&\le \frac{(n-2)M}{4}\int_{\mathbf{R^n}} e^{nw(y)} dy \\
&\quad +(n-2)M(\int_{B_2(x_0)} \frac{1}{|x_0-y|^{2p}} dy)^{1/p} \cdot (\int_{B_2(x_0)} e^{qnw(y)} dy)^{1/q},
\end{aligned}$$

where p and q are such that $1/p + 1/q = 1$. Since $n \ge 3$, choosing $p = \frac{4n-1}{8}$, then $p > 1$ and $q = \frac{4n-1}{4n-9} < 4$. Clearly with those choices of p and q, the first integral of the second term in the right side of above equation is bounded. Other two integrals are also bounded by (2.14) and the assumption (a).

Therefore $|\Delta w|$ is bounded almost everywhere on $\mathbf{R}^n \backslash B_{R+8}(0)$. But if $w \in C^2(\mathbf{R^n})$, then Δw is continuous and hence is bounded on $\bar{B}_{R+8}(0)$.

This finishes the proof of Lemma 2.2. □

With help of this lemma, we also have the following.

Lemma 2.3. *Suppose w is a C^2 function on $\mathbf{R^n}$ such that $0 \leq (-\Delta) w(x) \leq A$ almost everywhere on $\mathbf{R^n}$ for some constant A and $\int_{\mathbf{R^n}} Q(y) e^{nw(y)} dy = \alpha < \infty$ with $0 < m \leq Q \leq M$. Then there exists a constant B, depending on A, n, m, M and α, such that $w(x) \leq B$ almost everywhere on $\mathbf{R^n}$.*

Remark: Note that A depends on w. Hence the constant B also depends on w.

Proof: For any point x_0 in $\mathbf{R^n}$, let w_1 be the solution of the Poisson's problem

$$\begin{cases} (-\Delta) w_1 = (-\Delta w) := f & \text{in } B_1(x_0), \\ w_1 = 0 & \text{on } \partial B_1(x_0). \end{cases} \tag{2.15}$$

It follows from the elliptic estimate of Poisson's equation (for example, [20] on page 189, Theorem 8.16) that

$$|w_1(x)| \leq \sup_{B_1(x_0)} |w_1(x)| \leq C(n) \sup_{B_1(x_0)} |f| \leq CA, \tag{2.16}$$

since $w_1(x) = 0$ on $\partial B_1(x_0)$.

Now we set $w_2(x) = w(x) - w_1(x)$ in $B_1(x_0)$. Then it is obvious that $(-\Delta) w_2 = 0$ in the unit ball $B_1(x_0)$ in weak sense. By mean value property for harmonic functions, we reach at the estimate

$$\|w_2^+\|_{L^\infty(B_{1/2}(x_0))} \leq C_2(n) (\int_{B_1(x_0)} w_2^+ dx), \tag{2.17}$$

where w_2^+ is the positive part of w_2. However, by the definition of w_2, we have

$$w_2^+ \leq w^+ + |w_1|.$$

Notice that we have the obvious inequality

$$\int_{B_1(x_0)} nw^+ dx \leq \int_{B_1(x_0)} e^{nw} dx \leq \alpha.$$

Thus, combining those estimates, we get

$$\int_{B_1(x_0)} w_2^+ dx \leq \frac{\alpha}{2n} + \frac{A}{2n}\omega_n := C_3(n, A, \alpha),$$

where ω_n is the volume of solid unit ball in $\mathbf{R^n}$. Thus it follows from estimate (2.17) that

$$\|w_2^+\|_{L^\infty(B_{1/2}(x_0))} \leq C_2(n)C_3(n, A, \alpha).$$

Finally by definition again, we have

$$w = w_1 + w_2,$$

thus,

$$w^+ \leq w_2^+ + |w_1| \leq C_2(n)C_3(n, A, \alpha) + \frac{A}{2n},$$

which is independent of x_0. This holds for almost every $x \in B_1(x_0)$. Observe that the measure of countable union of measure zero sets is still zero, Lemma 2.3 follows by a simple covering argument. □

The next lemma is the so-called Pohozev's identity which implies some necessary conditions for the integral equation (2.1) to have a solution.

Lemma 2.4. *Suppose a C^2 function u satisfies the integral equation (2.1). Assume there exist constants $0 < m < M$ such that $m \leq Q \leq M$. Assume that $e^u \in L^n(\mathbf{R^n})$.*

(1) *If $| < x, \nabla Q(x) > | \leq C$ for some constant $C > 0$ and $|x|$ sufficiently large, then $\frac{2}{n\beta_n}\int_{\mathbf{R^n}} < x, \nabla Q(x) > e^{nu(x)}dx = \gamma(\gamma - 2)$;*
(2) *If there exists a constant $C > 0$ such that $|\nabla Q|(x) \leq C$ for $|x|$ sufficiently large, then $\int_{\mathbf{R^n}} \nabla Q(x)e^{nu(x)}dx = 0$,*

where γ is given by

$$\gamma = \frac{1}{\beta_n}\int_{\mathbf{R^n}} Q(x)e^{nu(x)}dx. \tag{2.18}$$

Proof: Part (1) has been shown in [31]. Notice that the fixed sign condition on Q plus the assumption on the integrability of the function e^u implies that the constant γ is finite and $Q(x)e^{nu(x)}$ is absolutely integrable over $\mathbf{R^n}$. Therefore Theorem 1 in [31] can be applied. We should point out that the proof provided there is not complete. It only provided the formal calculation. Notice that the proof of Lemma 2.1 in [31] depends on this part. But when Q is constant, the conditions in this lemma is clearly fulfilled. Here, for completeness of our argument, we fulfil the detail with those extra assumption as we have stated. The method for both cases is the same. We only deal with the second case. Since $Qe^{nu} \in L^{\infty}_{loc}(\mathbf{R^n})$ is absolutely integrable and u is of class C^2, both sides of equation (2.1) are C^2 functions and we can take twice weak derivative. This is to say that we can get:

$$\Delta u(x) = -\frac{n-2}{\beta_n}\int_{\mathbf{R^n}} \frac{Q(y)e^{nu(y)}}{|x-y|^2}dy, \tag{2.19}$$

in the sense of weak derivative. Therefore Lemmas 2.2 and 2.3 can be applied to conclude that Qe^{nu} is in $L^p(\mathbf{R^n})$ for any $p \geq 1$. Now through routine argument, we can see that we have the following:

$$\nabla u(x) = -\frac{1}{\beta_n}\int_{\mathbf{R^n}} \frac{(x-y)Q(y)e^{nu(y)}}{|x-y|^2}dy, \tag{2.20}$$

in the sense of weak derivative. By property of weak derivative, we also have

$$Q(x)\nabla e^{nu(x)} = -\frac{nQ(x)e^{nu(x)}}{\beta_n}\int_{\mathbf{R^n}} \frac{Q(y)e^{nu(y)}}{|x-y|^2}dy. \tag{2.21}$$

Now choose a smooth compact supported function $\eta(t)$ such that $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$ and also $-2 \leq \eta'(t) \leq 0$ for all t. Multiplying both sides of (2.21) by $\eta(\frac{|x|}{R})$ for all real number $R > 0$ and integrate over the ball $\mathbf{R^n}$, we obtain:

$$\begin{aligned} &\int_{\mathbf{R^n}} Q(x)\eta(\frac{|x|}{R})e^{nu(x)}\nabla u(x)dx \qquad (2.22)\\ = \; & -\frac{1}{\beta_n}\int_{\mathbf{R^n}}\{\int_{\mathbf{R^n}} \eta(\frac{|x|}{R})\frac{x-y}{|x-y|^2}Q(y)e^{nu(y)}dy\}Q(x)e^{nu(x)}dx. \end{aligned}$$

But the left hand side of the same equation will imply, by integration by parts,

$$\begin{aligned}&\int_{\mathbf{R^n}} Q(x)\eta(\frac{|x|}{R})e^{nu(x)}\nabla u(x)dx \\ = &-\frac{1}{n}\int_{\mathbf{R^n}} \nabla Q(x)[\eta(\frac{|x|}{R})]e^{nu(x)}dx \\ &+\frac{1}{nR}\int_{\mathbf{R^n}} \eta^{'}(\frac{|x|}{R})\frac{x}{|x|}Q(x)e^{nu(x)}dx.\end{aligned} \tag{2.23}$$

Notice that $|\eta^{'}(\frac{|x|}{R})|\leq 2$ for $R\leq|x|\leq 2R$ and otherwise it vanishes. Then integrability of $|Q|e^{nu}$ implies that the second integral approaches to zero as $R\to\infty$. Clearly the first integral goes to the integral of $(\nabla Q)e^{nu}$ with the help of the integrability of $(|\nabla Q|)e^{nu}$ over $\mathbf{R^n}$.

Now we consider the right hand side. For each fixed R, by Hardy-Littlewood-Sobolev inequality, the function $\frac{x-y}{|x-y|^2}Q(x)Q(y)e^{n(u(x)+u(y))}$ is absolutely integrable over $\mathbf{R^n}\times\mathbf{R^n}$ since we have $Q(x)e^{nu(x)}\in L^{2n/(2n-1)}(\mathbf{R^n})$. Hence we can take the limit under the integral sign as $R\to\infty$ by dominated convergence theorem. This implies that

$$\begin{aligned}&\frac{1}{n}\int_{\mathbf{R^n}}\nabla Q(x)e^{nu(x)}dx \\ = &-\frac{1}{\beta_n}\int_{\mathbf{R^n}}\{\int_{\mathbf{R^n}}\frac{x-y}{|x-y|^2}Q(y)e^{nu(y)}dy\}Q(x)e^{nu(x)}dx.\end{aligned} \tag{2.24}$$

Again the right hand side is absolutely integrable as a function over $\mathbf{R^n}\times\mathbf{R^n}$, thus we can conclude that the integral vanishes by interchange variables x and y. Hence the second part of Lemma 2.4 follows. □

Remark 2.5. *In Lemma 2.1, we may multiply the function Q by a suitable constant to make $\gamma=2$. Then the standard Pohozev's identity holds.*

Finally, we discuss the non-degeneracy for the linearized integral equation of (2.1) at standard solutions. For simplicity, we set:

$$U_{\Lambda,a}(x)=\log[\frac{2\Lambda}{\Lambda^2+|x-a|^2}], \tag{2.25}$$

for $\Lambda>0$ and $a\in\mathbf{R^n}$. By changing variables, we only need to prove

Theorem 2.6. *Suppose that the bounded function $\phi(x)$ satisfies the integral equation*

$$\phi(x) = \frac{n}{\beta_n} \int_{\mathbf{R^n}} \{\log[\frac{|y|}{|x-y|}]\} e^{nU_{1,0}(y)} \phi(y) dy + C. \tag{2.26}$$

Then there are constants C_j for $j = 0, 1, 2, \cdots, n$ such that the function $\phi(x)$ is given as

$$\phi(x) = \sum_{j=0}^{n} C_j \psi_j(x), \tag{2.27}$$

where

$$\psi_0(x) = \frac{|x|^2 - 1}{|x|^2 + 1}, \quad \psi_j(x) = \frac{2x_j}{1 + |x|^2}, \quad j = 1, 2, \cdots, n. \tag{2.28}$$

Proof: First of all, we want to show that if $\phi(x)$ satisfies the equation (2.26), then

$$h := \frac{n}{\beta_n} \int_{\mathbf{R^n}} e^{nU_{1,0}(x)} \phi(x) dx = 0. \tag{2.29}$$

In fact, it follows from the equation (2.26) and the definition of h that the following is true:

$$\begin{aligned} & \phi(\frac{x}{|x|^2}) - h \log |x| \\ = \;& \frac{n}{\beta_n} \int_{\mathbf{R^n}} \{\log[\frac{|y|}{|x||\frac{x}{|x|^2} - y|}]\} e^{nU_{1,0}(y)} \phi(y) dy \\ = \;& \frac{n}{\beta_n} \int_{\mathbf{R^n}} \{\log[\frac{1}{|\frac{y}{|y|^2} - x|}]\} e^{nU_{1,0}(y)} \phi(y) dy \\ = \;& \frac{n}{\beta_n} \int_{\mathbf{R^n}} \{\log[\frac{1}{|x-z|}]\} e^{nU_{1,0}(z)} \phi(\frac{z}{|z|^2}) dz, \end{aligned} \tag{2.30}$$

where we have used the special form for the function $U_{1,0}$. Since ϕ is bounded, the term $h \log |x|$ has to be bounded for x near zero which forces $h = 0$.

Next since the function $e^{nU_{1,0}(x)}\phi(x)$ is absolutely integrable over $\mathbf{R^n}$ and its decay behavior at ∞, we take the derivative under the integral sign. By doing so, we conclude that

$$\frac{1}{|x|^2}(-\Delta\phi)(\frac{x}{|x|^2}) = \frac{n(n-2)}{\beta_n} \int_{\mathbf{R^n}} [\frac{|z|^2}{|x-z|^2}] e^{nU_{1,0}(z)} \phi(\frac{z}{|z|^2}) dz. \tag{2.31}$$

Hence $-\Delta\phi$ is uniformly bounded for $|x| \geq 1$ which implies that ϕ has a limit at ∞. Then it follows from the equation (2.26) that the function $g(x) = \phi(\pi^{-1}(x)) - \log 2 + \log(1+|x|^2)$ is a well defined function on $\mathbf{S^n}$ where $\pi^{-1} : \mathbf{S}^n \longrightarrow \mathbf{R^n}$ is the standard stereographic projection of the sphere $\mathbf{S^n}$ into $\mathbf{R^n}$. And also it is well known that $\mathbf{P_n}$ is a conformal invariant operator, so

$$\begin{aligned}\int_{\mathbf{S^n}} (\mathbf{P}_n(g(x))g(x)d\sigma_x &= \int_{\mathbf{R^n}} [(-\Delta)^{n/2}\phi(y)]\phi(y)dy \\ &= n! \int_{\mathbf{R^n}} e^{nU_{1,0}(y)}\phi^2(y)dy \\ &= n! \int_{\mathbf{S^n}} g(x)^2 d\sigma_x, \qquad (2.32)\end{aligned}$$

where the second equality follows from (2.26). Notice that any function satisfying (2.26) also satisfies the following differential equation.

$$[(-\Delta)^{n/2}\phi(y)] = n! e^{nU_{1,0}(y)}\phi(y). \qquad (2.33)$$

We should point out that the normalization for the constant in calculating $(-\Delta)_x^{n/2}\log[\frac{|y|}{|x-y|}]$ has been absorbed in $U_{1,0}(0)$. Notice that this convention will be used throughout the whole article.

Now just observe that, by definition of the operator $\mathbf{P_n}$ it follows from Equation (2.33) that $\mathbf{P}_n g = n! g$. Thus g is a first eigenfunction with eigenvalue $n!$. It is well known that the first eigen-space of P_n on $\mathbf{S^n}$ is spanned by $\{\psi_0, \psi_1, \cdots \psi_n\}$ under stero-graphic projection. Theorem 2.6 follows. $\square$.

Remark 2.7. *We would like to point out that several facts, specially the properties of the operator* $\mathbf{P_n}$*, we have used in the proof of above theorem can be traced back to Chang and Yang's earlier paper* [6]*. The bound for* $\Delta\phi$ *can also be seen by direct calculation from the integral representation (2.26) and potential estimate.*

3. Some A Priori Estimates

In this section, we want to prove the a priori estimates for the solutions of the equation (2.1) with given bounded positive smooth function Q satisfying the following non-degeneracy condition

$$(\Delta Q)^2 + |\nabla Q|^2 \neq 0. \qquad (3.1)$$

.

Since $\mathbf{P_n}$ is a pseudo-differential operator, standard elliptic regularity estimates do not apply. We have to work with the integral equation (1.6).

With help of those lemmas in previous section, we simply conclude that those solutions of the integral equation (1.6) are C^∞ functions on $\mathbf{R^n}$ and the following estimates hold:

Lemma 3.1. *Let w satisfy (2.3). For each positive integer $0 < k \leq n-1$, there exists a constant $C(w)$ (depending on w) such that*

$$|\nabla^k w|(x) \leq C(w).$$

Proof: It follows from the integral representation of the function w, we obtain:

$$\begin{aligned} |\nabla^k w|(x) &\leq B_k \int_{\mathbf{R^n}} \frac{Q(y)e^{nw(y)}}{|x-y|^k} dy \\ &\leq B_k M e^{nB} \int_{B_1(x)} \frac{dy}{|x-y|^k} + B_k \int_{\mathbf{R^n}\setminus\mathbf{B_1(x)}} \frac{Q(y)e^{nw(y)}}{|x-y|^k} dy \\ &\leq C_k. \end{aligned} \tag{3.2}$$

In this estimate, B_k is just a constant depending only on n and k, and $B = B(w)$ is the constant giving in Lemma 2.3. □

In fact, we have more:

Lemma 3.2. *Let w satisfy (2.3). Then $w \in C^{n-1,\alpha}$ for any $0 \leq \alpha < 1$. Furthermore, there is a constant $C(w)$ (depending on w) such that $\|e^{nw}\|_{C^{n-1,\alpha}} \leq C(w)$.*

Proof: When n is even, it is clear since $(-\Delta)^{n/2} w = Q(x)e^{nw(x)}$ is bounded, standard elliptic estimate implies the result.

When n is odd, we write it as $n = 2k+1$. Then from the integral representation of w, we have

$$(-\Delta)^k w(x) = \frac{C_k}{\beta_n} \int_{\mathbf{R^n}} \frac{Q(y)e^{nw(y)}}{|x-y|^{2k}} dy.$$

By the inequality,

$$||x-y|^{-2k} - |x_1-y|^{-2k}| \leq 2k|x-x_1|^\alpha \{|x-y|^{-2k-\alpha} + |x_1-y|^{-2k-\alpha}\},$$

for any real number $0 \leq \alpha < 1$, (Equation (3) on page 225, [23]), we have

$$
\begin{aligned}
&|(-\Delta)^k w(x) - (-\Delta)^k w(x_1)| \\
\leq\ & \frac{2kC_k}{\beta_n}|x - x_1|^\alpha \{\int_{\mathbf{R^n}} \frac{Q(y)e^{nw(y)}}{|x-y|^{2k+\alpha}}dy + \int_{\mathbf{R^n}} \frac{Q(y)e^{nw(y)}}{|x_1-y|^{2k+\alpha}}dy\} \\
\leq\ & C|x - x_1|^\alpha, \qquad (3.3)
\end{aligned}
$$

where the last estimate is just same as the estimate (3.2). The last statement is just the summary of Lemmas 2.2, 2.3, 3.1 and the first part of Lemma 3.2. This completes the proof of Lemma 3.2. □

Now we are ready to state and prove the main result of this section.

Theorem 3.3. *Suppose Q is a positive smooth function defined on $\mathbf{S}^n$, $(n \geq 3)$, satisfying the non-degeneracy condition (3.1). Then there are constants C_1 and C_2 such that any solution u of the equation:*

$$\mathbf{P}_n u + (n-1)! = Qe^{nu} \ \textit{on} \ \mathbf{S^n}, \qquad (3.4)$$

satisfies the bounds

$$-C_1 \leq u \leq C_2. \qquad (3.5)$$

Here the constants C_1 and C_2 are **independent** *of the functions u.*

Proof: First let us show that there exists a constant C_2 such that $u \leq C_2$. Suppose this is not the case, there would exist a sequence $\{u_k\}$ such that $\max_{\mathbf{S^n}} u_k(x) = u_k(x_k) \to \infty$ and $x_k \to x_0$ as $k \to \infty$. Choose stereographic projection $\pi : \mathbf{S^n} \longrightarrow \mathbf{R^n}$ with north pole at $-x_0$ and set $w_k(x) = u_k(\pi^{-1}(x)) + \log\frac{2}{1+|x|^2}$. Then it is well known that w satisfies the integral representation

$$w_k(x) = \frac{1}{\beta_n}\int_{\mathbf{R^n}} \log[\frac{|y|}{|x-y|}]Q(y)e^{nw_k(y)}dy + C_k, \qquad (3.6)$$

where $C_k = w_k(0)$ and $Q(y) = Q(\pi(y))$. Translating w_k by a constant, and still denoting it by w_k, we get

$$w_k(x) = \frac{\rho_k}{\beta_n}\int_{\mathbf{R^n}} \log[\frac{|y|}{|x-y|}]Q(y)e^{nw_k(y)}dy, \qquad (3.7)$$

where $\rho_k \to 0$ as $k \to \infty$. An important fact is that

$$\rho_k \int_{\mathbf{R^n}} Q(y)e^{nw_k(y)}dy = 2\beta_n. \qquad (3.8)$$

Since $w_k(\pi^{-1}(x_k)) = u(x_k) + \log \frac{2}{1+|\pi^{-1}(x_k)|^2}$, when $k \to \infty$, $\pi^{-1}(x_k) \to 0$, we conclude that $w_k(y_k) \to \infty$ where $y_k = \pi^{-1}(x_k)$. Now set $v_k(z) = w_k(y_k + \epsilon_k z) - w_k(y_k)$ with ϵ_k to be determined later.

It follows from the integral representation of w_k that

$$\begin{aligned} v_k(z) &= \frac{\rho_k e^{nw_k(y_k)}}{\beta_n} \int_{\mathbf{R^n}} \log[\frac{|y_k - y|}{|y_k + \epsilon_k z - y|}] Q(y) e^{n[w_k(y) - w_k(y_k)]} dy \\ &= \frac{\epsilon_k^n \rho_k e^{nw_k(y_k)}}{\beta_n} \int_{\mathbf{R^n}} \log[\frac{|t|}{|z-t|}] Q(y_k + \epsilon_k t) e^{n[w_k(y_k + \epsilon_k t) - w_k(y_k)]} dt \\ &= \frac{\epsilon_k^n \rho_k e^{nw_k(y_k)}}{\beta_n} \int_{\mathbf{R^n}} \log[\frac{|t|}{|z-t|}] Q(y_k + \epsilon_k t) e^{nv_k(t)} dt. \end{aligned} \tag{3.9}$$

Now we choose $\epsilon_k > 0$ such that $\epsilon_k^n \rho_k e^{nw_k(y_k)} = 1$. The advantage for this choice of ϵ_k is that, by the equation (3.8), we have

$$\int_{\mathbf{R^n}} Q(y_k + \epsilon_k t) e^{nv_k(t)} dt = 2\beta_n. \tag{3.10}$$

Therefore the equation (3.9) implies that

$$v_k(z) = \frac{1}{\beta_n} \int_{\mathbf{R^n}} [\log \frac{|t|}{|z-t|}] Q(y_k + \epsilon_k t) e^{nv_k(t)} dt. \tag{3.11}$$

Apply Lemma 3.2 to conclude that there is a constant C, independent of k but depending on α such that $\|e^{nv_k(z)}\|_{C^{n-1,\alpha}} \le C$. Thus for some $0 < \alpha_0 < 1$, $v_k \to v_0$ in $C^{n-1,\alpha_0}_{loc}(\mathbf{R^n})$ as $k \to \infty$. By Lemma 2.1, $v_0 = \log[\frac{2\lambda}{\lambda^2 + |y-y_0|^2}]$. By the definition of v_k, we have $v_0 = U_{1,0}$.

We need the following lemma on the decay of $v_k(z)$:

Lemma 3.4. *For all $\delta \in (0,2)$, there exists $R_\delta, C_\delta > 0$ such that*

$$v_k(z) \le (2-\delta) \ln \frac{1}{|z|} + C_\delta, \quad \forall |z| \ge R_\delta. \tag{3.12}$$

Proof: The proof is standard. For the reader's convenience, we include it here.

Let $\delta \in (0,2)$ be fixed. Since $v_k \to U_{1,0}$ locally in $\mathbf{R^n}$, by (3.10), we may choose k large and R_δ such that

$$\int_{|t| \le \frac{R_\delta}{2}} Q(y_k + \epsilon_k t) e^{nv_k(t)} dt \ge (2 - \frac{\delta}{2})\beta_n.$$

We then compute

$$\begin{aligned}
v_k(z) &= \frac{1}{\beta_n}\Big(\int_{|t|\le\frac{R_\delta}{2}} + \int_{|t|\ge\frac{R_\delta}{2},|t|\le 2|z-t|} + \int_{|t|\ge\frac{R_\delta}{2},|t|\ge 2|z-t|}\Big) \\
&\quad [\log\frac{|t|}{|z-t|}]Q(y_k+\epsilon_k t)e^{nv_k(t)}dt \\
&\le \frac{1}{\beta_n}(2\beta_n - \frac{\delta}{2}\beta_n)\log\frac{1}{|z|} + \frac{1}{\beta_n}\int_{|t|\ge\frac{R_\delta}{2},|t|\ge 2|z-t|}(\log|t|)Qe^{nv_k(t)}dt \\
&+ \frac{1}{\beta_n}\int_{|t|\ge\frac{R_\delta}{2},|t|\ge 2|z-t|}\log\frac{1}{|z-t|}Qe^{nv_k}dt \\
&\le (2-\frac{\delta}{2})\log\frac{1}{|z|} + C - \delta + \frac{\delta}{2}\log|z| \\
&\le (2-\delta)\log\frac{1}{|z|} + C_\delta
\end{aligned}$$

which proves the lemma. □

Let us continue the proof of Theorem 3.3. Observe that in the definition of γ in equation (2.18), we have $\gamma = 2$ in our situation. Since Q is a smooth function on $\mathbf{S}^n$, $|\nabla Q|$ is clearly bounded, by second case of Lemma 2.4, we have

$$\begin{aligned}
0 &= \int_{\mathbf{R^n}} \nabla Q(y_k+\epsilon_k t)e^{nv_k(t)}dt \\
&= \int_{\mathbf{R^n}}[\nabla Q(y_k+\epsilon_k t) - \nabla Q(y_k)]e^{nv_k(t)}dt \\
&\quad + \int_{\mathbf{R^n}} \nabla Q(y_k)e^{nv_k(t)}dt. \qquad (3.13)
\end{aligned}$$

The first term, as $k \to +\infty$, approaches zero by the Lebesgue's dominated convergence theorem since ∇Q is bounded and v_k has decay (3.12). Thus we obtain:

$$\nabla Q(y_k) \to 0, \text{ that is, } \nabla Q(0) = 0. \qquad (3.14)$$

In fact, more can be concluded from Equation (3.13). Namely, we have

$$\nabla Q(y_k) = O(\epsilon_k). \qquad (3.15)$$

Next since Q is a smooth function on S^n, we have $| < x, \nabla Q(x) > |$ is bounded. Hence, by Lemma 2.2(1), we also have

$$\begin{aligned} 0 &= \int_{\mathbf{R^n}} < t, \nabla Q(y_k + \epsilon_k t) > e^{nv_k(t)} dt \\ &= \int_{\mathbf{R^n}} < t, \nabla Q(y_k + \epsilon_k t) - \nabla Q(y_k) > e^{nv_k(t)} dt \\ &\quad + < \nabla Q(y_k), \int_{\mathbf{R^n}} t e^{nv_k(t)} dt > \\ &= \int_{\mathbf{R^n}} \sum Q_{ij}(y_k) \epsilon_k t_i t_j e^{nv_k(t)} dt + o(\epsilon_k). \end{aligned} \tag{3.16}$$

To check the last equality above, we observe that v_0, the limit of the sequence $\{v_k\}$ is radially symmetric with respect to the point 0. It is because v_0 does satisfy the integral equation (2.1) with $Q(y) = Q(0)$. Hence we easily conclude that

$$\int_{\mathbf{R^n}} z e^{nv_0(z)} dz = 0. \tag{3.17}$$

Thus we obtain:

$$\begin{aligned} &< \nabla Q(y_k), \int_{\mathbf{R^n}} t e^{nv_k(t)} dt > \\ = &< \nabla Q(y_k), \int_{\mathbf{R^n}} t \{ e^{nv_k(t)} - e^{nv_0(t)} \} dt > \\ = &< \nabla Q(y_k), \int_{\mathbf{R^n}} t \{ e^{n(v_k(t) - v_0(t))} - 1 \} e^{nv_0(t)} dt > \\ = & \; o(\epsilon_k), \end{aligned} \tag{3.18}$$

where in the last step, we have used the estimate (3.15) and the decay (3.12).

Now, by similar reasons, the equation (3.16) implies that

$$0 = \epsilon_k \Delta Q(y_k) \int_{\mathbf{R^n}} e^{nv_k(t)} |t|^2 dt + o(\epsilon_k). \tag{3.19}$$

Thus we have $\Delta Q(0) = 0$ which contradicts the non-degeneracy assumption on Q (see Equation (3.1)).

Once the upper bound on the solution u is available, the lower bound is easy. Notice that every solution u can be written as $w(x) - \log \frac{2}{1+|x|^2}$ with w satisfying the integral equation (2.1). It will not be hard to see that $w(x) - \log \frac{2}{1+|x|^2}$ has a lower bound independent of w.

This finishes the proof of Theorem 3.3. □

4. Perturbation result

In this section, we use Liapunov-Schmidt reduction method to solve the equation (2.1) with $\|Q-(n-1)!\|_{C^2(\mathbf{R^n})} < \epsilon$ and ϵ sufficiently small. (Similar approach was used by Rey and Wei ([26], [27]).) This approach is different from the usual one adopted by Chang and Yang ([5], [6]). Here we don't use any type of Moser-Trudinger inequalities.

Let us rewrite the function Q as $Q=(n-1)!(1+\epsilon\hat{Q})$. Of course if Q is non-degenerate in the sense of (1.7), so is $\hat{Q}$. We consider the integral equation (2.1).To be more precise, we write the equation in a non-local operator form:

$$S[u] := (-\Delta)^{n/2}u - \frac{\beta_n(n-1)!}{2}\frac{Q(x)e^{nu(x)}}{\int_{\mathbf{R^n}} Q(x)e^{nu(x)}dx}. \tag{4.1}$$

In this section, we should construct a function u such that $S[u]=0$ and it can be lift to $\mathbf{S^n}$ so that this is a solution we are looking for. The solution will have the form

$$u(x) = U_{\Lambda,a}(x) + \phi(x), \tag{4.2}$$

where $(\Lambda, a) \in (0,1] \times \mathbf{S^n}$ will be chosen later and $\phi(x)$ is relatively small and $U_{\Lambda,a}(x)$ is given by

$$U_{\Lambda,a}(x) = \log\frac{2\Lambda}{\Lambda^2+|x-a|^2}. \tag{4.3}$$

Observe that if $u(x)$ takes the form (4.2) with ϕ uniformly bounded on R^n and $\lim_{|x|\to\infty}\phi(x)$ exists , clearly we can lift it to $\mathbf{S^n}$ by stereographic projection.

Now we substitute (4.2) into the equation (4.1) to obtain

$$S[U_{\Lambda,a}+\phi] = S[U_{\Lambda,a}] + L[\phi] + N[\phi], \tag{4.4}$$

where

$$S[U_{\Lambda,a}] = (-\Delta)^{n/2}U_{\Lambda,a} - \frac{\beta_n(n-1)!}{2}\frac{Q(x)e^{nU_{\Lambda,a}(x)}}{\int_{\mathbf{R^n}} Q(x)e^{nU_{\Lambda,a}(x)}dx}, \tag{4.5}$$

$$L[\phi] = (-\Delta)^{n/2}\phi - \frac{\beta_n n!}{2}e^{nU_{\Lambda,a}}\phi + \frac{\beta_n n!}{2}\frac{e^{nU_{\Lambda,a}}\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}\phi(x)dx}{(\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}dx)^2}, \tag{4.6}$$

and

$$N[\phi] = O(\epsilon < y >^{2n} |\phi| + |\phi|^2 < y >^{2n}), \tag{4.7}$$

where $< y >^2 = (1 + |y|^2)$.

Note that

$$\int_{\mathbf{R^n}} (S[U_{\lambda,a}] + N[\phi])dx = 0. \tag{4.8}$$

Now we begin with

Lemma 4.1. *Suppose the bounded function ϕ satisfies $\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}} \phi dx = 0$ and $\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}} \psi_{\Lambda,j}\phi = 0$ for $j = 0, 1, 2, \cdots, n$ with $\psi_{\Lambda,j}$ given by*

$$\psi_{\Lambda,0}(x) = \frac{|x-a|^2 - \Lambda^2}{\Lambda(\Lambda^2 + |x-a|^2)}, \tag{4.9}$$

and

$$\psi_{\Lambda,j}(x) = \frac{2(x_j - a_j)}{\Lambda^2 + |x-a|^2}. \tag{4.10}$$

If $\phi - \log \frac{2}{1+|x|^2}$ can be lifted to be a smooth function on $\mathbf{S^n}$, then there is a constant $c_0 > 0$ such that

$$\int_{\mathbf{R^n}} [(-\Delta)^{n/2}\phi]\phi dx - n!\frac{\beta_n}{2}(1 + c_0) \int_{\mathbf{R^n}} e^{n(U_{\Lambda,a}(x)}\phi^2(x))dx \geq 0. \tag{4.11}$$

Proof: When we consider it as the eigenvalue problem on $\mathbf{S^n}$ for generalized Paneitz operator $\mathbf{P_n}$, the inequality we stated above is clear since it is well know that the first non-zero eigenvalue of $\mathbf{P_n}$ on $\mathbf{S^n}$ is always equal to $n!$. □

Now we should adopt the following notation in the future argument:

$$\|\phi\|_* = \sup_{y \in \mathbf{R^n}} |\phi(y)|, \tag{4.12}$$

and

$$\|f\|_{**} = \sup_{y \in \mathbf{R^n}} < y >^{2n} |f(y)|. \tag{4.13}$$

Lemma 4.2. *Let f be a function on $\mathbf{R^n}$ such that $\|f\|_{**}$ is finite and $\int_{\mathbf{R^n}} f dy = 0$. Assume the bounded function ϕ is a solution of the equation*

$$L[\phi] + f + \sum_{j=0}^{n} C_j e^{nU_{\Lambda,a}} \psi_{\Lambda,j} = 0, \tag{4.14}$$

for some constants C_j such that $\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}}\phi dx = 0$ and $\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}}\psi_{\Lambda,j}\phi dx = 0$ for $j = 0, 1, 2, \cdots, n$. Then we have

$$\|\phi\|_* \le C\|f\|_{**}, \tag{4.15}$$

for some positive constant $C > 0$ which depends on the upper bound of $n + \Lambda + |a|$ only.

Proof: Since $\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}dx = 2\beta_n$ which is independent of Λ, a, by taking the derivative with respect to Λ and a, we find that

$$\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}\psi_{\Lambda,j}(x)dx = 0, \tag{4.16}$$

for $j = 0, 1, 2, \cdots, n$. Therefore, together with the fact that $L[\psi_{\Lambda,j}] = 0$ for every j, we have

$$\int_{\mathbf{R^n}} L[\phi]\psi_{\Lambda,j}dx = 0, \tag{4.17}$$

for all j by integration by parts and the assumption. Thus if we multiply the equation (4.14) by $\psi_{\Lambda,j}$ and integrate it over the space $\mathbf{R^n}$, we obtain the estimate

$$|C_j| = O(\|f\|_{**}), j = 0, 1, 2, \cdots, n. \tag{4.18}$$

By the integral representation of the equation (4.14) we have

$$\begin{aligned}
\phi(x) &= \frac{n}{\beta_n}\int_{\mathbf{R^n}}\{\log\left[\frac{|y|}{|x-y|}\right]\}e^{nU_{\Lambda,a}(y)}\phi(y)dy \\
&\quad +\frac{n}{\beta_n}\int_{\mathbf{R^n}}\{\log\left[\frac{|y|}{|x-y|}\right]\}f(y)dy \\
&\quad +\sum_{j=0}^{n} C_j\psi_{\Lambda,j} + \beta.
\end{aligned} \tag{4.19}$$

Since ϕ is bounded, it will not be hard to see that $\phi(x)$ has limit as $|x|$ tends to infinity. Therefore it can be lifted to be a smooth function on $\mathbf{S^n}$.

Next multiplying the equation (4.14) by ϕ and integrating the resulting equation, we obtain

$$\int_{\mathbf{R^n}}[(-\Delta)^{n/2}\phi]\phi dx - \frac{n!\beta_n}{2}\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}\phi^2(x)dx + \int_{\mathbf{R^n}} f\phi dx = 0. \tag{4.20}$$

It follows from the estimate of Lemma 4.1 and (4.20) that there is a constant B_0 such that

$$\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}\phi^2(x)dx \le B_0 \int_{\mathbf{R^n}} |f||\phi|dx. \tag{4.21}$$

This implies that

$$\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}\phi^2(x)dx \le B\|\phi\|_*\|f\|_{**}. \tag{4.22}$$

Now by taking the derivative of the equation (4.19), we have the estimate:

$$\begin{aligned} |\nabla\phi(x)| &\le C[\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(y)} \left\{|\phi(y)| + e^{-nU_{\Lambda,a}(y)}|f(y)|\right\} dy + \sum_{j=0}^{n} |C_j|] \\ &\le C\|f\|_{**}. \end{aligned} \tag{4.23}$$

by using the Hölder inequality together with (4.21) in the first term and definition for $\|\cdot\|_{**}$ in the second term. The last term follows from (4.18).

Then we have estimate on the function ϕ at 0 as follows:

$$\begin{aligned} 2\beta_n|\phi(0)| &= |\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(y)}[\phi(0)-\phi(y)]dy| \\ &\le \|\nabla\phi\|_{L^\infty}\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(y)}|y|dy \\ &\le C\|f\|_{**}. \end{aligned} \tag{4.24}$$

Here we have used the estimate (4.23).

Now it also follows from (4.19) that the following estimate holds true.

$$\begin{aligned} |\phi(x)-\phi(0)| &= |\frac{n}{\beta_n}\int_{\mathbf{R^n}}\{\log\left[\frac{|x||y|}{|x-y|}\right]\}e^{nU_{\Lambda,a}(y)}\phi(y)dy| \\ &\quad +|\frac{n}{\beta_n}\int_{\mathbf{R^n}}\{\log\left[\frac{|x||y|}{|x-y|}\right]\}f(y)dy| \\ &\quad +\sum_{j=0}^{n}|C_j||\psi_{\Lambda,j}(x)-\psi_{\Lambda,j}(0)| \\ &\le C\|f\|_{**}, \end{aligned} \tag{4.25}$$

where the first line was achieved by observation that $\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(y)}\phi(y)dy = 0$ and $\int_{\mathbf{R^n}} f(y)dy = 0$, while the second inequality follows the standard potential estimates and (4.18). Now clearly the required estimate follows from triangle inequality. □

Lemma 4.3. *For every function f with the property that $\|f\|_{**} < \infty$ and $\int_{\mathbf{R^n}} f dx = 0$ and every point (Λ, a), there exist constants $C_j(\Lambda, a, f)$ for $j = 0, 1, 2, \cdots, n$ such that the equation (4.14) has a unique solution.*

Proof: The uniqueness of ϕ has been shown in Lemma 4.2. Notice that for every fixed f, there is only one set of constants C_j for $j = 0, 1, 2, \cdots, n$ such that the equation (4.14) has a solution. This follows from the estimate (4.18).

Now for a fixed function f with the required property and for a given point (Λ, a), we choose constant C_j according to the equation

$$\int_{\mathbf{R^n}} \psi_{\Lambda,j} f dx + C_j \int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)} \psi^2_{\Lambda,j}(x) dx = 0. \tag{4.26}$$

Now we define the pre-Hilbert space H by

$$\begin{aligned} H: \ = \ & \Big\{ \phi | \int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)} \phi(x) dx = \int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)} \psi_{\Lambda,j}(x)\phi(x) dx = 0, \\ & < \phi, \phi >= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \{\log \frac{|x||y|}{|x-y|}\} e^{n(U_{\Lambda,a}(x) + U_{\Lambda,a}(y))} \phi(x)\phi(y) dx dy < \infty \Big\}. \end{aligned} \tag{4.27}$$

Notice that it is simple consequence of potential theory that H is a pre-Hilbert space. In fact, with the definition of the inner product by

$$< \phi, \psi >= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \{\log \frac{|x||y|}{|x-y|}\} e^{n(U_{\Lambda,a}(x) + U_{\Lambda,a}(y))} \phi(x)\psi(y) dx dy$$

for any ϕ, ψ in H, bi-linear and symmetric properties are easy to see. For non-negativity, we note that

$$\begin{aligned} < \phi, \phi > &= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \{\log \frac{|x||y|}{|x-y|}\} e^{n(U_{\Lambda,a}(x) + U_{\Lambda,a}(y))} \phi(x)\phi(y) dx dy \\ &= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \{\lim_{\delta \to 0^+} \frac{[\frac{|x||y|}{|x-y|}]^\delta - 1}{\delta}\} e^{n(U_{\Lambda,a}(x) + U_{\Lambda,a}(y))} \phi(x)\phi(y) dx dy \\ &= \lim_{\delta \to 0^+} \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \{\frac{[\frac{|x||y|}{|x-y|}]^\delta - 1}{\delta}\} e^{n(U_{\Lambda,a}(x) + U_{\Lambda,a}(y))} \phi(x)\phi(y) dx dy \\ &\geq 0. \end{aligned} \tag{4.28}$$

Now the equation (4.14) can be written as

$$\begin{aligned} &\phi - \frac{n}{\beta_n}\int_{\mathbf{R^n}}\{\log\frac{|x||y|}{|x-y|}\}e^{nu_{\Lambda,a}(y)}\phi(y)dy \\ = &\frac{n}{\beta_n}\int_{\mathbf{R^n}}\{\log\frac{|x||y|}{|x-y|}\}f(y)dy + \sum_{j=0}^{n} C_j\psi_{\Lambda,j}(x) + \beta, \quad (4.29)\end{aligned}$$

where the constants C_j are given by the equation (4.26). If we denote the second term on the left of the equation (4.29) by $T[\phi]$ and the right hand side of the same equation as $\hat{f}$, then the equation can be simplified to be

$$\phi - T[\phi] = \hat{f}. \quad (4.30)$$

Clearly T is linear operator which maps bounded functions into bounded functions, hence it is a compact operator on H. By Fredholm's alternative, (4.30) has a solution if and only if

$$\phi - T[\phi] = 0$$

has only trivial solution. However, the latter has trivial solution is just the consequence of previous Lemma with $f = 0$. Thus Lemma 4.3 follows. □

Let us denote the map $f \longrightarrow \phi$ in Lemma 4.3 by $A(\phi)$.

Lemma 4.4. *There exists a unique $\phi = \phi_{\Lambda,a}$ such that*

$$L[\phi] + S[U_{\Lambda,a}] + N[\phi] + \sum_{j=0}^{n} C_j e^{nU_{\Lambda,a}(x)}\psi_{\Lambda,j} = 0. \quad (4.31)$$

Moreover, there is a constant $C > 0$ such that $\|\phi\|_* \leq C\epsilon$ with $\epsilon = \|Q-(n-1)!\|_{C^2(\mathbf{R^n})}$. Also, the map $(\Lambda, a) \longrightarrow \phi_{\Lambda,a}$ is continuous.

Proof: The tool we are going to use is the contraction mapping principle. In order to do so, first of all, let us rewrite the equation (4.31) in its equivalent form:

$$\phi = A(S[U_{\Lambda,a}] + N[\phi]) := B[\phi]. \quad (4.32)$$

For a positive constant C_1, define a convex set in H by

$$Z := \{\phi | \|\phi\|_* < C_1\epsilon\}. \quad (4.33)$$

By definition of $S[U_{\Lambda,a}]$ and $N[\phi]$, we obtain

$$\int_{\mathbf{R^n}} (S[U_{\Lambda,a}] + N[\phi])dx = 0, \tag{4.34}$$

which makes the definition of operator B meaningful.

Next we have

$$\|B[\phi_1] - B[\phi_2]\|_* \leq C(\epsilon + \|\phi_1\|_*)\|\phi_1 - \phi_2\|_*. \tag{4.35}$$

Finally there exists a constant $C > 0$ such that the following inequality holds true.

$$\|S[U_{\Lambda,a}] + N[\phi]\|_* \leq C\epsilon(1 + \|\phi\|_*)\|\phi\|_*^2. \tag{4.36}$$

The equations (4.34), (4.35) and (4.36) together imply that the operator B is a contraction mapping from Z into Z, maybe with different constant C if we choose ϵ sufficiently small. Hence B has a fixed point.

The continuity of $\phi_{\Lambda,a}$ on parameters Λ, a follows from the integral representation formula. Hence Lemma 4.4 holds true. □

Lemma 4.5. *The solution given in previous Lemma 4.4 can always be lifted to be a smooth function on* $\mathbf{S^n}$.

Proof: It is not hard to see that $\phi_{\Lambda,a}$ as well as its derivatives has a limit at infinity by dominant convergence theorem. Keep in mind that $\phi_{\Lambda,a}$ is bounded by its nature. □

Let us now compute the asymptotic expansions of $C_j(\Lambda, a)$. Multiplying the equation (4.31) by $\psi_{\Lambda,l}$, we obtain,

$$\begin{aligned} & \sum_{j=0}^{n} \int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)} \psi_{\Lambda,j}(x)\psi_{\Lambda,l}(x)dx \\ = \quad & \int_{\mathbf{R^n}} (-L[\phi_{\Lambda,a}] - S[U_{\Lambda,a}] - N[\phi_{\Lambda,a}])\psi_{\Lambda,l}dx \\ = \quad & -\int_{\mathbf{R^n}} S[U_{\Lambda,a}]\psi_{\Lambda,l}dx + O(\epsilon^2). \end{aligned} \tag{4.37}$$

We should point out that in this calculation we have used the equation (4.17) as well as the fact that $\|N[\phi_{\Lambda,a}]\|_{**} \leq C\epsilon^2$. With this calculation, we have left just to compute the term $\int_{\mathbf{R^n}} S[U_{\Lambda,a}]\psi_{\Lambda,l}dx$.To do this, let us recall the definition of $S[U_{\Lambda,a}]$ to have

$$\begin{aligned} S[U_{\Lambda,a}] &= (-\Delta)^{n/2}U_{\Lambda,a} - \frac{\beta_n}{2}\frac{Q(x)e^{nU_{\Lambda,a}(x)}}{\int_{\mathbf{R^n}} Q(x)e^{nU_{\Lambda,a}(x)}dx} \\ &= (1-\frac{\beta_n}{2}\frac{1+\epsilon\hat{Q}(x)}{\int_{\mathbf{R^n}}(1+\epsilon\hat{Q}(x))e^{nU_{\Lambda,a}(x)}dx})e^{nU_{\Lambda,a}(x)}. \quad (4.38) \end{aligned}$$

Thus it follows from this equation that

$$\begin{aligned} \int_{\mathbf{R^n}} S[U_{\Lambda,a}]\psi_{\Lambda,l}dx &= \int_{\mathbf{R^n}}(1-\frac{\beta_n}{2}\frac{1+\epsilon\hat{Q}(x)}{\int_{\mathbf{R^n}}(1+\epsilon\hat{Q}(x))e^{nU_{\Lambda,a}(x)}dx})e^{nU_{\Lambda,a}(x)}\psi_{\Lambda,l}dx \\ &= \frac{\beta_n\epsilon}{2\int_{\mathbf{R^n}}(1+\epsilon\hat{Q}(x))e^{nU_{\Lambda,a}(x)}dx}\int_{\mathbf{R^n}}\hat{Q}(x)e^{nU_{\Lambda,a}(x)}\psi_{\Lambda,l}(x)dx \\ &= -\epsilon\int_{\mathbf{R^n}}\hat{Q}(x)e^{nU_{\Lambda,a}(x)}\psi_{\Lambda,l}(x)dx + O(\epsilon^2). \quad (4.39) \end{aligned}$$

Notice that the functions $\psi_{\Lambda,j}$ satisfy the relations:

$$\int_{\mathbf{R^n}} e^{nU_{\Lambda,a}(x)}\psi_{\Lambda,i}(x)\psi_{\Lambda,j}(x)dx = \gamma_j^{-1}\delta_{ij}. \quad (4.40)$$

Now equations (4.37) - (4.40) imply that

$$C_j(\Lambda,a) = -\epsilon\gamma_j\int_{\mathbf{R^n}}\hat{Q}(x)e^{nU_{\Lambda,a}(x)}\psi_{\Lambda,l}(x)dx + O(\epsilon^2). \quad (4.41)$$

Now we define a mapping G from $\mathbf{B^{n+1}}$ into $\mathbf{R^{n+1}}$ by

$$\hat{G}(z) = (G_0(z), G_1(z), \cdots, G_n(z)) \in \mathbf{R^{n+1}}$$

where

$$G_l(z) = \int_{\mathbf{R^n}}\hat{Q}(x)e^{nU_{\frac{1}{1-|z|},\pi(\frac{z}{|z|})}(x)}\psi_{\frac{1}{1-|z|},l}(x)dx, \quad (4.42)$$

for $l = 0, 1, 2, \cdots n$.

Next by the equation (4.41), we know that if the degree $deg(\hat{G}, \mathbf{B}^{n+1}, 0) \neq 0$, then there exists a point $z_0 \in \mathbf{B^{n+1}}$ such that $C_j(\Lambda, a) = 0$ for all $j = 0, 1, 2, \cdots, n$ where $\Lambda = (1-|z_0|)$ and $a = \pi^{-1}(z_0/|z_0|)$.

To see the fact that $deg(\hat{G}, B^{n+1}, 0) \neq 0$, we need to do several calculations. First of all, we note that for $j \geq 1$, we have

$$\begin{aligned}
\hat{G}_j(\Lambda, a) &= \int_{\mathbf{R^n}} \hat{Q}(x) e^{nU_{\Lambda,a}(x)} \psi_{\Lambda,j}(x) dx \\
&= \frac{1}{2\Lambda} \int_{\mathbf{R^n}} \hat{Q}(\Lambda y + a) y_j [\frac{2}{1+|y|^2}]^{n+1} dy \\
&= c \frac{\partial \hat{Q}}{\partial x_j}(a) + o(\Lambda), \qquad (4.43)
\end{aligned}$$

as $\Lambda \to 0$ for some constant $c > 0$ depending only on the dimension n.

Similarly for $j = 0$, we have

$$\begin{aligned}
\hat{G}_0(\Lambda, a) &= \int_{\mathbf{R^n}} \hat{Q}(x) e^{nU_{\Lambda,a}(x)} \psi_{\Lambda,0}(x) dx \\
&= \frac{1}{2\Lambda} \int_{\mathbf{R^n}} \hat{Q}(\Lambda y + a)(|y|^2 - 1)[\frac{2}{1+|y|^2}]^{n+1} dy \\
&= -c_1 \Lambda \Delta \hat{Q}(a) + o(\Lambda), \qquad (4.44)
\end{aligned}$$

as $\Lambda \to 0$ and some constant $c_1 > 0$ depending only on n which might be different from c in above.

Set $\delta = c_1 \Lambda / c$ to obtain:

$$\hat{G}(z) = c(\delta(-\Delta \hat{Q})(a), \nabla \hat{Q}(a)) + o(\Lambda), \qquad (4.45)$$

where $a = \pi^{-1}(z/|z|)$ with $|z| \leq 1$. Now we define other mapping as follows:

$$G_\delta(z) = (-\Delta \hat{Q}(a), \delta \nabla \hat{Q}(a)). \qquad (4.46)$$

Then it is clear that

$$\hat{G}(z) \cdot G_\delta(z) = c\{[\|\nabla \hat{Q}(a)\|^2 + (\Delta \hat{Q}(a)]^2]\delta + o(\delta)\}.$$

By non-degeneracy assumption on the function Q, hence on $\hat{Q}$, that if δ is sufficiently small, $\hat{G}(z) \cdot G_\delta(z) > 0$ on $\partial B_{1-\delta}(0)$. Now we fix δ small, then $\hat{G}(a) \cdot G_\delta(a) > 0$ on $\mathbf{S^n}$. By simple property of the degree theory (see Proposition 1.27 of [18]), we have

$$deg(\hat{G}, B^{n+1}_{1-\delta}(0), 0) = deg(\hat{G}/|\hat{G}|, \mathbf{S^n_{1-\delta}}).$$

Then we set $H(t,z) = t\hat{G}(z) + (1-t)G_\delta(z)$, we have shown that the degree of $\hat{G}/|\hat{G}|$ on $\mathbf{S^n_{1-\delta}}$ is same as that of $G_\delta/|G_\delta|$ on $\mathbf{S}^{n+1}_{1-\delta}$. However, by natural definition of G_δ, we can see that it is also well defined on the sphere $\mathbf{S^n}$. Finally for any real numbers, $\lambda > 0$ and $s > 0$, we define the map:

$$G_{\lambda,s}(z) = (-s\Delta\hat{Q}(a), \lambda\nabla\hat{Q}(a)),$$

since this map never vanishes for all $a \in \mathbf{S^n}$, the degree of the maps is well defined and that they all have same degree which implies that

$$\begin{aligned} & deg(\hat{G}/|\hat{G}|, \mathbf{S^n_{1-\delta}}) \\ = \ & deg(\hat{G}_\delta/|\hat{G}_\delta|, \mathbf{S}^n_{1-\delta}) \\ = \ & deg(G_{1-r_0,1}/|G_{1-r_0,1}|, \mathbf{S}^n_{1-\delta}) \\ = \ & deg(G_{1,1}/G_{1,1}|, \mathbf{S^n_{1-\delta}}) \\ = \ & deg(G^*/|G^*|, \mathbf{S^n}) \neq \mathbf{0} \end{aligned}$$

Keep in mind that here we have identified the domains $\mathbf{S^n}$ and $\mathbf{S^n_{1-\delta}}$ for our map G^* which is clearly true since they give the same values for G^*.

Thus we have finished the proof of the main theorem for the case $Q - (n-1)!$ is small in C^2 topology.

5. Proof of main theorem 1.1

The proof of our main theorem is exact same as in our previous paper [29], since all the eigenvalues and eigenfunctions of $\mathbf{P_n}$ are known [1], [2]. We will not reproduce it here again. For interested readers, we refer them to section 5 of [29]. □

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Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk

Department of Mathematics, National University of Singapore, 2 Science drive 2, Singapore 117543, Republic of Singapore

E-mail address: matxuxw@nus.edu.sg