INFINITELY MANY POSITIVE SOLUTIONS FOR AN ELLIPTIC PROBLEM WITH CRITICAL OR SUPER-CRITICAL GROWTH

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Abstract. We prove that for some supercritical exponents $p > \frac{N+2}{N-2}$ and for some smooth domains $\mathcal{D}$ in $\mathbb{R}^N$ there are infinitely many (distinct) positive solutions to the following Lane-Emden-Fowler equation

$$\begin{cases} -\Delta u = u^p, \ u > 0 & \text{in } \mathcal{D}, \\ u = 0, & \text{on } \partial\mathcal{D}. \end{cases}$$

This seems to be the first result for such type of equations.

1. Introduction

One of the earliest, and perhaps the simplest, nonlinear equations is the following Lane-Emden-Fowler equation,

$$-\Delta u = u^p\,, \ u > 0 \quad \text{in } \Omega\,, \ u = 0 \quad \text{on } \partial\Omega\,, \tag{1.1}$$

where $p > 1$ and Ω is a domain with smooth boundary in $\mathbb{R}^N$.

It is well-known that the critical exponent $p = \frac{N+2}{N-2}$ plays an important role in the solvability question. When $1 < p < \frac{N+2}{N-2}$, a solution can be found as an extremal for the best constant in the compact embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, namely a minimizer of the variational problem

$$\inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}\,.$$

When $p \geq \frac{N+2}{N-2}$, this minimization procedure fails, so does existence in general: Pohozaev [18] discovered that no solution exists in this case if the domain is strictly star-shaped. On the other hand Kazdan and Warner [12] observed that if Ω is an annulus, $\Omega = \{x : a < |x| < b\}$, compactness holds for any $p > 1$ within the class of radial functions, and a solution can again be found variationally without any constraint in p.

In the classical paper [3], Brezis and Nirenberg considered the critical case $p = \frac{N+2}{N-2}$ and proved that compactness, and hence solvability, is restored by the addition of a suitable linear term, that is, replacing $u^{\frac{N+2}{N-2}}$ by $u^{\frac{N+2}{N-2}} + \lambda u$. In case of pure nonlinearity $u^{\frac{N+2}{N-2}}$, Coron

[4] used a variational approach to prove that (1.1) is solvable if Ω exhibits *a small hole*. Rey [19] established existence of multiple solutions if Ω exhibits several small holes. Bahri and Coron [1] established that solvability holds for $p = \frac{N+2}{N-2}$ whenever Ω has a non-trivial topology. On the other hand, examples in [5, 11] shows that when $p = \frac{N+2}{N-2}$ (1.1) can still have a solution on some domains whose topology is trivial. Thus both the topology and the shape of the domain can affect the existence of solution for (1.1) in the critical case. It is pointed out in [2] that Rabinowitz asked whether the non-triviality of the domain topology can guarantee the existence of at least one positive solution for solvability in the supercritical case $p > \frac{N+2}{N-2}$. This was answered negatively by Passaseo [16, 17] by means of an example for $N \geq 4$ and $p > \frac{N+1}{N-3}$. If p is supercritical but close to critical, bubbling solutions are found, see [7, 8, 14, 15].

In the case of p being purely supercritical, there are very few existence results. Variational machinery no longer applies, due to lack of Sobolev inequality. In [10], del Pino and Wei extended Coron's result to supercritical problems (modulo some sequence of critical exponents) using perturbation methods. The role of the *second critical exponent* $p = \frac{N+1}{N-3}$, the Sobolev exponent in one dimension less, is investigated in the paper by del Pino, Musso and Pacard [9] in which they constructed solutions concentrating on a boundary geodesics for $p = \frac{N+1}{N-3} - \varepsilon$ with $\varepsilon \to 0+$. But the results in [10, 9] are for problems with perturbations either on the nonlinearities, or on the domain. As far as we know, except the obvious radial solutions when the domain is an annulus ([12]), there is no existence result for (1.1) if there is no perturbation on the problem.

In this paper, we explore the role of *lower-dimensional Sobolev exponents* on the existence and multiplicity of solutions to (1.1). Namely we consider the following equation with super-critical growth:

$$(1.2) \qquad \begin{cases} -\Delta u = u^{\frac{N-m+2}{N-m-2}}, \ u > 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $m > 1$ is a positive integer, Ω is a bounded domain in $\mathbb{R}^N$, and $N \geq 3 + m$. Note that $\frac{N-m+2}{N-m-2}$ is the critical Sobolev exponent in $\mathbb{R}^{N-m}$.

By the results in [16, 17], it is not sufficient to just assume that Ω has non-trivial topology to obtain an existence result for (1.2). The aim of this paper is to investigate the conditions on the domain Ω which ensure that (1.2) has infinitely many positive solutions. It is easy to see that (1.2) is equivalent to

$$
(1.3) \qquad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 & \text{in } \mathcal{D}, \\ u = 0, & \text{on } \partial\mathcal{D}, \end{cases}
$$

where $m \geq 1$ is a positive integer, $\mathcal{D}$ is a bounded domain in $\mathbb{R}^{N+m}$, and $N \geq 3$.

To simplify (1.2), we impose a partial symmetry condition on $\mathcal{D}$:

(R): Write $y = (y^*, y^{**})$, $y^* \in \mathbb{R}^{N-1}$ and $y^{**} \in \mathbb{R}^{m+1}$. Then $y \in \mathcal{D}$ if and only if $(y^*, |y^{**}|, 0, \cdots, 0) \in \mathcal{D}$.

For any $\mathcal{D}$ satisfying (R), we look for a solution of the form $u(y) = u(y^*, |y^{**}|)$ for (1.3). Let

$$
\Omega = \big\{(y^*, y_N) \in \mathbb{R}^N_+ : (y^*, y_N, 0, \cdots, 0) \in \mathcal{D}\big\},
$$

where $\mathbb{R}^N_+ = \{y : \ y \in \mathbb{R}^N, \ y_N > 0\}$. Then (1.3) is transformed to the following problem:

$$
(1.4) \qquad \begin{cases} -div(y_N^m Du) = y_N^m u^{2^*-1}, \ u > 0, & y \in \Omega, \\ u = 0, & \text{on } \partial\Omega \cap \mathbb{R}^N_+. \end{cases}
$$

Using the Pohozaev identity, we can easily find that (1.4) has no solution in some domains such as a half ball centered at the origin. On the other hand, if $\Omega \cap \mathbb{R}^N_+ \neq \emptyset$, (1.4) is degenerate. To avoid the difficulties caused by the possible degeneracy of (1.3), we impose that following condition on Ω:

(Ω_1): $\Omega \subset\subset \mathbb{R}^N_+$.

If Ω satisfies (Ω_1), the corresponding domain $\mathcal{D}$ is a torus-like domain and thus it has non-trivial homology. Passaseo's result suggests that more conditions on the domain be needed to obtain an existence result for (1.3).

Problem (1.4) is a critical problem in $\mathbb{R}^N$. Due to the non-compactness of this problem, it is not practical to use the variational techniques to obtain multiplicity result for (1.4). In this paper, we will prove that under some conditions on Ω, (1.4) has infinitely many positive solutions by constructing solutions with many bubbles. To achieve this goal, we impose further the following conditions on Ω.

(Ω_2): For any $\theta \in (0, 2\pi)$, $(r\cos\theta, r\sin\theta, y_3, \cdots, y_N) \in \Omega$, if $(r, 0, y_3, \cdots, y_N) \in \Omega$.

(Ω_3): $y \in \Omega$ if and only if $(y_1, y_2, y_3, \cdots, -y_i, \cdots, y_{N-1}, y_N) \in \Omega, \ i = 3, \cdots, N-1;$

(Ω_4): there is $x^* \in \partial\Omega$ with $x^* = (r^*, 0, \cdots, 0, l^*)$ for some $r^* > 0$ and $l^* > 0$, such that

$$\begin{aligned}&\partial\Omega \cap \big\{y_2 = \cdots = y_{N-1} = 0\big\} \cap B_\delta(x^*)\\ =&\big\{y_N = \psi(y_1),\ y_2 = \cdots = y_{N-1} = 0\big\} \cap B_\delta(x^*),\end{aligned}$$

and

$$\begin{aligned}&\Omega \cap \big\{y_2 = \cdots = y_{N-1} = 0\big\} \cap B_\delta(x^*)\\ =&\big\{y_N > \psi(y_1),\ y_2 = \cdots = y_{N-1} = 0\big\} \cap B_\delta(x^*),\end{aligned}$$

for some C^2 function ψ and small $\delta > 0$. Moreover, r^* is either a strict local minimum point, or strict local maximum point of ψ.

Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that $N \geq 5$. If Ω satisfies (Ω_1), (Ω_2), (Ω_2), and (Ω_4), then problem (1.4) has infinitely many distinct positive solutions.*

The domain in Figure 1 satisfies (Ω_1)–(Ω_4).

We make a comparison between our result and those in [9]. First we allow $m \geq 2$ while $m = 1$ in [9]. Now let $m = 1$. We prove the existence of solutions with large number of bubbles placed on a boundary geodesics for the purely critical exponent, while del Pino-Musso-Pacard [9] established the existence of a lower-dimensional bubble solution on a boundary geodesics for the slightly subcritical exponent.

As far as we know, the only other infinite multiplicity result is on Gelfand's problem in a unit ball

$$-\Delta u = \lambda(1+u)^p, u > 0 \text{ in } B_1, u = 0 \text{ on } \partial B_1. \tag{1.5}$$

Problem (1.5) can be reduced to an ODE problem by Gidas-Ni-Nirenberg theorem. Using ODE analysis, Joseph and Lundgren ([13]) showed that for some special values of $\lambda = \lambda_p$ and for $p \in (\frac{N+2}{N-2}, p_{JL})$ there are infinitely many positive solutions to (1.5). (Here p_{JL} is the so-called Joseph-Lundgren exponent.) For the purely Lane-Emden-Fowler equation, theorem 1.1 is the first result of infinite multiplicity.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

As we remarked earlier, our main idea is to glue bubbles together. Firstly, we construct an approximate solution, which is a bubble, for (1.4).

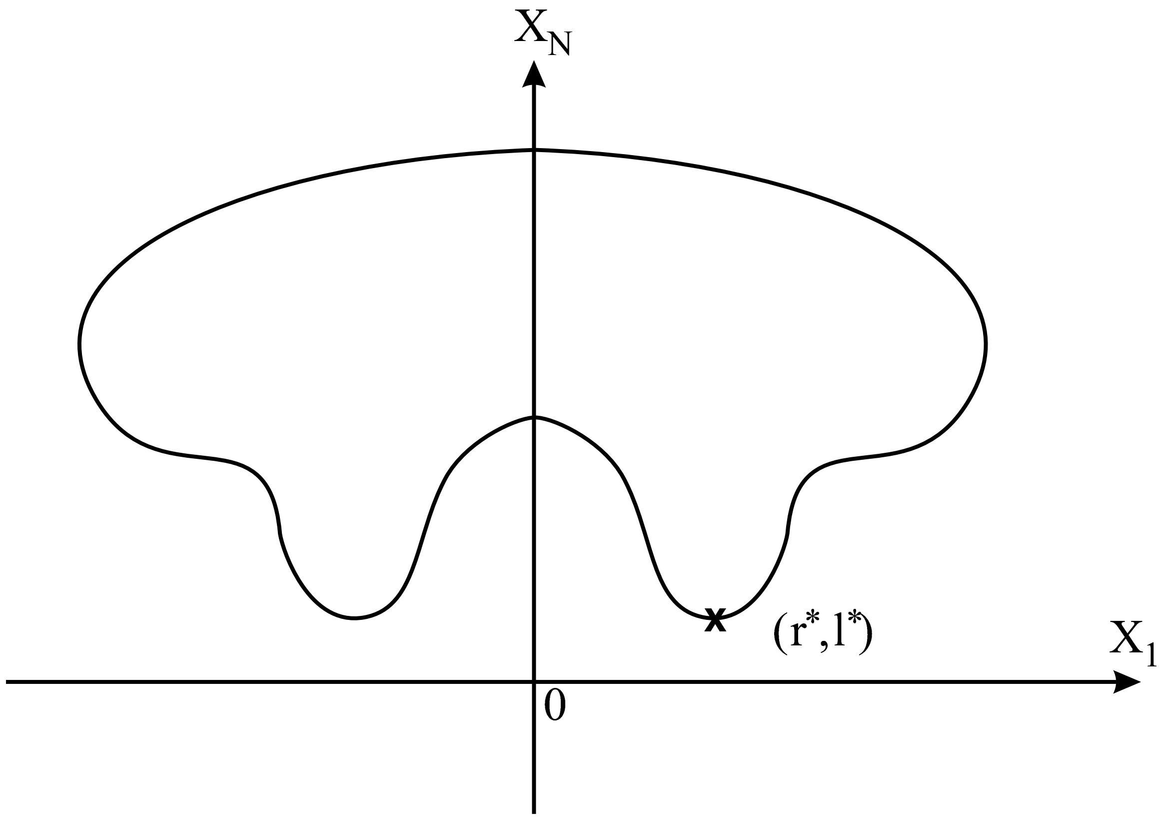

FIGURE 1. Domain Shape

Denote $2^* = \frac{2N}{N-2}$. It is well-known that the functions

$$U_{x,\mu}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \left(\frac{\mu}{1+\mu^2|y-x|^2}\right)^{\frac{N-2}{2}}, \ \mu > 0, \ x \in \mathbb{R}^N$$

are the only solutions to the following problem

$$-\Delta u = u^{2^*-1}, \ u > 0 \text{ in } \mathbb{R}^N.$$

Since $U_{x,\mu}$ does not vanish on $\partial\Omega$, we define $PU_{x,\mu}$ as the solution of the following problem:

$$-\Delta PU_{x,\mu} = U_{x,\mu}^{2^*-1}, \ \text{ in } \Omega, \quad PU_{x,\mu} = 0 \text{ on } \partial\Omega. \tag{1.6}$$

We use $PU_{x,\mu}$ as an approximate solution for (1.4). Our main task now is to determine the location x of the bubbles, as well as the concentration rate μ of the bubbles. In the singular perturbation problems, such as those in [7, 8, 9], the parameter plays a crucial role in determining the concentration rate of the bubbles. Though (1.4) is not a singular perturbation problem, it is well known now that we can use k, the number of bubbles, as our parameter, if k is large. This idea was first introduced by us [20] in the study of prescribing scalar curvature problem on S^N

$$-\Delta_{S^N} u + \frac{N(N-2)}{2} u = K u^{\frac{N+2}{N-2}}, u > 0 \text{ on } S^N. \tag{1.7}$$

Let us fix a positive integer $k \ge k_0$, where k_0 is large, which is to be determined later.

The calculations in Appendix A suggest that we should make the scaling parameter satisfy

$$\mu \in \big[\Lambda_0 k^{\frac{N-1}{N-2}},\ \Lambda_1 k^{\frac{N-1}{N-2}}\big],$$

for some large constant $\Lambda_1 > 0$ and some small constant $\Lambda_0 > 0$.

Using the symmetry conditions (Ω_2) and (Ω_3), we introduce the following space:

$$H_s = \big\{ u : u \in H_0^1(\Omega), u \text{ is even in } y_h, h = 2, \cdots, N-1,$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \frac{2\pi j}{k}), r\sin(\theta + \frac{2\pi j}{k}), y'') \big\},$$

where $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$.

We will look for solutions in the space H_s. So, we put k bubbles in an one-dimensional circle as follows: let

$$x_j = \big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0, \cdots, 0, l\big), \quad j = 1, \cdots, k,$$

and let

$$W_{r,l,\mu}(y) = \sum_{j=1}^{k} PU_{x_j,\mu}.$$

In this paper, we will prove that for any large k, (1.4) has a solution u_k with

$$u_k \approx W_{r_k, l_k, \mu_k}.$$

Now, we discuss the location of the bubbles. Due the weight y_N^m in (1.4), the energy of the bubble $PU_{x_1,\mu}$ will increase as $x_{1,N}$ increases. On the other hand, due to the Dirichlet boundary condition, the energy of the bubble $PU_{x_1,\mu}$ will also increase as x_1 moves toward the boundary. So we see that in the vertical direction, the energy achieves its minimum at a point near the bottom part of the boundary. This property comes directly from the equation and the boundary condition. In order to obtain a balance in the horizontal directions, we need to impose (Ω_4) on the domain. From this discussion, we know we should put x_1 close to the boundary point x^*. For such x_1, there is a unique (h,d), such that

$$x_1 = (h, 0, \cdots, 0, \psi(h)) + d\nu,$$

where ν is the unit inward normal of $\partial\Omega$ at $(h, 0, \cdots, 0, \psi(h))$. In the following, we will use h and d, instead of r and l, as the coordinates for x_1. So we will use the notation:

$$W_{h,d,\mu}(y) = \sum_{j=1}^{k} PU_{x_j,\mu}.$$

Theorem 1.1 is a direct consequence of the following result:

Theorem 1.2. *Suppose that $N \geq 5$. If Ω satisfies (Ω_1)–(Ω_4), then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.4) has a solution u_k of the form*

$$u_k = W_{h_k,d_k,\mu_k}(y) + \omega_k, \tag{1.8}$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\mu_k^{-\frac{N-2}{2}} \|\omega_k\|_{L^\infty} \to 0$, $\mu_k \in [\Lambda_0 k^{\frac{N-1}{N-2}},\ \Lambda_1 k^{\frac{N-1}{N-2}}]$, and $d_k \to 0$, $h_k \to r^*$.

Conditions (Ω_2) and (Ω_3) are symmetry conditions, which allow us to find a solution in the space H_s. It is condition (Ω_4) that makes the construction of the solution of the form (1.8) possible. We believe that these symmetry assumptions may be replaced by some kind of conditions on geodesics on the boundary. On the other hand, the weight y_N^m plays a crucial role in determining the location of the bubbles in the vertical direction. So the technique in this paper can not be used to obtain a multiplicity result for

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad u > 0 \ \text{ in } \Omega, \quad u = 0, \ \text{ on } \partial\Omega.$$

We will use a reduction argument to prove Theorem 1.2. In section 2, we will carry out the reduction in a weighted space. The main result Theorem 1.2 is proved in section 3, while all the technical estimates are put in the appendices.

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2. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction. Since this part is similar to [20], we shall only give a sketch of the proofs.

Let

$$\|u\|_* = \sup_{y\in\Omega}\Big(\sum_{j=1}^k \frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1}\mu^{-\frac{N-2}{2}}|u(y)|, \tag{2.1}$$

and

$$\|f\|_{**} = \sup_{y\in\Omega}\Big(\sum_{j=1}^k \frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1}\mu^{-\frac{N+2}{2}}|f(y)|, \tag{2.2}$$

where $\tau=\frac{N-2}{N-1}$. For this choice of τ, we find that

$$\sum_{j=2}^k \frac{1}{|\mu x_j-\mu x_1|^\tau} \le \frac{Ck^\tau}{\mu^\tau}\sum_{j=2}^k\frac{1}{j^\tau} \le \frac{Ck}{\mu^\tau}\le C'.$$

We will use ∂_1, ∂_2 and ∂_3 to denote $\frac{\partial}{\partial h}$, $\frac{\partial}{\partial d}$ and $\frac{\partial}{\partial \mu}$ respectively. Let

$$Z_{i,j} = -div(y_N^m\partial_j PU_{x_i,\mu}),\quad j=1,2,3.$$

Consider

$$\begin{cases} -div(y_N^m D\phi_k) - (2^*-1)y_N^m W_{h,d,\Lambda}^{2^*-2}\phi_k = & \xi + \sum_{j=1}^3 c_j\sum_{i=1}^k Z_{i,j}, \quad \text{in } \Omega,\\ \phi_k\in H_s, & \\ < Z_{i,l},\phi_k >= 0 & i=1,\cdots,k,\ l=1,2,3 \end{cases} \tag{2.3}$$

for some numbers c_i, where $< u, v >= \int_\Omega uv$.

Before we proceed, we need the following lemma. The proof may be known but we can not find a reference so we give a proof in Appendix C.

Lemma 2.1. *Let u be the solution of*

$$-div(y_N^m Du) = y_N^m f(y),\ \ y\in\Omega,\quad u=0,\ \ on\ \partial\Omega.$$

Then, there is a constant $C>0$, such that

$$|u(x)|\le C\int_\Omega \frac{|f(y)|}{|y-x|^{N-2}}\,dy.$$

Now we have the following *a priori estimates.*

Lemma 2.2. *Assume that ϕ_k solves (2.3) for $\xi = \xi_k$. If $\|\xi_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. The proof of this lemma is similar to the proof of Lemma 2.1 in [20]. Thus, we just sketch it.

We argue by contradiction. Suppose that there are $k \to +\infty$, $\xi = \xi_k$, $h_k \to r^*$, $d_k \to 0$, $\mu_k \in \big[\Lambda_0 k^{\frac{N-1}{N-2}}, \Lambda_1 k^{\frac{N-1}{N-2}}\big]$, and ϕ_k solving (2.3) for $\xi = \xi_k$, $\mu = \mu_k$, $d = d_k$, $h = h_k$, with $\|\xi_k\|_{**} \to 0$, and $\|\phi_k\|_* \geq c' > 0$. We may assume that $\|\phi_k\|_* = 1$. For simplicity, we drop the subscript k.

Using Lemma 2.1, we obtain

$$
\begin{aligned}
|\phi(y)| \leq & C \int_\Omega \frac{1}{|z-y|^{N-2}} W_{h,d,\mu}^{2^*-2} |\phi(z)|\, dz \\
& + C \int_\Omega \frac{1}{|z-y|^{N-2}} \Big(|\xi(z)| + \Big| \sum_{j=1}^{3} c_j \sum_{i=1}^{k} Z_{i,j}(z) \Big| \Big)\, dz.
\end{aligned}
\tag{2.4}
$$

Using Lemma B.3, we have

$$
\begin{aligned}
& \int_\Omega \frac{1}{|z-y|^{N-2}} W_{h,d,\mu}^{2^*-2} |\phi(z)|\, dz \\
\leq & C \|\phi\|_* \mu^{\frac{N-2}{2}} \int_{\Omega_\mu} \frac{1}{|z-y|^{N-2}} W_{h,d,\mu}^{2^*-2} \sum_{j=1}^{k} \frac{1}{(1+\mu|z-x_j|)^{\frac{N-2}{2}+\tau}}\, dz \\
\leq & C \|\phi\|_* \mu^{\frac{N-2}{2}} \sum_{j=1}^{k} \frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.
\end{aligned}
\tag{2.5}
$$

It follows from Lemma B.2 that

$$
\begin{aligned}
& \int_\Omega \frac{1}{|z-y|^{N-2}} |\xi(z)|\, dz \leq C \|\xi\|_{**} \int_\Omega \frac{1}{|z-y|^{N-2}} \sum_{j=1}^{k} \frac{\mu^{\frac{N+2}{2}}}{(1+\mu|z-x_j|)^{\frac{N-2}{2}+\tau}} \\
\leq & C \|\xi\|_{**} \mu^{\frac{N-2}{2}} \sum_{j=1}^{k} \frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}},
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
&\int_\Omega \frac{1}{|z-y|^{N-2}}\Big|\sum_{i=1}^k Z_{i,l}(z)\Big|\,dz\\
\le& C\int_{\mathbb{R}^N} \frac{1}{|z-\mu y|^{N-2}}\sum_{i=1}^k \frac{\mu^{\frac{N-2}{2}+m_i}}{(1+|z-\mu x_j|^2)^{\frac{N+2}{2}}}\,dz \qquad (2.7)\\
\le& C\mu^{\frac{N-2}{2}+m_i}\sum_{i=1}^k \frac{1}{(1+\mu|y-x_i|)^{\frac{N-2}{2}+\tau}},
\end{aligned}
$$

where $m_i = 1$ if $i = 1, 2$, and $m_3 = -1$.

Next, we estimate c_l, $l = 1, 2, 3$. Multiplying (2.3) by $Z_{i,l}$ and integrating, we see that c_t satisfies

$$
\sum_{t=1}^3\sum_{j=1}^k \langle Z_{j,t}, \partial_l U_{x_i,\mu}\rangle c_t = \big\langle -div(y_N^m D\phi) - (2^*-1)y_N^m W_{h,d,\mu}^{2^*-2}\phi, \partial_l U_{x_i,\mu}\big\rangle - \big\langle h, \partial_l U_{x_i,\mu}\big\rangle. \qquad (2.8)
$$

It follows from Lemma B.1 that

$$
\begin{aligned}
&\big|\langle h, \partial_l U_{x_i,\mu}\rangle\big| \le C\|h\|_{**}\int_\Omega \frac{\mu^{\frac{N-2}{2}+m_i}}{(1+\mu|z-x_i|)^{N-2}}\sum_{j=1}^k \frac{\mu^{\frac{N+2}{2}}}{(1+\mu|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz\\
\le& C\|h\|_{**}.
\end{aligned}
$$

On the other hand, using Lemma B.3, we can prove

$$
\begin{aligned}
&\big\langle -div(y_N^m D\phi) - (2^*-1)y_N^m W_{h,d,\mu}^{2^*-2}\phi, \partial_l U_{x_i,\mu}\big\rangle\\
=&\big\langle -div(y_N^m D\partial_l U_{x_i,\mu}) - (2^*-1)y_N^m W_{h,d,\mu}^{2^*-2}\partial_l U_{x_i,\mu}, \phi\big\rangle = o(\|\phi\|_*)\mu^{m_i}. \qquad (2.9)
\end{aligned}
$$

Thus we obtain from (2.8) that

$$
c_l = \frac{1}{\mu^{m_i}}\Big(o(\|\phi\|_*) + O(\|h\|_{**})\Big). \qquad (2.10)
$$

So,

$$
\|\phi\|_* \le \Big(o(1) + \|h_k\|_{**} + \frac{\sum_{j=1}^k \frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^k \frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big). \qquad (2.11)
$$

Since $\|\phi\|_* = 1$, we obtain from (2.11) that there is $R > 0$, such that

$$\|\mu^{\frac{N-2}{2}}\phi(y)\|_{B_{\mu^{-1}R}(x_i)} \geq a > 0, \tag{2.12}$$

for some i. But $\bar{\phi}(y) = \mu^{-\frac{N-2}{2}}\phi(\mu(y-x_i))$ converges uniformly in any compact set to a solution u of

$$-\Delta u - (2^*-1)U_{0,\Lambda}^{2^*-2}u = 0, \quad \text{in } \mathbb{R}^N, \tag{2.13}$$

for some $\Lambda \in [\Lambda_1, \Lambda_2]$, and u is perpendicular to the kernel of (2.13). So, $u = 0$. This is a contradiction to (2.12).

□

From Lemma 2.2, using the same argument as in the proof of Proposition 4.1 in [7], we can prove the following result :

Proposition 2.3. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \geq k_0$ and all $h \in L^\infty(\mathbb{R}^N)$, problem (2.3) has a unique solution $\phi \equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_* \leq C\|h\|_{**}. \tag{2.14}$$

Now, we consider

$$\begin{cases} -div\big(y_N^m D(W_{h,d,\mu}+\phi)\big) = y_N^m\big(W_{r,\mu}+\phi\big)^{2^*-1} + \sum_{t=1}^3 c_t \sum_{i=1}^k Z_{i,t}, & \text{in } \Omega, \\ \phi \in H_s, & \\ < Z_{i,l}, \phi >= 0, & i = 1, \cdots, k, \ l = 1,2,3. \end{cases} \tag{2.15}$$

The main result of this section is the following:

Proposition 2.4. *There is an integer $k_0 > 0$, such that for each $k \geq k_0$, (h,d) close to $(r^*, 0)$, $\mu \in \big[\Lambda_0 k^{\frac{N-1}{N-2}}, \Lambda_1 k^{\frac{N-1}{N-2}}\big]$, (2.15) has a unique solution $\phi = \phi(h,d,\mu)$, satisfying*

$$\|\phi\|_* \leq C\big(\frac{1}{\mu}\big)^{\frac{1}{2}+\sigma},$$

if $N \geq 5$, where $\sigma > 0$ is a small constant.

To prove Proposition 2.4, we need to prove two lemmas first. Rewrite (2.15) as

$$(2.16)\qquad \begin{cases} -div(y_N^m D\phi) - (2^*-1)y_N^m W_{h,d,\mu}^{2^*-2}\phi = & N(\phi) + l_k + \sum_{t=1}^3 c_i \sum_{i=1}^k Z_{i,t}, \text{ in } \Omega, \\ \phi \in H_s, \\ < Z_{i,l}, \phi >= 0, & i = 1, \cdots, k, \ l = 1, 2, \end{cases}$$

where

$$N(\phi) = y_N^m \Big(\big(W_{h,d,\mu} + \phi\big)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^*-1) W_{h,d,\mu}^{2^*-2}\phi \Big),$$

and

$$l_k = y_N^m \Big(W_{h,d,,\mu}^{2^*-1} - \sum_{j=1}^k U_{x_j,\mu}^{2^*-1} \Big) - m y_N^{m-1} \sum_{j=1}^k \frac{\partial PU_{x_j,\mu}}{\partial y_N}.$$

In order to use the contraction mapping theorem to prove that (2.16) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

Lemma 2.5. *If $N \geq 4$, then*

$$\|N(\phi)\|_{**} \leq C \|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|N(\phi)| \leq \begin{cases} C|\phi|^{2^*-1}, & N \geq 6; \\ C\big(W_{h,d,\mu}^{\frac{6-N}{N-2}}\phi^2 + |\phi|^{2^*-1}\big), & N = 4, 5. \end{cases}$$

Firstly, we consider $N \geq 6$.

Using

$$\sum_{j=1}^k a_j b_j \leq \big(\sum_{j=1}^k a_j^p\big)^{\frac{1}{p}} \big(\sum_{j=1}^k b_j^q\big)^{\frac{1}{q}}, \ \ \frac{1}{p} + \frac{1}{q} = 1, \ a_j, \ b_j \geq 0,$$

we obtain

$$
\begin{aligned}
|N(\phi)| \le\; & C\|\phi\|_*^{2^*-1}\Big(\sum_{j=1}^{k}\frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\
\le & C\|\phi\|_*^{2^*-1}\mu^{\frac{N+2}{2}}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\tau}}\Big)^{\frac{4}{N-2}} \qquad (2.17)\\
\le & C\|\phi\|_*^{2^*-1}\mu^{\frac{N+2}{2}}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

Thus, the result follows.

Suppose that $N=4,5$. Noting that $N-2\ge\frac{N-2}{2}+\tau$, we find

$$
\begin{aligned}
|N(\phi)| \le & C\|\phi\|_*^{2}\mu^{N-2}\Big(\sum_{i=1}^{k}\frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_i|)^{N-2}}\Big)^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2}\\
& + C\|\phi\|_*^{2^*-1}\mu^{\frac{N+2}{2}}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
\le & C\|\phi\|_*^{2}\mu^{\frac{N+2}{2}}\Big(\sum_{j=1}^{k}\frac{1}{1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\
& + C\|\phi\|_*^{2^*-1}\mu^{\frac{N+2}{2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
= & C\|\phi\|_*^{2}\mu^{\frac{N+2}{2}}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

So, we have proved that for $N\ge 4$,

$$
\|N(\phi)\|_{**}\le C\|\phi\|_*^{\min(2,2^*-1)}.
$$

□

Next, we estimate l_k.

Lemma 2.6. *If $N\ge 5$, then*

$$
\|l_k\|_{**}\le C\big(\frac{1}{\mu}\big)^{\frac{1}{2}+\sigma},
$$

where $\sigma>0$ is a small constant.

Proof. Define

$$\Omega_j=\{y:\ y\in\Omega,\ \langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\rangle\ge\cos\frac{\pi}{k}\},\quad y'=(y_1,y_2,,0,\cdots,0).$$

We have

$$\begin{aligned}l_k=&y_N^m\Big(W_{h,d,\mu}^{2^*-1}-\sum_{j=1}^k\big(PU_{x_j,\mu}\big)^{2^*-1}\Big)+y_N^m\Big(\sum_{j=1}^k\big(PU_{x_j,\mu}\big)^{2^*-1}-\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}\Big)\\&-my_N^{m-1}\sum_{j=1}^k\frac{\partial PU_{x_j,\mu}}{\partial y_N}\\=:&J_0+J_1+J_2.\end{aligned}$$

From the symmetry, we can assume that $y\in\Omega_1$. Then,

$$|y-x_j|\ge|y-x_1|,\quad\forall\, y\in\Omega_1.$$

Firstly, we claim

$$\frac{1}{1+|y-x_j|}\le\frac{C}{|x_j-x_1|},\quad\forall\, y\in\Omega_1,\ j\ne1.\tag{2.18}$$

In fact, if $|y-x_1|\le\frac12|x_1-x_j|$, then $|y-x_j|\ge\frac12|x_1-x_j|$. If $|y-x_1|\ge\frac12|x_1-x_j|$, then $|y-x_j|\ge|y-x_1|\ge\frac12|x_1-x_j|$, since $y\in\Omega_1$.

For the estimate of J_0, we have

$$|J_0|\le\frac{C\mu^2}{(1+\mu|y-x_1|)^4}\sum_{j=2}^k\frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_j|)^{N-2}}+C\Big(\sum_{j=2}^k\frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_j|)^{N-2}}\Big)^{2^*-1}.\tag{2.19}$$

Using (2.18), taking $1<\alpha\le N-2$, we obtain for any $y\in\Omega_1$,

$$\begin{aligned}&\frac{1}{(1+\mu|y-x_1|)^4}\frac{1}{(1+\mu|y-x_j|)^{N-2}}\\\le&C\frac{1}{(1+\mu|y-x_1|)^{N+2-\alpha}}\frac{1}{|\mu x_j-\mu x_1|^\alpha},\quad j>1.\end{aligned}\tag{2.20}$$

Take $\alpha>\max(\frac{N-1}{2},1)$ satisfying $N+2-\alpha\ge\frac{N+2}{2}+\tau$. Then

$$\begin{aligned}
&\frac{1}{(1+\mu|y-x_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+\mu|y-x_j|)^{N-2}}\\
&\le\frac{C}{(1+\mu|y-x_1|)^{N+2-\alpha}}\Big(\frac{k}{\mu}\Big)^{\alpha}=\frac{C}{(1+\mu|y-x_1|)^{N+2-\alpha}}\mu^{-\frac{\alpha}{N-1}}\\
&\le\frac{C}{(1+\mu|y-x_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{1}{\mu}\Big)^{\frac12+\sigma}.
\end{aligned}\tag{2.21}$$

Using the Hölder inequality, we obtain

$$\begin{aligned}
&\Big(\sum_{j=2}^{k}\frac{1}{(1+\mu|y-x_j|)^{N-2}}\Big)^{2^*-1}\\
\le&\sum_{j=2}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=2}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{\frac{4}{N-2}}.
\end{aligned}$$

Noting that $\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})>1$ if $N\ge 4$, we obtain

$$\begin{aligned}
&\Big(\sum_{j=2}^{k}\frac{1}{(1+\mu|y-x_j|)^{N-2}}\Big)^{2^*-1}\\
\le&C\Big(\sum_{j=2}^{k}\frac{1}{|\mu x_1-\mu x_j|^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
\le&C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
=&C\Big(\frac{1}{\mu}\Big)^{\frac{N+2}{N-1}(\frac12-\frac{\tau}{N+2})}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}\\
=&C\Big(\frac{1}{\mu}\Big)^{\frac12+\sigma}\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}},
\end{aligned}\tag{2.22}$$

since $\frac{N+2}{N-1}(\frac12-\frac{\tau}{N+2})>\frac12$. Thus, we have proved that if $N\ge 4$,

$$\|J_0\|_{**}\le C\Big(\frac{1}{\mu}\Big)^{\frac12+\sigma}.$$

Now, we estimate J_1. Let $H(y,x)$ be the regular part of the Green function for $-\Delta$ in Ω with the zero boundary condition. Let x_j^* be the reflection point of x_j with respect to $\partial\Omega$. Then

$$\frac{H(y,x_j)}{\mu^{N-2}}=\frac{C}{\mu^{N-2}|y-x_j^*|^{N-2}}\le\frac{C}{(1+\mu|y-x_j|)^{N-2}}, \tag{2.23}$$

since $\mu|y-x_j^*|\ge\mu d\to+\infty$. Using (A.1), we find

$$\begin{aligned}|J_1|\le&\sum_{j=1}^k\frac{C\mu^2}{(1+\mu|y-x_j|)^4}\frac{H(y,x_j)}{\mu^{\frac{N-2}{2}}}\\
\le&C\mu^{\frac{N+2}{2}}\sum_{j=1}^k\frac{1}{(1+\mu|y-x_j|)^{4+(1-t)(N-2)}}\Big(\frac{H(y,x_j)}{\mu^{N-2}}\Big)^t\\
\le&C\Big(\frac{1}{\mu d}\Big)^{t(N-2)}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{4+t(N-2)}}\\
\le&C\mu^{\frac{N+2}{2}}\Big(\frac{1}{\mu}\Big)^{t\frac{N-2}{N-1}}\sum_{j=1}^k\frac{1}{(1+\mu|y-x_j|)^{4+(1-t)(N-2)}}\\
\le&C\mu^{\frac{N+2}{2}}\Big(\frac{1}{\mu}\Big)^{\frac12+\sigma}\sum_{j=1}^k\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}},\end{aligned} \tag{2.24}$$

if we take $t\frac{N-2}{N-1}=\frac12+\sigma$ for some small$\sigma>0$. For such t, we can check that $4+(1-t)(N-2)\ge\frac{N+2}{2}+\tau$. Here, we have used $d\ge\frac{r_0}{k}$.

Finally, we estimate J_2. Similar to (2.23), we have

$$\begin{aligned}\frac{\partial PU_{x_j,\Lambda}}{\partial y_N}=&\frac{\partial U_{x_j,\Lambda}}{\partial y_N}+O\Big(\frac{1}{\mu^{\frac{N-2}{2}}}\Big|\frac{\partial H(y,x_j)}{\partial y_N}\Big|\Big)\\
=&\frac{\partial U_{x_j,\Lambda}}{\partial y_N}+O\Big(\frac{1}{\mu^{\frac{N-2}{2}}}\frac{1}{|y-x_j^*|^{N-1}}\Big)\\
=&O\Big(\frac{\mu^{\frac N2}}{(1+\mu|y-x_j|)^{N-1}}\Big)=\mu^{\frac{N+2}{2}}O\Big(\frac{1}{\mu^{\frac12+\sigma}(1+\mu|y-x_j|)^{N-\frac12-\sigma}}\Big),\end{aligned}$$

where x_j^* is the reflection point of x_j with respect to $\partial\Omega$.

If $N\ge5$ and $\sigma>0$ is small, then we have $N-\frac12-\sigma>\frac{N+2}{2}+\tau$. As a result

$$|J_2| \leq \frac{C}{\mu^{\frac{1}{2}+\sigma}} \sum_{j=1}^{k} \frac{\mu^{\frac{N+2}{2}}}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}. \tag{2.25}$$

□

Now, we are ready to prove Proposition 2.4.

Proof of Proposition 2.4. Let

$$E = \big\{u : u \in C(\Omega) \cap H_s, \|u\|_* \leq \big(\frac{1}{k}\big)^{\frac{1}{2}}, \int_\Omega Z_{i,l}\phi = 0,\ i = 1, \cdots, k,\ l = 1, 2, 3\big\}.$$

Then, (2.16) is equivalent to

$$\phi = A(\phi) =: L_k(N(\phi)) + L_k(l_k),$$

where L_k is defined in Proposition 2.3. We will prove that A is a contraction map from E to E.

We have

$$\begin{aligned} &\|A(\phi)\|_* \leq C\|N(\phi)\|_{**} + C\|l_k\|_{**} \\ \leq &C\|\phi\|_*^{\min(2^*-1,2)} + C\|l_k\|_{**} \leq \frac{C}{k^{\frac{1}{2}+\sigma}} \leq \frac{1}{k^{\frac{1}{2}}}. \end{aligned} \tag{2.26}$$

Thus, A maps E to E.

On the other hand,

$$\|A(\phi_1) - A(\phi_2)\|_* = \|L_k(N(\phi_1)) - L_k(N(\phi_2))\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**}.$$

If $N \geq 6$, then

$$|N'(t)| \leq C|t|^{2^*-2}.$$

As a result,

$$\begin{aligned} &|N(\phi_1) - N(\phi_2)| \leq C\big(|\phi_1|^{2^*-2} + |\phi_2|^{2^*-2}\big)|\phi_1 - \phi_2| \\ \leq &C\big(\|\phi_1\|_*^{2^*-2} + \|\phi_2\|_*^{2^*-2}\big)\|\phi_1 - \phi_2\|_* \Big(\sum_{j=1}^{k} \frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} \end{aligned}$$

As before, we have

$$\Big(\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\le C\sum_{j=1}^{k}\frac{1}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

So,

$$\begin{aligned}&\|A(\phi_1)-A(\phi_2)\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}\\ \le& C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\le\frac{1}{2}\|\phi_1-\phi_2\|_*.\end{aligned}$$

Thus, A is a contraction map.

For $N=5$,

$$|N'(t)|\le CW_{h,d,\Lambda}^{\frac{6-N}{N-2}}|t|+C|t|^{2^*-2}.$$

So,

$$\begin{aligned}&|N(\phi_1)-N(\phi_2)|\\ \le& C\big(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\big)|\phi_1-\phi_2|+C\big(|\phi_1|+|\phi_2|\big)W_{h,d,\Lambda}^{\frac{6-N}{N-2}}|\phi_1-\phi_2|\\ \le& C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^{k}\frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\ &+C\big(\|\phi_1\|_*+\|\phi_2\|_*\big)\|\phi_1-\phi_2\|_*W_{h,d,\Lambda}^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^{k}\frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2}\\ \le& C\big(\|\phi_1\|_*+\|\phi_2\|_*\big)\|\phi_1-\phi_2\|_*\sum_{j=1}^{k}\frac{\mu^{\frac{N+2}{2}}}{(1+\mu|y-x_j|)^{\frac{N+2}{2}+\tau}}.\end{aligned}\tag{2.27}$$

Thus, A is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi\in E$, such that

$$\phi=A(\phi).$$

Moreover, it follows from Proposition 2.3 that

$$\|\phi\|_*\le C\|l_k\|_{**}+C\|N(\phi)\|_{**}\le C\|l_k\|_{**}+C\|\phi\|_*^{\min(2^*-1,2)},$$

which gives

$$\|\phi\|_*\le C\big(\frac{1}{\mu}\big)^{\frac{1}{2}+\sigma},$$

if $N \geq 5$.

$\square$

3. Proof of Theorem 1.2

In this section, we will choose h, d and μ, such that the corresponding c_t is zero. For this purpose, we only need to solve the following problem:

$$\langle I'(W_{h,d,\mu} + \phi), \partial_i PU_{x_1,\mu} \rangle = 0, \quad i = 1, 2, 3,$$

where we use ∂_1, ∂_2 and ∂_3 to denote $\frac{\partial}{\partial h}$, $\frac{\partial}{\partial d}$ and $\frac{\partial}{\partial \mu}$ respectively.

We first prove the following result:

Proposition 3.1. *We have*

$$\langle I'(W_{h,d,\mu} + \phi), \frac{\partial PU_{x_1,\mu}}{\partial \mu} \rangle = -\frac{B_1}{\mu^{N-1} d^{N-2}} + \frac{B_4 k^{N-2}}{\mu^{N-1}} + O\Big(\frac{1}{\mu^{2+\sigma}}\Big), \tag{3.1}$$

$$\langle I'(W_{h,d,\mu} + \phi), \frac{\partial PU_{x_1,\mu}}{\partial h} \rangle = B_2 \psi'(h) + O\Big(\frac{1}{\mu^{\sigma}}\Big), \tag{3.2}$$

and

$$\langle I'(W_{h,d,\mu} + \phi), \frac{\partial PU_{x_1,\mu}}{\partial d} \rangle = B_3 - \frac{B_1}{\mu^{N-2} d^{N-1}} + O\Big(\frac{1}{\mu^{\sigma}}\Big). \tag{3.3}$$

The proof of Proposition 3.1 is similar to that of Lemma 2.6 and is quite technical. We leave it to the end of this section.

Proof of Theorem 1.2. Define

$$d = \frac{D}{k}, \quad \mu = \Lambda k^{\frac{N-1}{N-2}}.$$

Then, (3.1), (3.2) and (3.3) are equivalent to

$$-\frac{B_1}{\Lambda^{N-1} D^{N-2}} + \frac{B_4}{\Lambda^{N-1}} = o(1), \tag{3.4}$$

$$\psi'(h) = o(1), \tag{3.5}$$

and

$$B_3 - \frac{B_1}{\Lambda^{N-2} D^{N-1}} = o(1), \tag{3.6}$$

respectively.

Let

$$f_1(D, \Lambda) = -\frac{B_1}{\Lambda^{N-1} D^{N-2}} + \frac{B_4}{\Lambda^{N-1}},$$

and

$$f_2(D, \Lambda) = B_3 - \frac{B_1}{\Lambda^{N-2} D^{N-1}}.$$

Then, $f_1 = 0$ and $f_2 = 0$ have a unique solution

$$D_0 = \Big(\frac{B_1}{B_3}\Big)^{\frac{1}{N-2}}, \quad \Lambda_0 = \Big(\frac{B_1}{B_3 D_0^{N-1}}\Big)^{\frac{1}{N-2}}.$$

On the other hand, it is easy to see that

$$\frac{\partial f_1(D_0, \Lambda_0)}{\partial \Lambda} = 0, \quad \frac{\partial f_2(D_0, \Lambda_0)}{\partial D} > 0,$$

and

$$\frac{\partial f_1(D_0, \Lambda_0)}{\partial D} = \frac{\partial f_2(D_0, \Lambda_0)}{\partial \Lambda} > 0.$$

Thus

$$\deg\big(f^*, 0, B^*\big) \neq 0,$$

where $f^*(D, \Lambda) = \big(f_1(D, \Lambda), f_2(D, \Lambda)\big)$, and $B^* = \big\{(D, \Lambda) : |D - D_0| + |\Lambda - \Lambda_0| < \delta\big\}$. On the other hand, since r^* is either a strict local minimum point, or strict local maximum point of ψ, we know that for $\delta > 0$ small,

$$\deg\big(\psi', 0, (r^* - \delta . r^* + \delta)\big) \neq 0.$$

Let $\tilde f((h, D, \Lambda) = \big(\psi(h), f^*(D, \Lambda)\big)$, and let $\tilde B = (r^* - \delta . r^* + \delta) \times B^*$. Then

$$\deg\big(\tilde f, 0, \tilde B\big) \neq 0,$$

from which, we know that (3.4), (3.5) and (3.6) have a solution near (r^*, D_0, Λ_0). $\square$

Proof of Proposition 3.1. We have

$$\begin{aligned}&\big\langle I'(W_{h,d,\mu}+\phi),\partial_i PU_{x_1,\mu}\big\rangle\\ =&\big\langle I'(W_{h,d,\mu}),\partial_i PU_{x_1,\mu}\big\rangle+\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)U_{x_1,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}\\ &-m\int_\Omega y_N^{m-1}\frac{\partial(\partial_i PU_{x_1,\mu})}{\partial y_N}\phi.\end{aligned}$$

In view of Proposition A.1, we need to prove

$$\begin{aligned}&\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)U_{x_1,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}\\ &-m\int_\Omega y_N^{m-1}\frac{\partial(\partial_i PU_{x_1,\mu})}{\partial y_N}\phi\\ =&\mu^{m_i}O\big(\frac{1}{\mu^{1+\sigma}}\big),\end{aligned}\tag{3.7}$$

where $m_i=1$ if $i=1,2$, and $m_3=-1$.

The proof of (3.7) is similar to that of Lemma 2.6. Write

$$\begin{aligned}&\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)U_{x_1,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}\\ =&\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)W_{h,d,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}\\ &+(2^*-1)\int_\Omega y_N^m\Big(W_{h,d,\mu}^{2^*-2}-U_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}.\end{aligned}\tag{3.8}$$

If $N\geq 6$, then $2^*-1\leq 2$. In this case, we have the formula

$$(1+t)^{2^*-1}-1-(2^*-1)t=O(t^2).$$

As a result,

$$\begin{aligned}&\Big|\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)W_{h,d,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}\Big|\\ \le& C\int_\Omega W_{h,d,\mu}^{2^*-3}|\phi|^2|\partial_i PU_{x_1,\mu}|\le C\mu^{m_i}\int_\Omega U_{x_1,\mu}^{2^*-2}|\phi|^2 \qquad (\text{ since } 2^*-3\le 0)\\ \le& C\mu^{N+m_i}\|\phi\|_*^2\int_\Omega\sum_{j=1}^k\Big(\frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2\frac{1}{(1+\mu|y-x_1|)^4}\\ =& C\mu^{m_i}\|\phi\|_*^2\int_{\tilde\Omega}\sum_{j=1}^k\Big(\frac{1}{(1+|y-\mu x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2\frac{1}{(1+|y-\mu x_1|)^4},\end{aligned}$$

where $\tilde\Omega=\big\{y:\mu^{-1}y\in\Omega\big\}$.

Recall

$$\Omega_j=\big\{y:\ y\in\Omega,\ \big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\big\rangle\ge\cos\frac{\pi}{k}\big\},\quad y'=(y_1,y_2,,0,\cdots,0).$$

For any $y\in\tilde\Omega_n=\big\{y:\mu^{-1}y\in\Omega_n\big\}$,

$$\begin{aligned}&\sum_{j=1}^k\frac{1}{(1+|y-\mu x_j|)^{\frac{N-2}{2}+\tau}}\\ \le&\frac{1}{(1+|y-\mu x_n|)^{\frac{N-2}{2}+\tau}}+\frac{1}{(1+|y-\mu x_n|)^{\frac{N-2}{2}}}\sum_{j\ne n}\frac{C}{|\mu x_n-\mu x_j|^\tau}\\ \le&\frac{C}{(1+|y-\mu x_n|)^{\frac{N-2}{2}}},\end{aligned}$$

So,

$$\begin{aligned}&\int_{\tilde\Omega_n}\sum_{j=1}^k\Big(\frac{1}{(1+|y-\mu x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2\frac{1}{(1+|y-\mu x_1|)^4}\\ \le& C\int_{\tilde\Omega_n}\frac{1}{(1+|y-\mu x_n|)^{N-2}}\frac{1}{(1+|y-\mu x_1|)^4}\le C\frac{1}{|\mu x_n-\mu x_1|^{\frac32}}.\end{aligned}$$

So, for $N\ge 6$,

$$\begin{aligned}&\Big|\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)W_{h,d,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}\Big|\\ \le& C\mu^{m_i}\|\phi\|_*^2\le\frac{C\mu^{m_i}}{\mu^{1+\sigma}}.\end{aligned}$$

If $N = 5$, then

$$
\begin{aligned}
&\Big|\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)W_{h,d,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}\Big|\\
\le& C\int_\Omega\Big(W_{h,d,\mu}^{2^*-3}|\phi|^2+|\phi|^{2^*-1}\Big)|\partial_i PU_{x_1,\mu}|\\
\le& C\mu^{m_i}\big(\|\phi\|_*^2+\|\phi\|_*^{2^*-1}\big)\int_{\tilde\Omega}\sum_{j=1}^k\Big(\frac{1}{(1+|y-\mu x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
\le& C\mu^{m_i}\big(\|\phi\|_*^2+\|\phi\|_*^{2^*-1}\big)\le\frac{C\mu^{m_i}}{\mu^{1+\sigma}}.
\end{aligned}
$$

So, we have proved

$$
\int_\Omega y_N^m\Big((W_{h,d,\mu}+\phi)^{2^*-1}-W_{h,d,\mu}^{2^*-1}-(2^*-1)W_{h,d,\mu}^{2^*-2}\phi\Big)\partial_i PU_{x_1,\mu}=\mu^{m_i}O\Big(\frac{1}{\mu^{1+\sigma}}\Big). \tag{3.9}
$$

Now, we estimate the second term in the right hand side of (3.8):
Write

$$
\begin{aligned}
&\int_\Omega y_N^m\Big(W_{h,d,\mu}^{2^*-2}-U_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}\\
=&\int_\Omega y_N^m\Big(W_{h,d,\mu}^{2^*-2}-PU_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}+\int_\Omega y_N^m\Big(PU_{x_1,\mu}-U_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}
\end{aligned}
\tag{3.10}
$$

If $N \ge 6$, then using (2.21),

$$
\begin{aligned}
&\Big|\int_\Omega y_N^m\Big(W_{h,d,\mu}^{2^*-2}-PU_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}\Big|\le C\mu^{m_i}\int_\Omega\sum_{j=2}^k U_{x_j,\mu}U_{x_1,\mu}^{2^*-2}|\phi|\\
\le& C\mu^{m_i}\|\phi\|_*\int_{\tilde\Omega}\sum_{j=2}^k\frac{1}{(1+|y-\mu x_j|)^{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-\mu x_i|)^{\frac{N-2}{2}+\tau}}\frac{1}{(1+|y-\mu x_1|)^4}\\
\le& C\mu^{m_i}\|\phi\|_*\frac{1}{\mu^{\frac12+\sigma}}\int_{\tilde\Omega}\sum_{i=1}^k\frac{1}{(1+|y-\mu x_i|)^{\frac{N-2}{2}+\tau}}\frac{1}{(1+|y-\mu x_1|)^{\frac{N+2}{2}+\tau}}\\
\le& C\mu^{m_i}\frac{1}{\mu^{1+\sigma}}.
\end{aligned}
\tag{3.11}
$$

If $N = 5$, then $2^* - 2 = \frac{4}{3}$ and $\tau = \frac{3}{4}$. We have

(3.12)
$$
\begin{aligned}
&\Big|\int_\Omega y_N^m\Big(W_{h,d,\mu}^{2^*-2}-PU_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}\Big|\\
\le& C\mu^{m_i}\int_\Omega\Big(\sum_{j=2}^k U_{x_j,\mu}U_{x_1,\mu}^{2^*-2}+\big(\sum_{j=2}^k U_{x_j,\mu}\big)^{2^*-2}U_{x_1,\mu}\Big)|\phi|\\
\le& C\mu^{m_i}\frac{1}{\mu^{1+\sigma}}+C\mu^{m_i}\int_\Omega\big(\sum_{j=2}^k U_{x_j,\mu}\big)^{2^*-2}U_{x_1,\mu}|\phi|\\
=& C\mu^{m_i}\frac{1}{\mu^{1+\sigma}}\\
&+\mu^{m_i}\|\phi\|_*\int_{\tilde\Omega}\big(\sum_{j=2}^k\frac{1}{1+|y-\mu x_i|)^{N-2}}\big)^{2^*-2}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\sum_{i=1}^k\frac{1}{1+|y-\mu x_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
$$
But

$$
\begin{aligned}
&\sum_{n=2}^k\int_{\tilde\Omega_n}\big(\sum_{j=2}^k\frac{1}{1+|y-\mu x_i|)^{N-2}}\big)^{2^*-2}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\sum_{i=1}^k\frac{1}{1+|y-\mu x_i|)^{\frac{N-2}{2}+\tau}}\\
\le&\sum_{n=2}^k\int_{\tilde\Omega_n}\frac{1}{1+|y-\mu x_n|)^{4-\frac{4\tau}{N-2}+\frac{N-2}{2}}}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
\le&\ln\mu\sum_{n=2}^k\frac{C}{|\mu x_n-\mu x_1|^{\frac{N}{2}+1-\frac{4}{N-1}}}\le\frac{C}{\mu^{\frac12+\sigma}},
\end{aligned}
$$
and

$$
\begin{aligned}
&\int_{\tilde\Omega_1}\big(\sum_{j=2}^k\frac{1}{1+|y-\mu x_i|)^{N-2}}\big)^{2^*-2}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\sum_{i=1}^k\frac{1}{1+|y-\mu x_i|)^{\frac{N-2}{2}+\tau}}\\
\le&\frac{Ck^4}{\mu^4}\int_{\tilde\Omega_1}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\sum_{i=1}^k\frac{1}{1+|y-\mu x_i|)^{\frac{N-2}{2}+\tau}}\\
\le&\frac{C}{\mu^{\frac12+\sigma}}.
\end{aligned}
$$
So, we also prove that for $N=5$,

$$
\Big|\int_\Omega y_N^m\Big(W_{h,d,\mu}^{2^*-2}-PU_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}\Big|\le\frac{C\mu^{m_i}}{\mu^{1+\sigma}}\tag{3.13}
$$

Using (2.23), similar to (2.24), we can prove

$$
\begin{aligned}
&\Big|\int_\Omega y_N^m\Big(PU_{x_1,\mu}-U_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}\Big|\\
\le& C\mu^{m_i}\int_\Omega \frac{H(y,x_1)}{\mu^{\frac{N-2}{2}}}U_{x_1,\mu}^{2^*-2}|\phi|\\
\le& C\mu^{m_i}\mu^{\frac{N+2}{2}}\frac{1}{\mu^{\frac12+\sigma}}\int_\Omega\frac{1}{(1+\mu|y-x_1|)^{\frac{N+2}{2}+\tau}}|\phi|\\
\le& C\mu^{m_i}\frac{1}{\mu^{\frac12+\sigma}}\|\phi\|_*\le\frac{C\mu^{m_i}}{\mu^{1+\sigma}}.
\end{aligned}
\tag{3.14}
$$

Combining (3.10), (3.11), (3.13) and (3.14), we obtain that for $N\ge 5$,

$$
\int_\Omega y_N^m\Big(W_{h,d,\mu}^{2^*-2}-\chi U_{x_1,\mu}^{2^*-2}\Big)\phi\partial_i PU_{x_1,\mu}=\mu^{m_i}O\big(\frac{k\|\phi\|_*}{\mu^N}\big).
\tag{3.15}
$$

Finally,

$$
\begin{aligned}
&\Big|\int_\Omega y_N^{m-1}\frac{\partial(\partial_i PU_{x_1,\mu})}{\partial y_N}\phi\Big|\\
\le& C\mu^{-1+m_i}\|\phi\|_*\int_{\tilde\Omega}\sum_{j=1}^k\frac{1}{(1+|y-\mu x_j|)^{\frac{N-2}{2}+\tau}}\frac{1}{(1+|y-\mu x_1|)^{N-1}}\\
\le& C\mu^{-1+m_i}\|\phi\|_*\le\frac{C\mu^{m_i}}{\mu^{\frac32+\sigma}}.
\end{aligned}
\tag{3.16}
$$

So, (3.7) follows from (3.9), (3.15) and (3.16).

□

Appendix A. Energy Expansion

In the appendixes, we always assume that

$$
x_j=\big(r\cos\frac{2(j-1)\pi}{k},r\sin\frac{2(j-1)\pi}{k},0,l\big),\quad j=1,\cdots,k.
$$

where 0 is the zero vector in $\mathbb{R}^{N-3}$, and (r,l) is close to (r^*,l^*). Recall that we write

$$
x_1=(h,0,\cdots,0,\psi(h))+d\nu,
$$

where ν is the inward unit normal of $\partial\Omega$ at $(h,0,\cdots,0,\psi(h))$.

Let $G(y,z)$ be the Green function of $-\Delta$ in Ω with the Dirichlet boundary condition. Let $H(y,z)$ be the regular part of the Green function.

Let recall that

$$\mu\in\big[\Lambda_0 k^{\frac{N-1}{N-2}},\Lambda_1 k^{\frac{N-1}{N-2}}\big],$$

where $\Lambda_0>0$ is a small constant, and $\Lambda_1>0$ is a large constant.

Define

$$I(u)=\frac{1}{2}\int_\Omega y_N^m|Du|^2-\frac{1}{2^*}\int_\Omega y_N^m|u|^{2^*},$$

$$U_{x_j,\mu}(y)=\big(N(N-2)\big)^{\frac{N-2}{4}}\frac{\mu^{\frac{N-2}{2}}}{(1+\mu^2|y-x_j|^2)^{\frac{N-2}{2}}},$$

and

$$W_{h,d,\mu}(y)=\sum_{j=1}^k PU_{x_j,\mu}(y),$$

where $PU_{x,\mu}$ is the solution of (1.6). It is well known that

$$U_{x_j,\mu}(y)-PU_{x_j,\mu}(y)=\frac{\tilde BH(y,x)}{\mu^{\frac{N-2}{2}}}+O\Big(\frac{1}{d^N\mu^{\frac{N+2}{2}}}\Big),\tag{A.1}$$

where $\tilde B>0$ is a constant.

The main result of this section is the following estimates.

Proposition A.1. *We have*

$$\Big\langle I'(W_{h,d,\mu}),\frac{\partial PU_{x_1,\mu}}{\partial\mu}\Big\rangle=-\frac{B_1}{\mu^{N-1}d^{N-2}}+\frac{B_4k^{N-2}}{\mu^{N-1}}+O\Big(\frac{1}{\mu^{2+\sigma}}\Big),\tag{A.2}$$

$$\Big\langle I'(W_{h,d,\mu}),\frac{\partial PU_{x_1,\mu}}{\partial h}\Big\rangle=B_2\psi'(h)+O\Big(\frac{1}{\mu^{\sigma}}\Big),\tag{A.3}$$

and

$$\Big\langle I'(W_{h,d,\mu}),\frac{\partial PU_{x_1,\mu}}{\partial d}\Big\rangle=B_3-\frac{B_1}{\mu^{N-2}d^{N-1}}+O\Big(\frac{1}{\mu^{\sigma}}\Big),\tag{A.4}$$

where B_1, B_2, B_3 and B_4 are some positive constants, $\sigma>0$ is a small constant.

Proof. We use ∂_1, ∂_2 and ∂_3 to denote $\frac{\partial}{\partial h}$, $\frac{\partial}{\partial d}$, and $\frac{\partial}{\partial \mu}$ respectively. Then

(A.5)
$$
\begin{aligned}
&\big\langle I'(W_{h,d,\mu}), \partial_i PU_{x_1,\mu}\big\rangle\\
=&\int_\Omega y_N^m\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}-m\sum_{j=1}^k\int_\Omega y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\partial_i PU_{x_1,\mu}\\
=&l^m\int_\Omega y_N^m\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}+\int_\Omega (y_N^m-l^m)\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}\\
&-m\sum_{j=1}^k\int_\Omega y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\partial_i PU_{x_1,\mu}.
\end{aligned}
$$

On the other hand,

$$
\int_\Omega\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i U_{x_1,\mu}=\sum_{n=1}^k\int_{\Omega_n}\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}, \tag{A.6}
$$

where

$$
\Omega_j=\big\{y:\ y\in\Omega,\ \big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\big\rangle\ge\cos\frac{\pi}{k}\big\},\quad y'=(y_1,y_2,,0,\cdots,0).
$$

Let $m_i=1$ if $i=1,2$ and let $m_3=-1$. Then

$$
|\partial_i PU_{x_1,\mu}|\le C\mu^{m_i}U_{x_1,\mu}.
$$

So we have

(A.7)
$$
\begin{aligned}
&\Big|\int_{\Omega_n}\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}\Big|\le C\mu^{m_i}\int_{\Omega_n}\big|\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\big|U_{x_1,\mu}\\
\le& C\mu^{m_i}\int_{\Omega_n}\Big(U_{x_n,\mu}^{2^*-2}\sum_{j\ne n}U_{x_j,\mu}+\big(\sum_{j\ne n}U_{x_j,\mu}\big)^{2^*-1}\Big)U_{x_1,\mu}\\
\le& C\mu^{m_i}\int_{\tilde\Omega_n}\frac{1}{(1+|y-\mu x_n|)^4}\sum_{j\ne n}\frac{1}{(1+|y-\mu x_j|)^{N-2}}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
&+C\mu^{m_i}\int_{\tilde\Omega_n}\big(\sum_{j\ne n}\frac{1}{(1+|y-\mu x_j|)^{N-2}}\big)^{2^*-1}\frac{1}{(1+|y-\mu x_1|)^{N-2}},
\end{aligned}
$$

where $\tilde\Omega_n=\big\{y:\mu^{-1}y\in\Omega_n\big\}$.

It is easy to check that for any $y \in \tilde{\Omega}_n$, $n \neq j$, $|y-\mu x_j| \ge \frac{1}{2}|\mu x_j - \mu x_n|$. As a result,

$$\sum_{j\neq n}\frac{1}{(1+|y-\mu x_j|)^{N-2}} \le C\sum_{j\neq n}\frac{1}{|\mu x_j-\mu x_n|^{N-2}} \le \frac{Ck^{N-2}}{\mu^{N-2}}.$$

So, we obtain

$$\begin{aligned}
&\int_{\tilde{\Omega}_n}\frac{1}{(1+|y-\mu x_n|)^4}\sum_{j\neq n}\frac{1}{(1+|y-\mu x_j|)^{N-2}}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
\le&\frac{Ck^{N-2}}{\mu^{N-2}}\int_{\tilde{\Omega}_n}\frac{1}{(1+|y-\mu x_n|)^4}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
\le&\frac{Ck^{N-2}}{\mu^{N-2}}\frac{1}{\mu^2|x_n-x_1|^2}\int_{\tilde{\Omega}_n}\frac{1}{(1+|y-\mu x_n|)^N} \le \frac{Ck^{N-2}}{\mu^{N-2}}\frac{\ln\mu}{\mu^2|x_n-x_1|^2}.
\end{aligned} \tag{A.8}$$

On the other hand, let $t > \max(1, \frac{2(N-2)}{N+2})$. Then, for $y \in \tilde{\Omega}_n$,

$$\begin{aligned}
&\sum_{j\neq n}\frac{1}{(1+|y-\mu x_j|)^{N-2}}\\
\le& C\sum_{j\neq n}\frac{1}{|\mu x_j-\mu x_n|^t}\frac{1}{(1+|y-\mu x_n|)^{N-2-t}} \le \frac{Ck^t}{\mu^t}\frac{1}{(1+|y-\mu x_n|)^{N-2-t}}.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\int_{\tilde{\Omega}_n}\Big(\sum_{j\neq n}\frac{1}{(1+|y-\mu x_j|)^{N-2}}\Big)^{2^*-1}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
\le&\frac{Ck^{t(2^*-1)}}{\mu^{t(2^*-1)}}\int_{\tilde{\Omega}_n}\frac{1}{(1+|y-\mu x_n|)^{(N-2-t)(2^*-1)}}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
\le&\frac{Ck^{t(2^*-1)}}{\mu^{t(2^*-1)}}\frac{1}{|\mu x_n-\mu x_1|^{N-t(2^*-1)}}\int_{\tilde{\Omega}_n}\frac{1}{(1+|y-\mu x_n|)^{(N-2-t)(2^*-1)+N-2-(N-t(2^*-1))}}\\
=&\frac{Ck^{t(2^*-1)}}{\mu^{t(2^*-1)}}\frac{1}{|\mu x_n-\mu x_1|^{N-t(2^*-1)}}\int_{\tilde{\Omega}_n}\frac{1}{(1+|y-\mu x_n|)^N}\\
\le&\frac{Ck^{t(2^*-1)}}{\mu^{t(2^*-1)}}\frac{\ln\mu}{|\mu x_n-\mu x_1|^{N-t(2^*-1)}}.
\end{aligned} \tag{A.9}$$

Combining (A.7), (A.8) and (A.9), we find

$$
\begin{aligned}
&\Big|\sum_{n=2}^{k}\int_{\Omega_n}\big(\sum_{j=1}^{k}U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}\Big|\\
&\le\mu^{m_i}\frac{Ck^{N-2}}{\mu^{N-2}}\sum_{n=2}^{k}\frac{\ln\mu}{\mu^2|x_n-x_1|^2}+\mu^{m_i}\frac{Ck^{t(2^*-1)}}{\mu^{t(2^*-1)}}\sum_{n=2}^{k}\frac{\ln\mu}{|\mu x_n-\mu x_1|^{N-t(2^*-1)}}\\
&\le\mu^{m_i}\frac{Ck^N\ln\mu}{\mu^N}.
\end{aligned}
\tag{A.10}
$$

So, (A.6) and (A.10) yield

$$
\begin{aligned}
&\int_{\Omega}\big(\sum_{j=1}^{k}U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}\\
=&\int_{\Omega_1}\big(\sum_{j=1}^{k}U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}+O\Big(\mu^{m_i}\frac{k^N\ln\mu}{\mu^N}\Big).
\end{aligned}
\tag{A.11}
$$

It is standard to show that

$$
\begin{aligned}
&\int_{\Omega_1}\big(\sum_{j=1}^{k}U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}\\
=&\int_{\Omega_1}\big(U_{x_1,\mu}^{2^*-1}-(PU_{x_1,\mu})^{2^*-1}\big)\partial_i PU_{x_1,\mu}\\
&+\int_{\Omega_1}(PU_{x_1,\mu})^{2^*-2}\big(\sum_{j=2}^{k}PU_{x_j,\mu}\big)\partial_i PU_{x_1,\mu}+O\Big(\mu^{m_i}\frac{k^N\ln\mu}{\mu^N}\Big)\\
=&\partial_i\Big(\bar B_1\frac{H(x_1,x_1)}{\mu^{N-2}}-\sum_{i=2}^{k}\frac{\bar B_1G(x_1,x_i)}{\mu^{N-2}}\Big)+O\Big(\mu^{m_i}\frac{k^N\ln\mu}{\mu^N}\Big),
\end{aligned}
\tag{A.12}
$$

where $\bar B_1>0$ is a constant.

We now estimate the second term in (A.5). Similar to the above discussion, we have

$$
\begin{aligned}
&\int_{\Omega}(y_N^m-l^m)\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}\\
=&\int_{\Omega_1}(y_N^m-l^m)\big(\sum_{j=1}^k U_{x_j,\mu}^{2^*-1}-W_{h,d,\mu}^{2^*-1}\big)\partial_i PU_{x_1,\mu}+O\Big(\mu^{m_i}\frac{k^N\ln\mu}{\mu^N}\Big)\\
=&\int_{\Omega_1}(y_N^m-l^m)\big(U_{x_1,\mu}^{2^*-1}-(PU_{x_1,\mu})^{2^*-1}\big)\partial_i PU_{x_1,\mu}\\
&+\int_{\Omega_1}(y_N^m-l^m)(PU_{x_1,\mu})^{2^*-2}\big(\sum_{j=2}^k PU_{x_j,\mu}\big)\partial_i PU_{x_1,\mu}+O\Big(\mu^{m_i}\frac{k^N\ln\mu}{\mu^N}\Big)\\
=&O\Big(\frac{1}{\mu}|\partial_i\frac{H(x_1,x_1)}{\mu^{N-2}}|+\frac{1}{\mu}\sum_{j=2}^k|\partial_i\frac{G(x_j,x_1)}{\mu^{N-2}}|+\frac{k^N\ln\mu}{\mu^N}\Big)\mu^{m_i}.
\end{aligned}
\tag{A.13}
$$

Finally, we estimate the last terms in (A.5).

If $i=3$, then

$$
\begin{aligned}
&\Big|\sum_{j=2}^k\int_{\Omega}y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\partial_3 PU_{x_1,\mu}\Big|\\
\le& C\frac{1}{\mu^2}\sum_{j=2}^k\int_{\mathbb{R}^N}\frac{1}{(1+|y-\mu x_j|)^{N-1}}\frac{1}{(1+|y-\mu x_1|)^{N-2}}\\
\le& C\frac{1}{\mu^2}\sum_{j=2}^k\frac{1}{|\mu x_1-\mu x_j|^{N-3}}=O\Big(\frac{1}{\mu^2}\frac{k^{N-3}\ln\mu}{\mu^{N-3}}\Big).
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{\Omega}y_N^{m-1}\frac{\partial PU_{x_1,\mu}}{\partial y_N}\partial_3 PU_{x_1,\mu}\Big|\\
\le& C\int_{\Omega}\frac{\mu^{\frac{N}{2}}}{(1+\mu|y-x_1|)^{N-1}}\frac{\mu^{\frac{N-2}{2}}}{\mu(1+\mu|y-x_1|)^{N-2}}=O\Big(\frac{1}{\mu^{2+\sigma}}\Big).
\end{aligned}
$$

So, we obtain

$$
\sum_{j=1}^k\int_{\Omega}y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\partial_3 PU_{x_1,\mu}=O\Big(\frac{1}{\mu^{2+\sigma}}\Big).
$$

Thus, we have proved

(A.14)
$$\Big\langle I'(W_{h,d,\mu}), \frac{\partial PU_{x_1,\mu}}{\partial \mu}\Big\rangle = -\frac{(N-2)\bar{B}_1 H(x_1,x_1)}{\mu^{N-1}} + \sum_{i=2}^{k}\frac{(N-2)\bar{B}_1 G(x_1,x_i)}{\mu^{N-1}} + O\Big(\frac{1}{\mu^{2+\sigma}}\Big),$$

for some $\bar{B}_1 > 0$.

Let T be the unit tangent vector of $\partial\Omega$ at $(h,0,\cdots,0,\psi(h))$. Note that

$$T = \frac{\langle 1,0,\cdots,0,\psi'(h)\rangle}{\sqrt{1+|\psi'(h)|^2}}, \quad \nu = \frac{\langle -\psi'(h),0,\cdots,0,1\rangle}{\sqrt{1+|\psi'(h)|^2}}$$

As a result,

$$\partial_1 PU_{x_1,\mu} = \frac{\partial PU_{x_1,\mu}}{\partial x_{1,1}}\frac{1}{\sqrt{1+|\psi'(h)|^2}} + \frac{\partial PU_{x_1,\mu}}{\partial x_{1,N}}\frac{\psi'(h)}{\sqrt{1+|\psi'(h)|^2}}.$$

Noting that

$$\Big|\frac{\partial H(y,x_1)}{\partial x_{1,1}}\Big| \le \frac{C}{|y-x_1^*|^{N-1}},$$

where x_1^* is the reflection point of x_1 with respect to $\partial\Omega$, we can prove

$$\begin{aligned}
&\sum_{j=1}^{k}\int_\Omega y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\frac{\partial PU_{x_1,\mu}}{\partial x_{1,1}}\frac{1}{\sqrt{1+|\psi'(h)|^2}}\\
=&\int_\Omega y_N^{m-1}\frac{\partial PU_{x_1,\mu}}{\partial y_N}\frac{\partial PU_{x_1,\mu}}{\partial x_{1,1}}\frac{1}{\sqrt{1+|\psi'(h)|^2}} + O\Big(\frac{k^{N-2}\ln\mu}{\mu^{N-2}}\Big)\\
=&\int_\Omega y_N^{m-1}\frac{\partial U_{x_1,\mu}}{\partial y_N}\frac{\partial U_{x_1,\mu}}{\partial x_{1,1}}\frac{1}{\sqrt{1+|\psi'(h)|^2}}\\
&+O\Big(\int_\Omega \frac{\mu^{\frac{N}{2}}}{(1+\mu|y-x_1|)^{N-1}}\frac{1}{\mu^{\frac{N-2}{2}}}\frac{1}{|y-x_1^*|^{N-1}}\Big) + O\Big(\frac{k^{N-2}\ln\mu}{\mu^{N-2}}\Big)\\
=&\int_\Omega y_N^{m-1}\frac{\partial U_{x_1,\mu}}{\partial y_N}\frac{\partial U_{x_1,\mu}}{\partial x_{1,1}}\frac{1}{\sqrt{1+|\psi'(h)|^2}}\\
&+O\Big(\frac{1}{\mu^{N-2}d^{N-2-\theta}}\int_\Omega\frac{1}{|y-x_1|^{N-\theta}} + \frac{k^{N-2}\ln\mu}{\mu^{N-2}}\Big)\\
=&O\Big(\frac{1}{\mu^{N-2}d^{N-2-\theta}}\Big) = O\Big(\frac{1}{\mu^{\sigma}}\Big),
\end{aligned}$$

where $\theta > 0$ is any small constant and $\sigma > 0$ is a small constant. As a result,

$$
\begin{aligned}
&\sum_{j=1}^{k}\int_\Omega y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\partial_1 PU_{x_1,\mu}\\
=&\sum_{j=1}^{k}\int_\Omega y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\frac{\partial PU_{x_1,\mu}}{\partial x_{1,N}}\frac{\psi'(h)}{\sqrt{1+|\psi'(h)|^2}}+O\Big(\frac{1}{\mu^\sigma}\Big)\\
=&\int_\Omega y_N^{m-1}\frac{\partial U_{x_1,\mu}}{\partial y_N}\frac{\partial U_{x_1,\mu}}{\partial x_{1,N}}\frac{\psi'(h)}{\sqrt{1+|\psi'(h)|^2}}+O\Big(\frac{1}{\mu^\sigma}\Big)\\
=&-B'\psi(h)+O\Big(\frac{1}{\mu^\sigma}\Big).
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
&\Big\langle I'(W_{h,d,\mu}),\frac{\partial PU_{x_1,\mu}}{\partial h}\Big\rangle\\
=&B_2\psi'(h)+\frac{B_1}{\mu^{N-2}}\frac{\partial H(x_1,x_1)}{\partial h}-\frac{B_1}{\mu^{N-2}}\sum_{i=2}^{k}\frac{\partial G(x_i,x_1)}{\partial h}+O\Big(\frac{1}{\mu^\sigma}\Big),
\end{aligned}
\tag{A.15}
$$

where $B_2>0$ is a constant.

From

$$
\partial_2 PU_{x_j,\mu}=-\frac{\partial PU_{x_j,\mu}}{\partial x_{1,1}}\frac{\psi'(h)}{\sqrt{1+|\psi'(h)|^2}}+\frac{\partial PU_{x_j,\mu}}{\partial x_{1,N}}\frac{1}{\sqrt{1+|\psi'(h)|^2}}
$$

we can prove

$$
\begin{aligned}
&\sum_{j=1}^{k}\int_\Omega y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\partial_2 PU_{x_1,\mu}\\
=&\sum_{j=1}^{k}\int_\Omega y_N^{m-1}\frac{\partial PU_{x_j,\mu}}{\partial y_N}\frac{\partial PU_{x_1,\mu}}{\partial x_{1,N}}\frac{1}{\sqrt{1+|\psi'(h)|^2}}+O\Big(\frac{1}{\mu^\sigma}\Big)\\
=&\int_\Omega y_N^{m-1}\frac{\partial U_{x_1,\mu}}{\partial y_N}\frac{\partial U_{x_1,\mu}}{\partial x_{1,N}}\frac{1}{\sqrt{1+|\psi'(h)|^2}}+O\Big(\frac{1}{\mu^\sigma}\Big)\\
=&-B''+O\Big(\frac{1}{\mu^\sigma}\Big),
\end{aligned}
$$

where $B''>0$ is a constant. As a result,

$$\tag{A.16}\begin{aligned}&\Big\langle I'(W_{h,d,\mu}), \frac{\partial PU_{x_1,\mu}}{\partial d}\Big\rangle\\ =&B_3+\frac{\bar B_1}{\mu^{N-2}}\frac{\partial H(x_1,x_1)}{\partial d}-\frac{\bar B_1}{\mu^{N-2}}\sum_{i=2}^k\frac{\partial G(x_i,x_1)}{\partial d}+O\Big(\frac{1}{\mu^\sigma}\Big),\end{aligned}$$

where $B_3>0$ is a constant.

To finish the proof of Proposition A.1, we need estimate the Green function.

Recall that

$$x_1=(h,0,\cdots,0,\psi(h))+d\nu=(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}},0,\cdots,0,\psi(h)+\frac{d}{\sqrt{1+|\psi'(h)|^2}}).$$

Then, we have

$$\begin{aligned}x_j=\big((h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})\cos\frac{2(j-1)\pi}{k},(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})\sin\frac{2(j-1)\pi}{k},\\ 0,\cdots,0,\psi(h)+\frac{d}{\sqrt{1+|\psi'(h)|^2}}\big).\end{aligned}$$

We have

$$\tag{A.17} |x_j-x_1|^2=(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})^2(2-2\cos\frac{2(j-1)\pi}{k})=(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})^2\sin^2\frac{(j-1)\pi}{k}.$$

For the Green function $G(y,x_1)$,

$$G(y,x_1)=\frac{c_0}{|y-x_1|^{N-2}}-H(y,x_1),$$

where $c_0>0$ is a constant, and $H(y,x_1)$ is the regular part.

Let x_1^* be the reflection of x_1 with respect to $\partial\Omega$. Then

$$x_1^*=(h,0,\cdots,0,\psi(h))-d\nu,$$

and

$$H(y,x_1)=\frac{c_0}{|y-\bar x_1^*|^{N-2}}\big(1+O(d)\big).$$

So, we obtain

$$H(x_1,x_1)=\frac{c_0}{2^{N-2}d^{N-2}}\big(1+O(d)\big).$$

On the other hand,

$$x_j-x_1^*=\Big((h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})(1-\cos\frac{2(j-1)\pi}{k})+\frac{2d}{\sqrt{1+|\psi'(h)|^2}},\\ (h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})\sin\frac{2(j-1)\pi}{k},0,\cdots,0,\frac{2d}{\sqrt{1+|\psi'(h)|^2}}\Big).$$

So

$$\begin{aligned}\mid x_j-x_1^*|^2=&(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})^2(2-2\cos\frac{2(j-1)\pi}{k})\\ &+2(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})(1-\cos\frac{2(j-1)\pi}{k})\frac{2d}{\sqrt{1+|\psi'(h)|^2}}\\ &+\frac{8d^2}{1+|\psi'(h)|^2}\\ =&|x_j-x_1|^2+2(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})\sin^2\frac{(j-1)\pi}{k})\frac{d}{\sqrt{1+|\psi'(h)|^2}}\\ &+\frac{8d^2}{1+|\psi'(h)|^2}.\end{aligned}\tag{A.18}$$

Since $dk\to c>0$ and

$$0<c'\le\frac{\sin\frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}}\le c'',\quad j=2,\cdots,[\frac{k}{2}],$$

we can deduce

$$\frac{a_0}{j^2}\le\frac{1}{|x_j-x_1|^2}\Big(2(h-\frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})\sin^2\frac{(j-1)\pi}{k})\frac{d}{\sqrt{1+|\psi'(h)|^2}}+\frac{8d^2}{1+|\psi'(h)|^2}\Big)\le\frac{a_1}{j^2}$$

for some constant $a_1\ge a_0>0$. From

$$G(x_j,x_1)=\frac{c_0}{|x_j-x_1|^{N-2}}\big(1-\frac{c_0|x_j-x_1|^{N-2}}{|x_j-x_1^*|^{N-2}}\big(1+O(d)\big)\big),$$

we obtain

$$\sum_{j=2}^{k} G(x_j, x_1) = \frac{k^{N-2}}{(h - \frac{d\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})^{N-2}}\Big(B_4' + O(d)\Big) = B_4 k^{N-2} + O(k^{N-2}d),$$

where B_4' and B_4 are some positive constants.

So, (A.14) can be rewritten to (A.2).

From (A.17), we find

$$\frac{\partial |x_j - x_1|}{\partial h} = O\big(\frac{dk}{j}\big), \quad \frac{\partial |x_j - x_1|}{\partial d} = O\big(\frac{|\psi'(h)|k}{j}\big),$$

while from (A.18), we find

$$\frac{\partial |x_j - x_1^*|}{\partial h} = O\big(d + |\psi'(h)|\frac{k}{j}\big), \quad \frac{\partial |x_j - x_1^*|}{\partial d} = O\big(d + |\psi'(h)|\frac{k}{j}\big).$$

So, we can prove

$$\sum_{j=2}^{k} \frac{1}{\mu^{N-2}} \frac{\partial G(x_j, x_1)}{\partial h}, \quad \sum_{j=2}^{k} \frac{1}{\mu^{N-2}} \frac{\partial G(x_j, x_1)}{\partial d} = O\Big(\frac{1}{\mu^{N-2} d^{N-2}}\Big).$$

Thus, (A.15) and (A.16) are equivalent to (A.3) and (A.4) respectively.

□

Appendix B. Basic Estimates

In this section, we list some lemmas, whose proof can be found in [20].

For each fixed i and j, $i \neq j$, consider the following function

$$g_{ij}(y) = \frac{1}{(1 + |y - x_j|)^{\alpha}} \frac{1}{(1 + |y - x_i|)^{\beta}}, \tag{B.19}$$

where $\alpha \geq 1$ and $\beta \geq 1$ are two constants.

Lemma B.1. *For any constant $0 < \sigma \leq \min(\alpha, \beta)$, there is a constant $C > 0$, such that*

$$g_{ij}(y) \leq \frac{C}{|x_i - x_j|^{\sigma}} \Big(\frac{1}{(1 + |y - x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1 + |y - x_j|)^{\alpha+\beta-\sigma}} \Big).$$

Lemma B.2. *For any constant $0 < \sigma < N - 2$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z|)^{2+\sigma}} \, dz \leq \frac{C}{(1 + |y|)^{\sigma}}.$$

Let recall that

$$W_{h,d,\mu}(y) = \sum_{j=1}^{k} PU_{x_j,\mu}.$$

Lemma B.3. *Suppose that $N \geq 4$. Then there is a small $\theta > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{h,d,\mu}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz \\ \leq C \sum_{j=1}^{k} \frac{1}{(1+\mu|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.$$

Proof. The proof can be found in [20]. We just need to use

$$W_{h,r,\mu}(y) \leq C \sum_{i=1}^{k} \frac{\mu^{\frac{N-2}{2}}}{(1+\mu|y-x_i|)^{N-2}},$$

and

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{h,d,\mu}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz \\ \leq \int_{\mathbb{R}^N} \frac{1}{|\mu y-z|^{N-2}} \Big(\sum_{i=1}^{k} \frac{1}{(1+|z-\mu x_i|)^{N-2}}\Big)^{\frac{4}{N-2}} \sum_{j=1}^{k} \frac{1}{(1+|z-\mu x_j|)^{\frac{N-2}{2}+\tau}}\,dz.$$

□

Appendix C. Proof of Lemma 2.1

In this appendix, we prove Lemma 2.1.

First, for any $x \in \Omega$, we need to find a solution u, satisfying

$$-div(y_N^m Du) = \delta_x, \quad y \in \Omega. \tag{C.20}$$

For this purpose, we take a domain Ω_1, satisfying $\Omega \subset\subset \Omega_1 \subset\subset R_+^N$.

Let $u_1 = u - \frac{c_0}{x_N^m |y-x|^{N-2}}$, where $c_0 > 0$ is chosen in such a way that

$$-\Delta \frac{c_0}{|y-x|^{N-2}} = \delta_x.$$

Then

$$-\Delta u_1 - \frac{m}{y_N}\frac{\partial u_1}{\partial y_N} = \frac{m}{y_N}\frac{(N-2)(y_N - x_N)}{|y-x|^{N-1}}.$$

So, we consider the following problem:

$$\text{(C.21)}\qquad \begin{cases} -\Delta u - \frac{m}{y_N}\frac{\partial u}{\partial y_N} = \frac{a}{y_N}\frac{y_N - x_N}{|y-x|^{\alpha}}, & \text{in } \Omega_1; \\ u(y) = \frac{1}{|y-x|^{\alpha-2}}, & \text{on } \partial\Omega_1, \end{cases}$$

where $a > 0$ and $\alpha \le N$ are some constants.

Note that for any $x \in \Omega$, $\frac{1}{|y-x|^{\alpha-2}} \le C$ for any $y \in \partial\Omega_1$.

By the L^p estimate, it is easy to see that if $\alpha < 3$, then $|u(y)| \le C$.

Suppose now that $\alpha \ge 3$. Then the solution u of (C.21) satisfies $|u| \le \tilde{u}$, where $\tilde{u}$ is the solution of

$$\text{(C.22)}\qquad \begin{cases} -\Delta \tilde{u} - \frac{m}{y_N}\frac{\partial u}{\partial y_N} = \frac{a}{y_N}\frac{1}{|y-x|^{\alpha-1}}, & \text{in } \Omega_1; \\ \tilde{u}(y) = \frac{1}{|y-x|^{\alpha-2}}, & \text{on } \partial\Omega_1, \end{cases}$$

Let $u_1 = \tilde{u} - \frac{a}{(\alpha-2)(N-\alpha)}\frac{1}{|y-x|^{\alpha-3}}$. Then

$$\text{(C.23)}\qquad \begin{cases} -\Delta u_1 - \frac{m}{y_N}\frac{\partial u_1}{\partial y_N} = \frac{m(\alpha-3)}{y_N}\frac{y_N - x_N}{|y-x|^{\alpha-1}}, & \text{in } \Omega_1; \\ u_1(y) = \frac{1}{|y-x|^{\alpha-2}} - \frac{a}{(\alpha-2)(N-\alpha)}\frac{1}{|y-x|^{\alpha-3}}, & \text{on } \partial\Omega_1, \end{cases}$$

By the L^p estimate, it is easy to see that if $\alpha < 4$, then $|u_1(y)| \le C$. So, we have proved that if $\alpha < 4$, the solution of (C.21) satisfies $|u(y)| \le \frac{C}{|y-x|^{\alpha-3}}$ Now we can continue this procedure to prove that for any $\alpha \le N$, the solution of (C.21) satisfies $|u(y)| \le \frac{C}{|y-x|^{\alpha-3}}$.

From the above discussion, we know that (C.20) has a solution $S(y,x)$, satisfying

$$S(y,x) = \frac{a}{|y-x|^{N-2}} + O\Big(\frac{1}{|y-x|^{N-3}}\Big), \quad \text{as } y \to x.$$

By adding a constant to S, we can always assume that $S(y,x) > 0$. So, the Green function $G(y,x)$ for (C.20) satisfies

$$0 < G(y,x) < S(y,x) \le \frac{C}{|y-x|^{N-2}},$$

and the result follows.

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