INFINITELY MANY SOLUTIONS FOR THE PRESCRIBED SCALAR CURVATURE PROBLEM ON $\mathbb{S}^N$

JUNCHENG WEI AND SHUSEN YAN

ABSTRACT. We consider the following prescribed scalar curvature problem on $\mathbb{S}^N$

$$(*) \quad \begin{cases} -\Delta_{\mathbb{S}^N} u + \frac{N(N-2)}{2} u = \tilde{K} u^{\frac{N+2}{N-2}} \text{ on } \mathbb{S}^N, \\ u > 0 \end{cases}$$

where $\tilde{K}$ is positive and rotationally symmetric. We show that if $\tilde{K}$ has a local maximum point between the poles then equation (*) has **infinitely many non-radial positive** solutions, whose energy can be made arbitrarily large.

1. INTRODUCTION

Consider the standard N-sphere $(\mathbb{S}^N, g_0)$, $N \geq 3$. Let $\tilde{K}$ be a fixed smooth function. The prescribed curvature problem asks if one can find a conformally invariant metric g such that the scalar curvature becomes $\tilde{K}$. The problem consists in solving the following equation on $\mathbb{S}^N$:

$$(1.1) \qquad \begin{cases} -\Delta_{\mathbb{S}^N} u + \frac{N(N-2)}{2} u + \tilde{K} u^{\frac{N+2}{N-2}} = 0 \text{ on } \mathbb{S}^N, \\ u > 0. \end{cases}$$

Problem (1.1) does not always admit a solution. A first necessary condition for the existence is that $\max_{\mathbb{S}^N} \tilde{K} > 0$, but there are also some obstructions, which are said of *topological type*. For example, a necessary condition is the following Kazdan-Warner condition:

$$(1.2) \qquad \int_{\mathbb{S}^N} \nabla \tilde{K} \cdot \nabla x u^{\frac{2N}{N-2}} = 0.$$

The problem of determining which $\tilde{K}$ admits a solution to (1.1) has been studied extensively. See [1], [4]–[12], [14]–[22], [24, 25, 31] and the

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.

references therein. Some existence results have been obtained under some assumptions involving the Laplacian at the critical point of $\tilde{K}$, see Chang-Yang [9], Bahri-Coron [4] and Schoen-Zhang [30] for the case $N = 3$, and Y. Li [20] for the case $N \geq 4$. For example, in Bahri and Coron [4], it is assumed that $\tilde{K}$ is a positive Morse function with $\Delta\tilde{K}(x) \neq 0$ if $\nabla\tilde{K}(x) = 0$, then if $m(x)$ denotes the Morse index of the critical point x of K, (1.1) has a solution provided that

$$\sum_{\nabla\tilde{K}=0, \Delta\tilde{K}(x)<0} (-1)^{m(x)} \neq -1.$$

The result has been extended to any $\mathbb{S}^N, N \geq 3$ by Y.Li in [19]-[20]. Roughly, it is assumed that there exists $\beta, N-2 < \beta < N$ such that

$$\tilde{K}(\xi) = \tilde{K}(\xi_0) + \sum_{j=1}^{N} a_j |\xi_j - \xi_{0,j}|^\beta + h.o.t. \tag{1.3}$$

where $a_j \neq 0, \sum_{j=1}^N a_j \neq 0$. Let $\Sigma = \{\xi : \nabla\tilde{K}(\xi) = 0, \sum_{j=1}^N a_j < 0\}$ and $i(\xi)$ be the number of a_j such that $\tilde{K}(\xi) = 0, a_j < 0$. Then (1.1) has a solution provided

$$\sum_{\xi\in\Sigma} (-1)^{i(\xi)} \neq (-1)^N. \tag{1.4}$$

By using the stereo-graphic projection, the prescribed scalar curvature problem (1.1) can be reduced to (1.5)

$$\begin{cases} -\Delta u = K(y) u^{\frac{N+2}{N-2}}, u > 0, \quad y \in \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N) \end{cases} \tag{1.5}$$

where $D^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\int_{\mathbb{R}^N} |\nabla u|^2$.

Much less is known about the multiplicity of the solutions of (1.5). Amrosetti, Azorero and Peral [1], and Cao, Noussair and Yan [8] proved the existence of two or many solutions if K is a perturbation of the constant, i.e.

$$\tilde{K} = K_0 + \varepsilon h(x), 0 < \varepsilon << 1. \tag{1.6}$$

On the other hand, Y. Li proved in [17] that (1.5) has infinitely many solutions if $K(x)$ is periodic, while similar result was obtained in [31] if

$K(x)$ has a sequence of strict local maximum points tending to infinity. Note that this condition for $K(x)$ at the infinity implies that the corresponding function $\tilde{K}$ defined on $\mathbb{S}^N$ has a singularity at the south pole.

In this paper, we consider the simplest case, i.e., $\tilde{K}$ is rotationally symmetric, $K = K(r), r = |y|$. It follows from the Pohozaev identity (1.2) that (1.5) has no solution if $K'(r)$ has fixed sign. Thus we assume that K *is positive and not monotone.* On the other hand, Bianchi [6] showed that any solution of (1.5) is radially symmetric if there is a $r_0 > 0$, such that $K(r)$ is non-increasing in $(0, r_0]$, and non-decreasing in $[r_0, +\infty)$. Moreover, in [7], it was proved that (1.5) has no solutions for some function $K(r)$, which is non-increasing in $(0, 1]$, and non-decreasing in $[1, +\infty)$. Therefore, we see that to obtain a solution for (1.5), it is natural to assume that $K(r)$ *has a local maximum at* $r_0 > 0$. The purpose of this paper is to answer the following two questions:

Q1: Does the existence of a local maximum of K guarantee the existence of a solution to (1.5)?

Q2: Are there non-radially symmetric solutions to (1.5)?

(Question $Q2$ has been asked by Bianchi [6].)

The aim of this paper is to show that if $K(r)$ has a local maximum at $r_0 > 0$, then (1.5) has **infinitely many non-radial solutions**. This answers Q1 and Q2 affirmatively. As far as we know, we believe our result is the first on the existence of infinitely many solution for (1.5).

We assume that $K(r)$ satisfies the following condition:

(K): There is a constant $r_0 > 0$, such that

$$K(r) = K(r_0) - c_0|r - r_0|^m + O(|r - r_0|^{m+\theta}), \quad r \in (r_0 - \delta, r_0 + \delta),$$

where $c_0 > 0$, $\theta > 0$ are some constants, and the constant m satisfies $m \in [2, N - 2)$.

Without loss of generality, we assume that

$$K(r_0) = 1.$$

Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that $N \geq 5$. If $K(r)$ satisfies (K), then problem (1.5) has infinitely many non-radial solutions.*

Remark 1.2. *Theorem 1.1 shows that the condition in* [6] *is optimal. We shall prove Theorem 1.1 by constructing solutions with large number of bubbles lying near the sphere $|y| = r_0$. So the energy of these solutions can be made arbitrary large and the distance between different bubbles can be made arbitrary small. When $N = 3, 4$, we know that the energy of the solutions to (1.5) is uniformly bounded and the distance between bubbles is uniformly bounded from below. See* [30] *(for $N = 3$) and Theorem 0.10 of* [20] *(for $N = 4$). On the other hand, if $K(y) = K(y_0) + O(|y - y_0|^m)$ where $m \in [N-2, N)$ for $N \geq 5$, the energy of solutions is also be bounded. See* [20]. *So our assumptions on N and m are almost optimal in the construction of the solutions in this paper.*

Remark 1.3. *The radial symmetry can be replaced by the following weaker symmetry assumption: after suitably rotating the coordinate system,*

- (K1) $K(y) = K(y', y'') = K(|y'|, |y_{N_0+1}|, ..., |y_N|)$, *where* $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$,
- (K2) $K(y) = K(y_0) - c_0|y - y_0|^m + O(|y - y_0|^{m+\theta}), |y'| \in (|y_0'| - \delta, |y_0'| + \delta), |y''| \leq \delta$, *where* $y_0 = (y_0', 0)$.

Remark 1.4. *Theorem 1.1 exhibits a* **new phenomena** *for the prescribed scalar curvature problem. It suggests that if the critical points of K are not isolated, new solutions to (1.5) may bifurcate. We formulate the following conjecture in the general case.*

Conjecture: *Assume that the set $\{K(x) = \max_{x \in \mathbb{R}^N} K(x)\}$ is an l-dimensional smooth manifold without boundary, where $1 \leq l \leq N - 1$. The problem (1.5) admits infinitely many positive solutions.*

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

Let us fix a positive integer

$$k \geq k_0,$$

where k_0 is large, to be determined later.

Set

$$\mu = k^{\frac{N-2}{N-2-m}},$$

to be the scaling parameter.

Let $2^* = \frac{2N}{N-2}$. Using the transformation $u(y) \mapsto \mu^{-\frac{N-2}{2}} u\big(\frac{y}{\mu}\big)$, we find that (1.5) becomes

$$\begin{cases} -\Delta u = K\big(\frac{|y|}{\mu}\big) u^{2^*-1}, u > 0, \quad y \in \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.7}$$

It is well-known that the functions

$$U_{x,\Lambda}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \left(\frac{\Lambda}{1+\Lambda^2|y-x|^2}\right)^{\frac{N-2}{2}}, \ \mu > 0, \ x \in \mathbb{R}^N$$

are the only solutions to the problem

$$-\Delta u = u^{\frac{N+2}{N-2}}, \ u > 0 \text{ in } \mathbb{R}^N.$$

Let $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \Big\{u : u \in D^{1,2}(\mathbb{R}^N), u \text{ is even in } y_h, h = 2, \cdots, N,$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \frac{2\pi j}{k}), r\sin(\theta + \frac{2\pi j}{k}), y'')\Big\}.$$

Let

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and let

$$W_r(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \sum_{j=1}^{k} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-x_j|^2)^{\frac{N-2}{2}}}.$$

In this paper, we always assume that

$$r \in [r_0\mu - \frac{1}{\mu^{\bar\theta}}, r_0\mu + \frac{1}{\mu^{\bar\theta}}], \qquad \text{for some small } \bar\theta > 0,$$

and

$$L_0 \le \Lambda \le L_1, \qquad \text{for some constants } L_1 > L_0 > 0.$$

Theorem 1.1 is a direct consequence of the following result:

Theorem 1.5. *Suppose that $N \ge 5$. If $K(r)$ satisfies (K), then there is an integer $k_0 > 0$, such that for any integer $k \ge k_0$, (1.7) has a solution u_k of the form*

$$u_k = W_{r_k}(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\|\omega\|_{L^\infty} \to 0$, $r_k \in [r_0\mu - \frac{1}{\mu^{\bar\theta}}, r_0\mu + \frac{1}{\mu^{\bar\theta}}]$
.

We will use the techniques in the singularly perturbed elliptic problems to prove Theorem 1.5. We know that there is always a small parameter in a singularly perturbed elliptic problem. Although there is no parameter in (1.5), we use k, **the number of the bubbles** of the solutions, as the parameter in the construction of bubbles solutions for (1.5). This is the **new idea** of this paper. This is partly motivated by recent paper of Lin-Ni-Wei [23] where they constructed multiple spikes to a singularly perturbed problem. There they allowed the number of spikes to depend on the small parameter.

The main difficulty in constructing solution with k-bubbles is that we need to obtain a better control of the error terms. Since the number of the bubbles is large, it is very hard to carry out the reduction procedure by using the standard norm as in [3, 26]. Noting that the maximum norm will not be affected by the number of the bubbles, we will carry out the reduction procedure in a space with weighted maximum norm. Similar weighted maximum norm has been used in [13],[27]–[29]. But the estimates in the reduction procedure in this paper are much more complicated than those in [13],[27]–[29], because the number of the bubbles is large.

Acknowledgment. The first author is supported by an Earmarked Grant from RGC of Hong Kong. The second author is partially supported by ARC.

2. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction.

Let

$$\|u\|_* = \sup_{y\in\mathbb{R}^N}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1}|u(y)|, \tag{2.1}$$

and

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^N}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1}|f(y)|, \tag{2.2}$$

where $\tau = 1+\bar\eta$ and $\bar\eta>0$ is small.

Let

$$Z_{i,1} = \frac{\partial U_{x_i,\Lambda}}{\partial r}, \quad Z_{i,2} = \frac{\partial U_{x_i,\Lambda}}{\partial \Lambda}.$$

Consider

(2.3)

$$\begin{cases} -\Delta\phi_k - (2^*-1)K\big(\frac{|y|}{\mu}\big)W_r^{2^*-2}\phi_k = & h_k + c_1\sum_{i=1}^{k}U_{x_i,\Lambda}^{2^*-2}Z_{i,1} + c_2\sum_{i=1}^{k}U_{x_i,\Lambda}^{2^*-2}Z_{i,2}, \quad \text{in } \mathbb{R}^N,\\ \phi_k\in H_s, & \\ < U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\phi_k >= 0 & i=1,\cdots,k,\ l=1,2 \end{cases}$$

for some numbers c_i, where $< u,v >= \int_{\mathbb{R}^N} uv$.

Lemma 2.1. *Assume that ϕ_k solves (2.3) for $h=h_k$. If $\|h_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. We argue by contradiction. Suppose that there are $k\to+\infty$, $h=h_k$, $\Lambda_k\in[L_1,L_2]$, $r_k\in[r_0\mu-\frac{1}{\mu^\theta}, r_0\mu+\frac{1}{\mu^\theta}]$, and ϕ_k solving (2.3) for $h=h_k$, $\Lambda=\Lambda_k$, $r=r_k$, with $\|h_k\|_{**}\to0$, and $\|\phi_k\|_*\ge c'>0$. We may assume that $\|\phi_k\|_*=1$. For simplicity, we drop the subscript k.

We rewrite (2.3) as

(2.4)
$$\phi(y)=(2^*-1)\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}K\big(\frac{|z|}{\mu}\big)W_r^{2^*-2}\phi(z)\,dz$$
$$+\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\big(h(z)+c_1\sum_{i=1}^k Z_{i,1}(z)U_{x_i,\Lambda}^{2^*-2}(z)+c_2\sum_{i=1}^k Z_{i,2}(z)U_{x_i,\Lambda}^{2^*-2}(z)\big)\,dz.$$

Using Lemma B.3, we have

(2.5)
$$\Big|(2^*-1)\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}K\big(\frac{|z|}{\mu}\big)W_r^{2^*-2}\phi(z)\,dz\Big|$$
$$\le C\|\phi\|_*\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}W_r^{2^*-2}\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz$$
$$\le C\|\phi\|_*\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}+o(1)\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big).$$

It follows from Lemma B.2 that

$$\Big|\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}h(z)\,dz\Big|$$
$$\le C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz \tag{2.6}$$
$$\le C\|h\|_{**}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}},$$

and

(2.7)
$$\Big|\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\sum_{i=1}^k Z_{i,l}(z)U_{x_i,\Lambda}^{2^*-2}(z)\,dz\Big|$$
$$\le C\sum_{i=1}^k\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\frac{1}{(1+|z-x_i|)^{N+2}}\,dz\le C\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.$$

Next, we estimate c_l, $l=1,2$. Multiplying (2.3) by $Z_{1,l}$ and integrating, we see that c_t satisfies

(2.8)
$$\sum_{t=1}^{2}\sum_{i=1}^{k}\big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,t},Z_{1,l}\big\rangle c_t=\big\langle -\Delta\phi-(2^*-1)K\big(\frac{|y|}{\mu}\big)W_r^{2^*-2}\phi,Z_{1,l}\big\rangle-\big\langle h,Z_{1,l}\big\rangle.$$

It follows from Lemma B.1 that

$$\begin{aligned}&\big|\big\langle h,Z_{1,l}\big\rangle\big|\le C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz\\ \le& C\|h\|_{**}.\end{aligned}$$

On the other hand,

(2.9)
$$\begin{aligned}&\big\langle -\Delta\phi-(2^*-1)K\big(\frac{|z|}{\mu}\big)W_r^{2^*-2}\phi,Z_{1,l}\big\rangle\\ =&\big\langle -\Delta Z_{1,l}-(2^*-1)K\big(\frac{|z|}{\mu}\big)W_r^{2^*-2}Z_{1,l},\phi\big\rangle\\ =&(2^*-1)\big\langle\big(1-K\big(\frac{|z|}{\mu}\big)W_r^{2^*-2}Z_{1,l},\phi\big\rangle\\ =&\|\phi\|_*O\Big(\int_{\mathbb{R}^N}\big|K\big(\frac{|z|}{\mu}\big)-1\big|W_r^{2^*-2}(z)\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\Big).\end{aligned}$$

Similar to the proof of Lemma B.3, we obtain

$$\begin{aligned}&\int_{||z|-\mu r_0|\le\sqrt{\mu}}\big|K\big(\frac{|z|}{\mu}\big)-1\big|W_r^{2^*-2}(z)\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\ \le&\frac{C}{\sqrt{\mu}}\int_{\mathbb{R}^N}W_r^{2^*-2}(z)\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\ \le&\frac{C}{\sqrt{\mu}},\end{aligned}$$

and

$$
\begin{aligned}
&\int_{||z|-\mu r_0|\ge\sqrt{\mu}}\big|K\big(\frac{|y|}{\mu}\big)-1\big|W_r^{2^*-2}(z)\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\
\le&\frac{C}{\mu^\sigma}\int_{\mathbb{R}^N}W_r^{2^*-2}(z)\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau-2\sigma}}\,dz\\
\le&\frac{C}{\mu^\sigma},
\end{aligned}
$$

since if $||z|-\mu r_0|\ge\sqrt{\mu}$, then

$$
||z|-|x_1||\ge||z|-\mu r_0|-||x_1|-\mu r_0|\ge\sqrt{\mu}-\frac{1}{\mu^{\bar\theta}}\ge\frac{1}{2}\sqrt{\mu}.
$$

Thus,

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\big|K\big(\frac{|z|}{\mu}\big)-1\big|W_r^{2^*-2}(z)\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\
\le&\frac{C}{\mu^\sigma},
\end{aligned}
$$

which, together with (2.9), gives

$$
\Big\langle-\Delta\phi-(2^*-1)K\big(\frac{|z|}{\mu}\big)W_{r_k}^{2^*-2}\phi,Z_{1,l}\Big\rangle=\|\phi\|_*O\Big(\frac{1}{\mu^\sigma}\Big). \tag{2.10}
$$

But there is a constant $\bar c>0$,

$$
\sum_{i=1}^k\big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,t},Z_{1,l}\big\rangle=(\bar c+o(1))\delta_{tl}.
$$

Thus we obtain from (2.8) that

$$
c_l=O\Big(\frac{1}{\mu^\sigma}\|\phi\|_*+\|h\|_{**}\Big).
$$

So,

$$
\|\phi\|_*\le\Big(o(1)+\|h_k\|_{**}+\frac{\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big). \tag{2.11}
$$

Since $\|\phi\|_*=1$, we obtain from (2.11) that there is $R>0$, such that

$$\|\phi(y)\|_{B_R(x_i)} \geq a > 0, \tag{2.12}$$

for some i. But $\bar\phi(y)=\phi(y-x_i)$ converges uniformly in any compact set to a solution u of

$$-\Delta u-(2^*-1)U_{0,\Lambda}^{2^*-2}u=0, \quad \text{in } \mathbb{R}^N, \tag{2.13}$$

for some $\Lambda\in[L_1,L_2]$, and u is perpendicular to the kernel of (2.13). So, $u=0$. This is a contradiction to (2.12).

□

From Lemma 2.1, using the same argument as in the proof of Proposition 4.1 in [13], we can prove the following result :

Proposition 2.2. *There exists $k_0>0$ and a constant $C>0$, independent of k, such that for all $k\geq k_0$ and all $h\in L^\infty(\mathbb{R}^N)$, problem* (2.3) *has a unique solution $\phi\equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_* \leq C\|h\|_{**}, \qquad |c_l|\leq C\|h\|_{**}. \tag{2.14}$$

Now, we consider

(2.15)
$$\begin{cases} -\Delta\big(W_r+\phi\big)=K\big(\frac{y}{\mu}\big)\big(W_r+\phi\big)^{2^*-1}+ & \sum_{t=1}^2 c_t\sum_{i=1}^k U_{x_i,\Lambda}^{2^*-2}Z_{i,t}, \text{ in } \mathbb{R}^N,\\ \phi_k\in H_s, & \\ <U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\phi_k>=0, & i=1,\cdots,k,\ l=1,2. \end{cases}$$

We have

Proposition 2.3. *There is an integer $k_0>0$, such that for each $k\geq k_0$, $L_0\leq\Lambda\leq L_1$, $|r-\mu r_0|\leq\frac{1}{\mu^{\bar\theta}}$, where $\bar\theta>0$ is a fixed small constant,* (2.15) *has a unique solution $\phi=\phi(r,\Lambda)$, satisfying*

$$\|\phi\|_*\leq C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}, \qquad |c_t|\leq C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}.$$

Rewrite (2.15) as

(2.16)
$$\begin{cases}-\Delta\phi-(2^*-1)K\big(\frac{|y|}{\mu}\big)W_r^{2^*-2}\phi= & N(\phi)+l_k+\sum_{t=1}^2 c_i\sum_{i=1}^k U_{x_i,\Lambda}^{2^*-2}Z_{i,t},\ \text{in }\mathbb{R}^N,\\ \phi\in H_s, & \\ < U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\phi >=0, & i=1,\cdots,k,\ l=1,2,\end{cases}$$

where

$$N(\phi)=K\big(\frac{|y|}{\mu}\big)\Big(\big(W_r+\phi\big)^{2^*-1}-W_r^{2^*-1}-(2^*-1)W_r^{2^*-2}\phi\Big),$$

and

$$l_k=K\big(\frac{|y|}{\mu}\big)W_r^{2^*-1}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-1}.$$

In order to use the contraction mapping theorem to prove that (2.16) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

Lemma 2.4. *If $N\ge 6$, then*

$$\|N(\phi)\|_{**}\le Ck^{\frac{4}{N-2}}\|\phi\|_*^{2^*-1}.$$

If $N=5$,

$$\|N(\phi)\|_{**}\le C\|\phi\|_*^2.$$

Proof. We have

$$|N(\phi)|\le\begin{cases}C|\phi|^{2^*-1}, & N\ge 6;\\ CW_r^{\frac{1}{3}}\phi^2, & N=5.\end{cases}$$

Firstly, we consider $N\ge 6$. For any $p>1$, the function t^p is convex in $t>0$. Thus

$$\Big(\sum_{j=1}^k\frac{|a_j|}{k}\Big)^p\le\sum_{j=1}^k\frac{|a_j|^p}{k}. \tag{2.17}$$

Using (2.17), we obtain

$$
\begin{aligned}
|N(\phi)| \leq & C\|\phi\|_*^{2^*-1}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} \\
\leq & C\|\phi\|_*^{2^*-1}k^{\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\frac{N+2}{N-2}\tau}} \qquad (2.18)\\
\leq & C\|\phi\|_*^{2^*-1}k^{\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

Thus, the result follows.

It remains to prove the result for $N=5$. We have

$$
|N(\phi)| \leq C\|\phi\|_*^2\sum_{i=1}^{k}\frac{1}{1+|y-x_i|}\Big(\sum_{j=1}^{k}\frac{1}{1+|y-x_j|)^{\frac{3}{2}+\tau}}\Big)^2.
$$

Define

$$
\Omega_j=\Big\{y:\ y=(y',y'')\in\mathbb{R}^2\times\mathbb{R}^{N-2},\Big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\Big\rangle\geq\cos\frac{\pi}{k}\Big\}.
$$

Without loss of generality, we assume $y\in\Omega_1$. Then

$$
|y-x_j|\geq|y-x_1|,\quad j=2,\cdots,k.
$$

So,

$$
\begin{aligned}
&\sum_{i=2}^{k}\frac{1}{1+|y-x_i|}\leq\frac{1}{(1+|y-x_1|)^{\frac{1}{2}}}\sum_{i=2}^{k}\frac{1}{(1+|y-x_i|)^{\frac{1}{2}}}\\
\leq&\frac{C}{(1+|y-x_1|)^{\frac{2}{3}}}\sum_{i=2}^{k}\frac{1}{|x_1-x_i|^{\frac{1}{3}}}\\
\leq&\frac{C}{(1+|y-x_1|)^{\frac{2}{3}}}\frac{k}{\mu^{\frac{1}{3}}}\leq\frac{C}{(1+|y-x_1|)^{\frac{2}{3}}},
\end{aligned}
$$

since

$$
\frac{k}{\mu^{\frac{1}{3}}}\leq\frac{Ck}{k^{\frac{3}{3-m}\frac{1}{3}}}\leq C.
$$

Similarly

$$\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{3}{2}+\tau}}\le\frac{C}{(1+|y-x_1|)^{\frac{3}{2}+\tau-\frac{1}{3}}}.$$

So, we have proved

$$\begin{aligned}&\sum_{i=1}^{k}\frac{1}{1+|y-x_i|}\Big(\sum_{j=1}^{k}\frac{1}{1+|y-x_j|)^{\frac{3}{2}+\tau}}\Big)^2\\ \le&\frac{C}{(1+|y-x_1|)^{3+2\tau}}\le\frac{C}{(1+|y-x_1|)^{\frac{7}{2}+\tau}},\quad y\in\Omega_1,\end{aligned}$$

since $\tau>1$. Thus,

$$\|N(\phi)\|_{**}\le C\|\phi\|_*^2.$$

□

Next, we estimate l_k.

Lemma 2.5. *Assume that* $\big||x_1|-\mu r_0\big|\le\frac{1}{\mu^{\bar\theta}}$*, where* $\bar\theta>0$ *is a fixed small constant.*

If $N\ge 5$*, then*

$$\|l_k\|_{**}\le C\big(\frac{k}{\mu}\big)^{\frac{N+2}{2}-\tau}.$$

Proof. Define

$$\Omega_j=\big\{y:\ y=(y',y'')\in\mathbb{R}^2\times\mathbb{R}^{N-2},\big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\big\rangle\ge\cos\frac{\pi}{k}\big\}.$$

We have

$$\begin{aligned}l_k=&K\big(\frac{|y|}{\mu}\big)\Big(W_r^{2^*-1}-\sum_{j=1}^{k}U_{x_j,\Lambda}^{2^*-1}\Big)\\&+\sum_{j=1}^{k}U_{x_j,\Lambda}^{2^*-1}\Big(K\big(\frac{|y|}{\mu}\big)-1\Big)\\=:&J_1+J_2.\end{aligned}$$

From the symmetry, we can assume that $y\in\Omega_1$. Then,

$$|y-x_j|\ge|y-x_1|,\quad\forall\, y\in\Omega_1.$$

Thus,

(2.19)
$$|J_1|\le C\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N-2}}+C\Big(\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}.$$

Using Lemma B.1, we obtain

(2.20)
$$\begin{aligned}&\frac{1}{(1+|y-x_1|)^4}\frac{1}{(1+|y-x_j|)^{N-2}}\\ \le&C\Big(\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}+\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)\frac{1}{|x_j-x_1|^{\frac{N+2}{2}-\tau}}\\ \le&C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\frac{1}{|x_j-x_1|^{\frac{N+2}{2}-\tau}},\quad j>1.\end{aligned}$$

Since $\tau<2$, we see $\frac{N+2}{2}-\tau>\frac{N-2}{2}>1$. Thus

$$\begin{aligned}&\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{N-2}}\\ \le&C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}.\end{aligned}\tag{2.21}$$

On the other hand, for $y\in\Omega_1$, using Lemma B.1 again,

$$\begin{aligned}&\frac{1}{(1+|y-x_j|)^{N-2}}\le\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}}}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}}}\\ \le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}}\Big(\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}+\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}\Big)\\ \le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}.\end{aligned}$$

If $N\ge5$, $\tau=1+\bar\eta$ and $\bar\eta>0$ is small, then $\frac{N-2}{2}-\frac{N-2}{N+2}\tau>1$. Thus

$$\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\le C\Big(\frac{k}{\mu}\Big)^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}},$$

which, gives

$$\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}\le C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.$$

Thus, we have proved

$$\|J_1\|_{**}\le C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}.$$

Now, we estimate J_2. For $y\in\Omega_1$, and $j>1$, using Lemma B.1, we have

$$\begin{aligned}U_{x_j,\Lambda}^{2^*-1}(y)\le & C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}}}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}}}\\ \le & C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\frac{1}{|x_1-x_j|^{\frac{N+2}{2}-\tau}},\end{aligned}$$

which implies

$$\begin{aligned}&\Big|\sum_{j=2}^{k}\Big(K\big(\frac{|y|}{\mu}\big)-1\Big)U_{x_j,\Lambda}^{2^*-1}\Big|\\ \le & C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{\frac{N+2}{2}-\tau}}\\ \le & C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}.\end{aligned}\tag{2.22}$$

For $y\in\Omega_1$ and $||y|-\mu r_0|\ge\delta\mu$, where $\delta>0$ is a fixed constant, then

$$||y|-|x_1||\ge||y|-\mu r_0|-||x_1|-\mu r_0|\ge\frac{1}{2}\delta\mu.$$

As a result,

$$\begin{aligned}&\Big|U_{x_1,\Lambda}^{2^*-1}\Big(K\big(\frac{|y|}{\mu}\big)-1\Big)\Big|\\ \le& C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\frac{1}{\mu^{\frac{N+2}{2}-\tau}}.\end{aligned}\tag{2.23}$$

If $y\in\Omega_1$ and $||y|-\mu r_0|\le\delta\mu$, then

$$\begin{aligned}&\Big|K\big(\frac{|y|}{\mu}\big)-1\Big|\le C\big|\frac{|y|}{\mu}-r_0\big|^m\\ \le&\frac{C}{\mu^m}\Big((||y|-|x_1||)^m+||x_1|-\mu r_0|)^m\Big)\\ \le&\frac{C}{\mu^m}||y|-|x_1||^m+\frac{C}{\mu^{m+\bar\theta}},\end{aligned}$$

and

$$||y|-|x_1||\le||y|-\mu r_0|+|\mu r_0-|x_1||\le2\delta\mu.$$

But

$$\begin{aligned}&\frac{||y|-|x_1||^m}{\mu^m}\frac{1}{(1+|y-x_1|)^{N+2}}\\ =&\frac{1}{\mu^{\frac{N+2}{2}-\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\frac{||y|-|x_1||^m}{\mu^{m-\frac{N+2}{2}+\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}-\tau}}\\ \le&\frac{C}{\mu^{\frac{N+2}{2}-\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\frac{||y|-|x_1||^{\frac{N+2}{2}-\tau}}{(1+|y-x_1|)^{\frac{N+2}{2}-\tau}}\\ \le&\frac{C}{\mu^{\frac{N+2}{2}-\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\Big|U_{x_1,\Lambda}^{2^*-1}\Big(K\big(\frac{|y|}{\mu}\big)-1\Big)\Big|\\ \le&\frac{C}{\mu^{\frac{N+2}{2}-\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}},\quad ||y|-\mu r_0|\le\delta\mu.\end{aligned}\tag{2.24}$$

Combining (2.22), (2.23) and (2.24), we reach

$$\|J_2\|_{**} \le \frac{C}{\mu^{\frac{N+2}{2}-\tau}} + C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau} \le C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}.$$

□

Now, we are ready to prove Proposition 2.3.

Proof of Proposition 2.3. Let us recall that

$$\mu = k^{\frac{N-2}{N-m-2}}.$$

Let

$$E = \Big\{u : u \in C(\mathbb{R}^N), \|u\|_* \le \Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau-\eta}, \int_{\mathbb{R}^N} U_{x_i,\Lambda}^{2*-2} Z_{i,l}\phi = 0,\ i = 1, \cdots, k,\ l = 1, 2\Big\},$$

where $\eta > 0$ is a fixed small constant. Then, (2.16) is equivalent to

$$\phi = A(\phi) =: L(N(\phi)) + L(l_k).$$

We will prove that A is a contraction map from E to E.

In fact, if $N \ge 6$,

$$\begin{aligned} \|\phi\|_* \le{}& C\|N(\phi)\|_{**} + C\|l_k\|_{**} \\ \le{}& Ck^{\frac{4}{N-2}}\|\phi\|_*^{2^*-1} + C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau} \\ \le{}& Ck^{\frac{4}{N-2}}\Big(\frac{k}{\mu}\Big)^{(\frac{N+2}{2}-\tau-\eta)\frac{N+2}{N-2}} + C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau} \\ ={}& \frac{Ck^{\frac{4}{N-2}+\frac{N+2}{N-2}(2-\eta)-\frac{4\tau}{N-2}}}{\mu^{\frac{N+2}{N-2}(2-\eta)-\frac{4\tau}{N-2}}}\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau} \le \Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau-\eta}, \end{aligned} \tag{2.25}$$

since

$$\begin{aligned} &\frac{N-2}{N-m-2}\Big(\frac{N+2}{N-2}(2-\eta) - \frac{4\tau}{N-2}\Big) \\ \ge{}& \frac{N-2}{N-4}\Big(\frac{N+2}{N-2}(2-\eta) - \frac{4\tau}{N-2}\Big) > \frac{4}{N-2} + \frac{N+2}{N-2}(2-\eta) - \frac{4\tau}{N-2}, \end{aligned}$$

if we take $\eta > 0$ is small and τ is close to 1. Thus, A maps E to E.

On the other hand,

$$\|A(\phi_1)-A_(\phi_2)\|_* = \|L(N(\phi_1))-L(N(\phi_2))\|_* \le C\|N(\phi_1)-N(\phi_2)\|_{**}.$$

If $N\ge 6$, then

$$|N'(t)|\le C|t|^{2^*-2}.$$

As a result,

$$\begin{aligned}&|N(\phi_1)-N(\phi_2)|\le C\big(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\big)|\phi_1-\phi_2|\\ \le& C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\end{aligned}$$

As before, we have

$$\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\le Ck^{\frac{4}{N-2}}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

So,

$$\begin{aligned}&\|A(\phi_1)-A_(\phi_2)\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}\\ \le& Ck^{\frac{4}{N-2}}\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\\ \le&\frac{Ck^{\frac{4}{N-2}+\frac{2(N+2)}{N-2}-\frac{4(\tau+\eta)}{N-2}}}{\mu^{\frac{2(N+2)}{N-2}-\frac{4(\tau+\eta)}{N-2}}}\|\phi_1-\phi_2\|_*\le\frac12\|\phi_1-\phi_2\|_*.\end{aligned}$$

Thus, A is a contraction map.

The case $N=5$ can be discussed in a similar way.

It follows from the contraction mapping theorem that there is a unique $\phi\in E$, such that

$$\phi=A(\phi).$$

Moreover, it follows from Proposition 2.2 that

$$\|\phi\|_*\le C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau}.$$

□

3. Proof of Theorem 1.5

Let

$$F(r,\Lambda)=I\big(W_r+\phi\big),$$

where $r=|x_1|$, ϕ is the function obtained in Proposition 2.3, and

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}|Du|^2-\frac{1}{2^*}\int_{\mathbb{R}^N}K\big(\frac{|y|}{\mu}\big)|u|^{2^*}.$$

Proposition 3.1. *We have*

$$\begin{aligned}F(r,\Lambda)=&I(W_r)+O\Big(\frac{k}{\mu^{m+\sigma}}\Big)\\ =&k\Big(A+\frac{B_1}{\Lambda^m\mu^m}+\frac{B_2}{\Lambda^{m-2}\mu^m}(\mu r_0-|x_1|)\big)^2\\ &-\sum_{i=2}^{k}\frac{B_3}{\Lambda^{N-2}|x_1-x_j|^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-|x_1||^3\Big)\Big),\end{aligned}$$

where $\sigma>0$ is a fixed constant, $B_i>0$, $i=1,2,3$, is some constant.

Proof. Since

$$\big\langle I'\big(W_r\big),\phi\big\rangle=0,$$

there is $t\in(0,1)$ such that

$$\begin{aligned}F(r,\Lambda)&=I(W_r)+\frac{1}{2}D^2I\big(W_r+t\phi\big)(\phi,\phi)\\ =&I(W_r)+\int_{\mathbb{R}^N}\big(|D\phi|^2-(2^*-1)K\big(\frac{|y|}{\mu}\big)\big(W_r+t\phi\big)^{2^*-2}\phi^2\big)\\ =&I(W_r)+(2^*-1)\int_{\mathbb{R}^N}K\big(\frac{|y|}{\mu}\big)\Big(\big(W_r+t\phi\big)^{2^*-2}-W_r^{2^*-2}\Big)\phi^2\\ &+\int_{\mathbb{R}^N}\big(N(\phi)+l_k\big)\phi\\ =&I(W_r)+O\Big(\int_{\mathbb{R}^N}\big(|\phi|^{2^*}+|N(\phi)||\phi|+|l_k||\phi|\big)\Big).\end{aligned}$$

But

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \big(|N(\phi)||\phi| + |l_k||\phi|\big) \\
\le & C\Big(\|N(\phi)\|_{**} + \|l_k\|_{**}\Big)\|\phi\|_* \int_{\mathbb{R}^N} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
$$

Using Lemma B.1

$$
\begin{aligned}
&\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
=&\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+2\tau}} + \sum_{j=1}^{k}\sum_{i\ne j} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
\le&\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+2\tau}} + C\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+\tau}} \sum_{j=2}^{k} \frac{1}{|x_j-x_1|^{\tau}} \\
\le & C\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+\tau}},
\end{aligned}
$$

since $\tau > 1$. Thus, we obtain

$$
\int_{\mathbb{R}^N} \big(|N(\phi)||\phi| + |l_k||\phi|\big) \le Ck\Big(\|N(\phi)\|_{**} + \|l_k\|_{**}\Big)\|\phi\|_* \le Ck\Big(\frac{k}{\mu}\Big)^{N+2-2\tau}.
$$

On the other hand,

$$
\int_{\mathbb{R}^N} |\phi|^{2^*} \le C\|\phi\|_*^{2^*} \int_{\mathbb{R}^N} \Big(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}.
$$

But using Lemma B.1, if $y \in \Omega_1$,

$$
\begin{aligned}
&\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\\
\le&\sum_{j=2}^{k}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{4}+\frac{1}{2}\tau}}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{4}+\frac{1}{2}\tau}}\\
\le&C\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{1}{2}\bar\eta}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\tau-\frac{1}{2}\bar\eta}}\le C\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{1}{2}\bar\eta}},
\end{aligned}
$$

Thus,

$$
\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le\frac{C}{(1+|y-x_1|)^{N+2^*\frac{1}{2}\bar\eta}},\quad y\in\Omega_1.
$$

Thus,

$$
\int_{\mathbb{R}^N}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le Ck.
$$

So, we have proved

$$
\int_{\mathbb{R}^N}|\phi|^{2^*}\le Ck\|\phi\|_*^{2^*}\le Ck\big(\frac{k}{\mu}\big)^{2^*(\frac{N+2}{2}-\tau)}.
$$

□

Proposition 3.2. *We have*

$$
\begin{aligned}
&\frac{\partial F(r,\Lambda)}{\partial\Lambda}\\
=&k\Big(-\frac{B_1m}{\Lambda^{m+1}\mu^m}+\sum_{i=2}^{k}\frac{B_3(N-2)}{\Lambda^{N-1}|x_1-x_j|^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-|x_1||^2\Big)\Big),
\end{aligned}
$$

where $\sigma>0$ is a fixed constant.

Proof. We have

$$
\begin{aligned}
\frac{\partial F(r,\Lambda)}{\partial \Lambda} &= \Big\langle I'(W_r+\phi), \frac{\partial W_r}{\partial \Lambda} + \frac{\partial \phi}{\partial \Lambda}\Big\rangle \\
=&\Big\langle I'(W_r+\phi), \frac{\partial W_r}{\partial \Lambda}\Big\rangle + \sum_{l=1}^{2}\sum_{i=1}^{k} c_l \Big\langle U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \frac{\partial \phi}{\partial \Lambda}\Big\rangle.
\end{aligned}
$$

But

$$
\Big\langle U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \frac{\partial \phi}{\partial \Lambda}\Big\rangle = -\Big\langle \frac{\partial (U_{x_i,\Lambda}^{2^*-2} Z_{i,l})}{\partial \Lambda}, \phi\Big\rangle
$$

Thus, using Proposition 2.3,

$$
\begin{aligned}
&\Big|\sum_{i=1}^{k} c_l \Big\langle U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \frac{\partial \phi}{\partial \Lambda}\Big\rangle\Big| \\
\le& C|c_l|\|\phi\|_* \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{N+2}} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \\
\le& \frac{C}{\mu^{m+\sigma}}.
\end{aligned}
$$

On the other hand,

$$
\int_{\mathbb{R}^N} D(W_r+\phi) D\frac{\partial W_r}{\partial \Lambda} = \int_{\mathbb{R}^N} DW_r D\frac{\partial W_r}{\partial \Lambda},
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)(W_r+\phi)^{2^*-1}\frac{\partial W_r}{\partial \Lambda} \\
=&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big) W_r^{2^*-1}\frac{\partial W_r}{\partial \Lambda} + (2^*-1)\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big) W_r^{2^*-2}\frac{\partial W_r}{\partial \Lambda}\phi + O\Big(\int_{\mathbb{R}^N} |\phi|^{2^*}\Big).
\end{aligned}
$$

Moreover, from $\phi \in E$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big) W_r^{2^*-2}\frac{\partial W_r}{\partial \Lambda}\phi\\
=&\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)\big(W_r^{2^*-2}\frac{\partial W_r}{\partial \Lambda}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\big)\phi+\sum_{j=1}^k\int_{\mathbb{R}^N}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\phi\\
=&k\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)\big(W_r^{2^*-2}\frac{\partial W_r}{\partial \Lambda}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\big)\phi+k\int_{\mathbb{R}^N}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi,
\end{aligned}
$$

$$
\begin{aligned}
&\Big|\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)\big(W_r^{2^*-2}\frac{\partial W_r}{\partial \Lambda}-\sum_{j=1}^k U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\big)\phi\Big|\\
\le& C\int_{\Omega_1}\Big(U_{x_1,\Lambda}^{2^*-2}\sum_{j=2}^k U_{x_j,\Lambda}+\sum_{j=2}^k U_{x_j,\Lambda}^{2^*-1}\Big)|\phi|\\
\le&\frac{C}{\mu^{m+\sigma}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi\Big|\\
\le&\Big|\int_{||y|-\mu r_0|\le\sqrt{\mu}}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi\Big|+\Big|\int_{||y|-\mu r_0|\ge\sqrt{\mu}}\big(K\big(\frac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi\Big|\\
\le&\frac{C}{\mu^{m+\sigma}}.
\end{aligned}
$$

Thus, we have proved

$$
\frac{\partial F(r,\Lambda)}{\partial \Lambda}=\frac{\partial I(W_r)}{\partial \Lambda}+O\Big(\frac{1}{\mu^{m+\sigma}}\Big),
$$

and the result follows from Proposition A.2. □

Since

$$
|x_j-x_1|=2|x_1|\sin\frac{(j-1)\pi}{k},\quad j=2,\dots,k,
$$

we have

$$\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{N-2}}=\frac{1}{(2|x_1|)^{N-2}}\sum_{j=2}^{k}\frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}$$
$$=\begin{cases}\frac{1}{(2|x_1|)^{N-2}}\sum_{j=2}^{\frac{k}{2}}\frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}+\frac{1}{(2|x_1|)^{N-2}}, & \text{if } k \text{ is even;}\\ \frac{1}{(2|x_1|)^{N-2}}\sum_{j=2}^{[\frac{k}{2}]}\frac{1}{(\sin\frac{(j-1)\pi}{k})^{N-2}}, & \text{if } k \text{ is old.}\end{cases}$$

But

$$0<c'\le\frac{\sin\frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}}\le c'',\quad j=2,\cdots,[\frac{k}{2}].$$

So, there is a constant $B_4>0$, such that

$$\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{N-2}}=\frac{B_4k^{N-2}}{|x_1|^{N-2}}+O\Big(\frac{k}{|x_1|^{N-2}}\Big).$$

Thus, we obtain

$$\begin{aligned}F(r,\Lambda)=&k\Big(A+\frac{B_1}{\Lambda^m\mu^m}+\frac{B_2}{\Lambda^{m-2}\mu^m}(\mu r_0-r)^2\\&-\frac{B_4k^{N-2}}{\Lambda^{N-2}r^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-r|^3+\frac{k}{r^{N-2}}\Big)\Big),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial F(r,\Lambda)}{\partial\Lambda}\\=&k\Big(-\frac{B_1m}{\Lambda^{m+1}\mu^m}+\frac{B_4(N-2)k^{N-2}}{\Lambda^{N-1}r^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-r|^2+\frac{k}{r^{N-2}}\Big)\Big).\end{aligned}$$

Let Λ_0 be the solution of

$$-\frac{B_1m}{\Lambda^{m+1}}+\frac{B_4(N-2)}{\Lambda^{N-1}r_0^{N-2}}=0.$$

Then

$$\Lambda_0=\Big(\frac{B_4(N-2)}{B_1mr_0^{N-2}}\Big)^{\frac{1}{N-2-m}}.$$

Define

$$D=\Big\{(r,\Lambda):r\in[\mu r_0-\frac{1}{\mu^{\bar\theta}},\mu r_0+\frac{1}{\mu^{\bar\theta}}],\quad \Lambda\in[\Lambda_0-\frac{1}{\mu^{\frac{3}{2}\bar\theta}},\Lambda_0+\frac{1}{\mu^{\frac{3}{2}\bar\theta}}]\Big\},$$

where $\bar\theta>0$ is a small constant.

For any $(r,\Lambda)\in D$, we have

$$\frac{r}{\mu}=r_0+O\big(\frac{1}{\mu^{1+\bar\theta}}\big).$$

Thus,

$$r^{N-2}=\mu^{N-2}\big(r_0^{N-2}+O\Big(\frac{1}{\mu^{1+\bar\theta}}\big)\Big).$$

So,

(3.26)

$$\begin{aligned}F(r,\Lambda)=&k\Big(A+\big(\frac{B_1}{\Lambda^m}-\frac{B_4}{\Lambda^{N-2}r_0^{N-2}}\big)\frac{1}{\mu^m}\\&+\frac{B_2}{\Lambda^{m-2}\mu^m}(\mu r_0-r)^2+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-r|^3+\frac{k}{\mu^{N-2}}\Big)\Big),\quad (r,\Lambda)\in D,\end{aligned}$$

and

(3.27)

$$\begin{aligned}&\frac{\partial F(r,\Lambda)}{\partial\Lambda}\\=&k\Big(\big(-\frac{B_1m}{\Lambda^{m+1}}+\frac{B_4(N-2)}{\Lambda^{N-1}r_0^{N-2}}\big)\frac{1}{\mu^m}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-r|^2+\frac{k}{\mu^{N-2}}\Big)\Big),\quad (r,\Lambda)\in D.\end{aligned}$$

Now, we define

$$\bar F(r,\Lambda)=-F(r,\Lambda),\quad (r,\Lambda)\in D.$$

Let

$$\alpha_2=k(-A+\eta),\quad ,\alpha_1=k\Big(-A-\big(\frac{B_1}{\Lambda_0^m}-\frac{B_4}{\Lambda_0^{N-2}r_0^{N-2}}\big)\frac{1}{\mu^m}-\frac{1}{\mu^{m+\frac{5}{2}\bar\theta}}\Big),$$

where $\eta>0$ is a small constant.

Let

$$\bar{F}^{\alpha} = \big\{(r,\Lambda) \in D, \bar{F}(r,\Lambda) \le \alpha\big\}.$$

Consider

$$\begin{cases} \frac{dr}{dt} = -D_r \bar{F}, & t > 0; \\ \frac{d\Lambda}{dt} = -D_\Lambda \bar{F}, & t > 0; \\ (r,\Lambda) \in F^{\alpha_2}. \end{cases}$$

Then

Proposition 3.3. *The flow $(r(t),\Lambda(t))$ does not leave D before it reaches F^{α_1}.*

Proof. If $\Lambda = \Lambda_0 + \frac{1}{\mu^{\frac{3}{2}\bar{\theta}}}$, noting that $|r - \mu r_0| \le \frac{1}{\mu^{\bar{\theta}}}$, we obtain from (3.27) that

$$\frac{\partial \bar{F}(r,\Lambda)}{\partial \Lambda} = k\Big(c' \frac{1}{\mu^{m+\frac{3}{2}\bar{\theta}}} + O\Big(\frac{1}{\mu^{m+2\bar{\theta}}}\Big)\Big) > 0.$$

So, the flow does not leave D.

Similarly, if $\Lambda = \Lambda_0 - \frac{1}{\mu^{\frac{3}{2}\bar{\theta}}}$, then we obtain from (3.27) that

$$\frac{\partial \bar{F}(r,\Lambda)}{\partial \Lambda} = k\Big(-c' \frac{1}{\mu^{m+\frac{3}{2}\bar{\theta}}} + O\Big(\frac{1}{\mu^{m+2\bar{\theta}}}\Big)\Big) < 0.$$

So, the flow does not leave D.

Suppose now $|r - \mu r_0| = \frac{1}{\mu^{\bar{\theta}}}$. Since $|\Lambda - \Lambda_0| \le \frac{1}{\mu^{\frac{3}{2}\bar{\theta}}}$, we see

$$\begin{aligned} &\frac{B_1}{\Lambda^m} - \frac{B_4}{\Lambda^{N-2} r_0^{N-2}} = \frac{B_1}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^{N-2} r_0^{N-2}} + O\big(|\Lambda - \Lambda_0|^2\big) \\ =&\frac{B_1}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^{N-2} r_0^{N-2}} + O\big(\frac{1}{\mu^{3\bar{\theta}}}\big). \end{aligned}$$

So, using (3.26), we obtain

(3.28)

$$
\begin{aligned}
&\bar F(r,\Lambda)\\
=&k\Big(-A-\big(\frac{B_1}{\Lambda_0^m}-\frac{B_4}{\Lambda_0^{N-2}r_0^{N-2}}\big)\frac{1}{\mu^m}-\frac{B_2}{\Lambda_0^{m-2}\mu^m}(\mu r_0-r)^2+O\Big(\frac{1}{\mu^{m+3\bar\theta}}\Big)\Big)\\
\le&k\Big(-A-\big(\frac{B_1}{\Lambda_0^m}-\frac{B_4}{\Lambda_0^{N-2}r_0^{N-2}}\big)\frac{1}{\mu^m}-\frac{B_2}{\Lambda_0^{m-2}\mu^{m+2\bar\theta}}+O\Big(\frac{1}{\mu^{m+3\bar\theta}}\Big)\Big)<\alpha_1.
\end{aligned}
$$

□

Proof of Theorem 1.5. We will prove that $\bar F$, and thus F, has a critical point in D.

Define

$$
\Gamma=\Big\{h:h(r,\Lambda)=(h_1(r,\Lambda),h_2(r,\Lambda))\in D,(r,\Lambda)\in D\\
h(r,\Lambda)=(r,\Lambda),\ \text{if } |r-\mu r_0|=\frac{1}{\mu^{\bar\theta}}\Big\}.
$$

Let

$$
c=\inf_{h\in\Gamma}\max_{(r,\Lambda)\in D}\bar F(h(r,\Lambda)).
$$

We claim that c is a critical value of $\bar F$. To prove this, we need to prove

(i) $\alpha_1<c<\alpha_2$;

(ii) $\sup_{|r-\mu r_0|=\frac{1}{\mu^{\bar\theta}}}\bar F(h(r,\Lambda))<\alpha_1,\ \forall\, h\in\Gamma$.

To prove (ii), let $h\in\Gamma$. Then for any $\bar r$ with $|\bar r-\mu r_0|=\frac{1}{\mu^{\bar\theta}}$, we have $h(\bar r,\Lambda)=(\bar r,\tilde\Lambda)$ for some $\tilde\Lambda$. Thus, by (3.28),

$$
\bar F(h(r,\Lambda))=\bar F(\bar r,\tilde\Lambda)<\alpha_1.
$$

Now we prove (i). It is easy to see that

$$
c<\alpha_2.
$$

For any $h=(h_1,h_2)\in\Gamma$. Then $h_1(r,\Lambda)=r$, if $|r-\mu r_0|=\frac{1}{\mu^{\bar\theta}}$. Define

$$
\tilde h_1(r)=h_1(r,\Lambda_0).
$$

Then $\tilde{h}_1(r) = r$, if $|r - \mu r_0| = \frac{1}{\mu^{\bar\theta}}$. So, there is a $\bar r \in (\mu r_0 - \frac{1}{\mu^{\bar\theta}}, \mu r_0 + \frac{1}{\mu^{\bar\theta}})$, such that

$$\tilde{h}_1(\bar r) = \mu r_0.$$

Let $\bar\Lambda = h_2(\bar r, \Lambda_0)$. Then from (3.26)

$$\begin{aligned}
&\max_{(r,\Lambda)\in D} \bar F(h(r,\Lambda)) \ge \bar F(h(\bar r, \Lambda_0)) = \bar F(\mu r_0, \bar\Lambda)\\
=&k\Big(-A - \big(\frac{B_1}{\bar\Lambda^m} - \frac{B_4}{\bar\Lambda^{N-2} r_0^{N-2}}\big)\frac{1}{\mu^m} + O\Big(\frac{1}{\mu^{m+\sigma}} + \frac{k}{\mu^{N-2}}\Big)\Big)\\
=&k\Big(-A - \big(\frac{B_1}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^{N-2} r_0^{N-2}}\big)\frac{1}{\mu^m} + O\Big(\frac{1}{\mu^{m+3\bar\theta}}\Big)\Big) > \alpha_1.
\end{aligned}$$

□

Appendix A. Energy Expansion

In all of the appendixes, we always assume that

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and $r \in [r_0\mu - \frac{1}{\mu^{\bar\theta}}, r_0\mu + \frac{1}{\mu^{\bar\theta}}]$ for some small $\bar\theta > 0$.

Let recall that

$$\mu = k^{\frac{N-2}{N-2-m}},$$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |Du|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K\big(\frac{|y|}{\mu}\big)|u|^{2^*},$$

$$U_{x_j,\Lambda}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-x_j|^2)^{\frac{N-2}{2}}},$$

and

$$W_r(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \sum_{j=1}^k \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-x_j|^2)^{\frac{N-2}{2}}}.$$

In this section, we will calculate $I(W_r)$.

Proposition A.1. *We have*

$$\begin{aligned}I(W_r) =&k\Big(A+\frac{B_1}{\Lambda^m\mu^m}+\frac{B_2}{\Lambda^{m-2}\mu^m}(\mu r_0-r))^2\\&-\sum_{i=2}^k\frac{B_3}{\Lambda^{N-2}|x_1-x_j|^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-r|^3\Big)\Big),\end{aligned}$$

where B_i, $i=1,2,3$, *is some positive constant,* $A>0$ *is a constant, and* $r=|x_1|$.

Proof. By using the symmetry, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|DW_r|^2=\sum_{j=1}^k\sum_{i=1}^k\int_{\mathbb{R}^N}U_{x_j,\Lambda}^{2^*-1}U_{x_i,\Lambda}\\=&k\Big(\int_{\mathbb{R}^N}U_{0,1}^{2^*}+\sum_{i=2}^k\int_{\mathbb{R}^N}U_{x_1,\Lambda}^{2^*-1}U_{x_i,\Lambda}\Big)\\=&k\Big(\int_{\mathbb{R}^N}U_{0,1}^{2^*}+\sum_{i=2}^k\frac{B_0}{\Lambda^{N-2}|x_1-x_j|^{N-2}}+O\Big(\sum_{i=2}^k\frac{1}{|x_1-x_j|^{N-2+\sigma}}\Big)\Big).\end{aligned}$$

Let

$$\Omega_j=\Big\{y:\ y=(y',y'')=\mathbb{R}^2\times\mathbb{R}^{N-2},\ \Big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\Big\rangle\ge\cos\frac{\pi}{k}\Big\}.$$

Then,

$$\begin{aligned}&\int_{\mathbb{R}^N}K\Big(\frac{|y|}{\mu}\Big)|W_r|^{2^*}=k\int_{\Omega_1}K\Big(\frac{|y|}{\mu}\Big)|W_r|^{2^*}\\=&k\Big(\int_{\Omega_1}K\Big(\frac{|y|}{\mu}\Big)U_{x_1,\Lambda}^{2^*}-2^*\int_{\Omega_1}K\Big(\frac{|y|}{\mu}\Big)\sum_{i=2}^kU_{x_1,\Lambda}^{2^*-1}U_{x_i,\Lambda}\\&+O\Big(\int_{\Omega_1}U_{x_1,\Lambda}^{2^*/2}\Big(\sum_{i=2}^kU_{x_i,\Lambda}\Big)^{2^*/2}\Big)\Big).\end{aligned}$$

Note that for $y\in\Omega_1$, $|y-x_i|\ge|y-x_1|$. Using Lemma B.1, we find

$$
\begin{aligned}
&\sum_{i=2}^{k} U_{x_i,\Lambda} \le C \sum_{i=2}^{k} \frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}}} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}}} \\
\le & \frac{1}{(1+|y-x_1|)^{N-2-\alpha}} \sum_{i=2}^{k} \frac{1}{|x_i-x_1|^{\alpha}}.
\end{aligned}
$$

If we take the constant α with $\max(1, \frac{(N-2)^2}{N}) < \alpha < N-2$, then

$$
\int_{\Omega_1} U_{x_1,\Lambda}^{2^*/2} \Big(\sum_{i=2}^{k} U_{x_i,\Lambda}\Big)^{2^*/2} = O\Big(\big(\frac{k}{\mu}\big)^{N-2+\sigma}\Big).
$$

On the other hand, it is easy to show

$$
\begin{aligned}
&\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big) \sum_{i=2}^{k} U_{x_1,\Lambda}^{2^*-1} U_{x_i,\Lambda} \\
=&\int_{\Omega_1} \sum_{i=2}^{k} U_{x_1,\Lambda}^{2^*-1} U_{x_i,\Lambda} + \int_{\Omega_1} \Big(K\big(\frac{|y|}{\mu}\big) - 1\Big) \sum_{i=2}^{k} U_{x_1,\Lambda}^{2^*-1} U_{x_i,\Lambda} \\
=&\sum_{i=2}^{k} \frac{B_0}{\Lambda^{N-2}|x_1-x_j|^{N-2}} + O\Big(\big(\frac{k}{\mu}\big)^{N-2+\sigma}\Big).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big) U_{x_1,\Lambda}^{2^*} \\
=&\int_{\mathbb{R}^N} U_{0,1}^{2^*} - \frac{c_0}{\mu^m} \int_{\Omega_1} ||y| - \mu r_0|^m U_{x_1,\Lambda}^{2^*} \\
&+ O\Big(\mu^{-m-\theta} \int_{\Omega_1} ||y| - \mu r_0|^{m+\theta} U_{x_1,\Lambda}^{2^*}\Big) \\
=&\int_{\mathbb{R}^N} U_{0,1}^{2^*} - \frac{c_0}{\mu^m} \int_{\Omega_1} ||y| - \mu r_0|^m U_{x_1,\Lambda}^{2^*} + O\Big(\frac{1}{\mu^{m+\theta}}\Big) \\
=&\int_{\mathbb{R}^N} U_{0,1}^{2^*} - \frac{c_0}{\mu^m} \int_{\mathbb{R}^N} ||y-x_1| - \mu r_0|^m U_{0,\Lambda}^{2^*} + O\Big(\frac{1}{\mu^{m+\theta}}\Big).
\end{aligned}
$$

But

$$\begin{aligned}
&||y-x_1|-\mu r_0|^m = ||x_1|-y_1+O(\frac{1}{|x_1|})-\mu r_0|^m\\
=&|y_1|^m + m|y_1|^{m-2}y_1(\mu r_0-|x_1|+O(\frac{1}{|x_1|}))\\
&+\frac{1}{2}m(m-1)|y_1|^{m-2}(\mu r_0-|x_1|+O(\frac{1}{|x_1|}))^2+O\Big((\mu r_0-|x_1|+O(\frac{1}{|x_1|}))^{2+\sigma}\Big)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N}||y-x_1|-\mu r_0|^m U_{0,\Lambda}^{2^*}\\
=&\int_{\mathbb{R}^N}|y_1|^m U_{0,\Lambda}^{2^*}+\frac{1}{2}m(m-1)\int_{\mathbb{R}^N}|y_1|^{m-2}U_{0,\Lambda}^{2^*}(\mu r_0-|x_1|))^2\\
&+O\Big(|\mu r_0-|x_1||^{2+\sigma}\Big).
\end{aligned}$$

Thus, we have proved

$$\begin{aligned}
&\int_{\mathbb{R}^N}K\big(\frac{|y|}{\mu}\big)|W_r|^{2^*}\\
=&k\Big(\int_{\mathbb{R}^N}|U_{0,1}|^{2^*}-\frac{c_0}{\Lambda^m\mu^m}\int_{\mathbb{R}^N}|y_1|^m U_{0,1}^{2^*}\\
&-\frac{c_0}{\Lambda^{m-2}\mu^m}\frac{1}{2}m(m-1)\int_{\mathbb{R}^N}|y_1|^{m-2}U_{0,1}^{2^*}(\mu r_0-|x_1|))^2\\
&+2^*\sum_{i=2}^k\frac{B_0}{\Lambda^{N-2}|x_1-x_j|^{N-2}}+O\Big(\frac{1}{\mu^{m+\sigma}}\Big)\Big).
\end{aligned}$$

□

We also need to calculate $\frac{\partial I(W_r)}{\partial\Lambda}$.

Proposition A.2. *We have*

$$\begin{aligned}
\frac{\partial I(W_r)}{\partial\Lambda}=&k\Big(-\frac{mB_1}{\Lambda^{m+1}\mu^m}+\sum_{i=2}^k\frac{B_3(N-2)}{\Lambda^{N-1}|x_1-x_j|^{N-2}}\\
&+O\Big(\frac{1}{\mu^{m+\sigma}}+\frac{1}{\mu^m}|\mu r_0-|x_1||^2\Big)\Big),
\end{aligned}$$

where B_i, $i=1,2,3$, is same positive constant in Proposition A.1

Proof. The proof of this proposition is similar to the proof of Proposition A.1. So we just sketch it.

We have

$$\frac{\partial I(W_r)}{\partial \Lambda} = k\Big((2^*-1)\sum_{i=2}^{k}\int_{\mathbb{R}^N} U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}U_{x_i,\Lambda} - \int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)W_r^{2^*-1}\frac{\partial W_r}{\partial \Lambda}\Big).$$

It is easy to check that for $y \in \Omega_1$,

$$\Big|\frac{\partial}{\partial \Lambda}\Big(W_r^{2^*} - U_{x_1,\Lambda}^{2^*} - 2^* U_{x_1,\Lambda}^{2^*-1}\sum_{i=2}^{k} U_{x_i,\Lambda}\Big)\Big| \le C U_{x_1,\Lambda}^{2^*/2}\big(\sum_{i=2}^{k} U_{x_i,\Lambda}\big)^{2^*/2}.$$

Thus,

$$\frac{\partial}{\partial \Lambda}W_r^{2^*} = \frac{\partial}{\partial \Lambda}U_{x_1,\Lambda}^{2^*} + 2^*\frac{\partial}{\partial \Lambda}\big(U_{x_1,\Lambda}^{2^*-1}\sum_{i=2}^{k} U_{x_i,\Lambda}\big) + O\Big(U_{x_1,\Lambda}^{2^*/2}\big(\sum_{i=2}^{k} U_{x_i,\Lambda}\big)^{2^*/2}\Big).$$

As a result, we have

$$\begin{aligned} &2^*\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)W_r^{2^*-1}\frac{\partial W_r}{\partial \Lambda} \\ &= \int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)\frac{\partial}{\partial \Lambda}U_{x_1,\Lambda}^{2^*} + 2^*\int_{\Omega_1} K\big(\frac{|y|}{\mu}\big)\frac{\partial}{\partial \Lambda}\big(U_{x_1,\Lambda}^{2^*-1}\sum_{i=2}^{k} U_{x_i,\Lambda}\big) \\ &\quad + O\Big(\int_{\Omega_1} U_{x_1,\Lambda}^{2^*/2}\big(\sum_{i=2}^{k} U_{x_i,\Lambda}\big)^{2^*/2}\Big). \end{aligned}$$

So, we obtain the desired result. □

Appendix B. Basic Estimates

For each fixed i and j, $i \neq j$, consider the following function

$$g_{ij}(y) = \frac{1}{(1+|y-x_j|)^{\alpha}}\frac{1}{(1+|y-x_i|)^{\beta}}, \tag{2.29}$$

where $\alpha \ge 1$ and $\beta \ge 1$ are two constants.

Lemma B.1. *For any constant* $0<\sigma\le\min(\alpha,\beta)$*, there is a constant* $C>0$*, such that*

$$g_{ij}(y)\le\frac{C}{|x_i-x_j|^{\sigma}}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}}+\frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big).$$

Proof. Let $d_{ij}=|x_i-x_j|$. If $y\in B_{\frac{1}{2}d_{ij}}(x_i)$, then

$$|y-x_j|\ge\frac{1}{2}|x_j-x_i|,\quad |y-x_j|\ge\frac{1}{2}|y-x_i|,$$

which gives

$$g_{ij}\le\frac{C}{|x_i-x_j|^{\sigma}}\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}},\quad y\in B_{\frac{1}{2}d_{ij}}(x_i).$$

Similarly, we can prove

$$g_{ij}\le\frac{C}{|x_i-x_j|^{\sigma}}\frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}},\quad y\in B_{\frac{1}{2}d_{ij}}(x_j).$$

Now we consider $y\in\mathbb{R}^N\setminus\big(B_{\frac{1}{2}d_{ij}}(x_i)\cup B_{\frac{1}{2}d_{ij}}(x_j)\big)$. Then we have

$$|y-x_i|\ge\frac{1}{2}|x_j-x_i|,\quad |y-x_j|\ge\frac{1}{2}|x_j-x_i|.$$

If $|y-x_i|\ge 2|x_i-x_j|$, then

$$|y-x_j|\ge|y-x_i|-|x_i-x_j|\ge\frac{1}{2}|y-x_i|.$$

As a result,

$$g_{ij}\le\frac{C}{(1+|y-x_i|)^{\alpha+\beta}}\le\frac{C_1}{|x_i-x_j|^{\sigma}}\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}},$$

because $|y-x_i|\ge\frac{1}{2}|x_j-x_i|$.

If $|y-x_i|\le 2|x_i-x_j|$, then

$$g_{ij}\le\frac{1}{(1+|y-x_i|)^{\alpha}}\frac{C}{|x_i-x_j|^{\beta}}\le\frac{C}{|x_i-x_j|^{\sigma}}\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}},$$

because $|y-x_j|\ge\frac{1}{2}|x_j-x_i|$.

□

Lemma B.2. *For any constant* $0 < \sigma < N-2$, *there is a constant* $C > 0$, *such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \le \frac{C}{(1+|y|)^{\sigma}}.$$

Proof. The result is well known. For the sake of completeness, we give the proof.

We just need to obtain the estimate for $|y| \ge 2$. Let $d = \frac{1}{2}|y|$. Then, we have

$$\int_{B_d(0)} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \le \frac{C}{d^{N-2}} \int_{B_d(0)} \frac{1}{(1+|z|)^{2+\sigma}}\, dz$$
$$\le \frac{C}{d^{N-2}} d^{N-2-\sigma} \le \frac{C}{d^{\sigma}},$$

and

$$\int_{B_d(y)} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \le \frac{C}{d^{2+\sigma}} \int_{B_d(y)} \frac{1}{|z-y|^{N-2}}\, dz \le \frac{C}{d^{\sigma}}.$$

Suppose that $z \in \mathbb{R}^N \setminus \big(B_d(0) \cup B_d(y)\big)$. Then

$$|z-y| \ge \frac{1}{2}|y|, \quad |z| \ge \frac{1}{2}|y|.$$

If $|z| \ge 2|y|$, then $|z-y| \ge |z| - |y| \ge \frac{1}{2}|z|$. As a result,

$$\frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} \le \frac{C}{|z|^{N-2}(1+|z|)^{2+\sigma}}.$$

If $|z| \le 2|y|$, then

$$\frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} \le \frac{C}{|y|^{N-2}(1+|z|)^{2+\sigma}} \le \frac{C_1}{|z|^{N-2}(1+|z|)^{2+\sigma}}.$$

Thus, we have proved that

$$\frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}} \le \frac{C}{|z|^{N-2}(1+|z|)^{2+\sigma}}, \quad z \in \mathbb{R}^N \setminus \big(B_d(0) \cup B_d(y)\big),$$

which, give

$$\int_{\mathbb{R}^N\setminus\big(B_d(0)\cup B_d(y)\big)}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z|)^{2+\sigma}}\,dz\le\frac{C}{d^\sigma}.$$

□

Let recall that

$$W_r(y)=\big(N(N-2)\big)^{\frac{N-2}{4}}\sum_{j=1}^k\frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-x_j|^2)^{\frac{N-2}{2}}}.$$

Lemma B.3. *Suppose that $N\ge5$ and $\tau\in(0,2)$. Then there is a small $\theta>0$, such that*

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}W_r^{\frac{4}{N-2}}(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\ \le& C\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}+o(1)\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}},\end{aligned}$$

where $o(1)\to0$ as $k\to+\infty$.

Proof. Firstly, we consider $N\ge6$. Then $\frac{4}{N-2}\le1$. Thus

$$W_r^{\frac{4}{N-2}}(z)\le\sum_{i=1}^k\frac{1}{(1+|z-x_i|)^4}.$$

So, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}W_r^{\frac{4}{N-2}}(z)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\ \le&\sum_{j=1}^k\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z-x_j|)^{4+\frac{N-2}{2}+\tau}}\,dz\\ &+\sum_{j=1}^k\sum_{i\ne j}\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z-x_i|)^4}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz.\end{aligned}$$

By Lemma B.2, if $\theta>0$ is so small that $\frac{N-2}{2}+\tau+\theta<N-2$, then

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_j|)^{4+\frac{N-2}{2}+\tau}}\,dz\\
\le &\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_j|)^{2+\frac{N-2}{2}+\tau+\theta}}\,dz \le \frac{C}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.
\end{aligned}
$$

On the other hand, it follows from Lemmas B.1 and B.2 that for $i \neq j$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_i|)^4} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\
\le &\frac{C}{|x_i-x_j|^2} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \Big(\frac{1}{(1+|z-x_i|)^{2+\frac{N-2}{2}+\tau}} + \frac{1}{(1+|z-x_j|)^{2+\frac{N-2}{2}+\tau}} \Big)\,dz\\
\le &\frac{C}{|x_i-x_j|^2} \Big(\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big).
\end{aligned}
$$

Noting that

$$
\sum_{j\neq i} \frac{1}{|x_i-x_j|^2} \le \frac{Ck^2}{\mu^2} \sum_{j=1}^{k} \frac{1}{j^2} = o(1),
$$

we obtain

$$
\begin{aligned}
&\sum_{j=1}^{k} \sum_{i\neq j} \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z-x_i|)^4} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\
= &o(1) \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}
$$

Suppose now that $N=5$. Recall that

$$
\Omega_j = \Big\{ y:\ y=(y',y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2},\ \Big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \Big\rangle \ge \cos\frac{\pi}{k} \Big\}.
$$

For $z \in \Omega_1$, we have $|z-x_j| \ge |z-x_1|$. Using Lemma B.1, we obtain

$$
\begin{aligned}
&\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^3}\le\frac{1}{(1+|z-x_1|)^2}\sum_{j=2}^{k}\frac{1}{1+|z-x_j|}\\
\le&\frac{C}{(1+|z-x_1|)^2}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|}\le\frac{C}{(1+|z-x_1|)^2}.
\end{aligned}
$$

Thus,

$$
W_r^{\frac{4}{3}}(z)\le\frac{C}{(1+|z-x_1|)^{\frac{8}{3}}}.
$$

As a result, for $z\in\Omega_1$, using Lemma B.1 again, we find

$$
\begin{aligned}
&W_r^{\frac{4}{3}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\\
\le&\frac{C}{(1+|z-x_1|)^{\frac{8}{3}+\frac{3}{2}+\tau}}+\frac{C}{(1+|z-x_1|)^{\frac{1}{3}+2+\frac{3}{2}+\tau}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\frac{1}{3}}}\\
\le&\frac{C}{(1+|z-x_1|)^{\frac{8}{3}+\frac{3}{2}+\tau}}+\frac{k}{\mu^{\frac{1}{3}}}\frac{C}{(1+|z-x_1|)^{\frac{1}{3}+2+\frac{3}{2}+\tau}}\\
\le&\frac{C}{(1+|z-x_1|)^{\frac{1}{3}+2+\frac{3}{2}+\tau}},
\end{aligned}
$$

since

$$
\frac{k}{\mu^{\frac{1}{3}}}\le\frac{Ck}{k^{\frac{3}{3-m}\frac{1}{3}}}\le C.
$$

So, we obtain

$$
\begin{aligned}
&\int_{\Omega_1}\frac{1}{|y-z|^3}W_r^{\frac{4}{3}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\,dz\\
\le&\int_{\Omega_1}\frac{1}{|y-z|^3}\frac{C}{(1+|z-x_1|)^{\frac{1}{3}+2+\frac{3}{2}+\tau}}\,dz\le\frac{C}{(1+|y-x_1|)^{\frac{1}{3}+\frac{3}{2}+\tau}}.
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\int_{\Omega} \frac{1}{|y-z|^3} W_r^{\frac{4}{3}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\, dz \\
&= \sum_{i=1}^{k} \int_{\Omega_i} \frac{1}{|y-z|^3} W_r^{\frac{4}{3}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{3}{2}+\tau}}\, dz \\
&\le \sum_{i=1}^{k} \frac{C}{(1+|y-x_i|)^{\frac{1}{3}+\frac{3}{2}+\tau}}.
\end{aligned}
$$

□

References

[1] A. Ambrosetti, G. Azorero, J. Peral, Perturbation of $-\Delta u - u^{\frac{N+2}{N-2}} = 0$, the scalar curvature problem in R^N and related topics. J. Funct. Anal. 165(1999), 117–149.

[2] A. Ambrosetti and A. Malchiodi, A multiplicity result for the Yamabe problem on S^N. J. Funct. Anal. 168(1999), 529-561.

[3] A. Bahri, Critical points at infinity in some variational problems, Research Notes in Mathematics. Vol 82, Longman-Pitman, 1989

[4] A. Bahri and J. Coron, The scalar-curvature problem on the standard three-dimensional sphere. J. Funct. Anal. 95(1991), 106-172.

[5] G. Bianchi,The scalar curvature equation on R^N and S^N. Adv. Diff. Equats. 1(1996), 857–880.

[6] G. Bianchi, Non-existence and symmetry of solutions to the scalar curvature equation. Comm. Partial Differential Equations 21 (1996), 229–234.

[7] G. Bianchi, H. Egnell, An ODE approach to the equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$, in R^n. Math. Z. 210 (1992), 137–166.

[8] D.Cao, E. Noussair, S. Yan, On the scalar curvature equation $-\Delta u = (1 + \varepsilon K)u^{(N+2)/(N-2)}$ in R^N. Calc. Var. Partial Differential Equations 15 (2002), 403–419.

[9] S.A. Chang, P. Yang, A perturbation result in prescribing scalar curvature on S^n. Duke Math. J. 64(1991), 27–69.

[10] S-Y A Chang, M. Gursky and P. C. Yang, Prescribing scalar curvature on S^2 and S^3, Calculus of Variation and Partial Differential Equation, 1(1993), 205-229.

[11] C.-C. Chen and C.-S. Lin, Estimate of the conformal scalar curvature equation via the method of moving planes, II. J. Diff. Geom. 49(1998), 115-178.

[12] C.-C. Chen and C.-S. Lin, Prescribing scalar curvature on S^N. I. A priori estimates. J. Diff. Geom. 57(2001), 67-171.

[13] M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem. Calc. Var. Partial Differential Equations 16 (2003), 113–145.

[14] W.Y. Ding, W.M. Ni, On the elliptic equation $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ and related topics. Duke Math. J. 52(1985), 485–506 (1985).

[15] Z.C. Han, Prescribing Gaussian curvature on S^2, Duke Math. J. 61(1990), 679-703.
[16] M. Ji, On positive scalar curvature on D^2, Calc. Var. PDE, 19(2004), 165-182.
[17] Y.Y. Li, On $-\Delta u = K(x)u^5$ in R^3. Comm. Pure Appl. Math. 46(1993), 303–340.
[18] Y.Y.Li, Prescribed scalar curvature on S^3, S^4 and related problems. J, Funct. Anal. 118(1993), 43–118 (1993).
[19] Y.Y.Li, Prescribed scalar curvature on S^n and related problems, I. J. Diff. Equations 120(1995), 319–410.
[20] Y.Y. Li, Prescribed scalar curvature on S^n and related problems. II. Existence and compactness. Comm. Pure Appl. Math. 49(1996), 541–597 (1996)
[21] Y. Li, W.M. Ni, On the conformal scalar curvature equation in R^n. Duke Math. J.57(1988), 859–924.
[22] C.S. Lin, S.S. Lin, Positive radial solutions for $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$ in R^n and related topics. Appl. Anal. 38(1990), 121–159.
[23] F.H. Lin, W.M. Ni, J. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem. Comm. Pure Appl. Math. 60 (2007), 252–281.
[24] W.M. Ni, On the elliptic equation $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$, its generalizations, and applications in geometry. Indiana Univ. Math. J. 31(1982), 493–529.
[25] E. Noussair, S. Yan, The scalar curvature equation on R^n. Nonlinear Analysis 45(2001), 483–514.
[26] O.Rey, The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent. J. Funct. Anal. 89(1990), 1–52.
[27] O. Rey, J. Wei, Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity. I. $N = 3$. J. Funct. Anal. 212 (2004), 472–499.
[28] O. Rey, J. Wei, Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity. II. $N \geq 4$. Ann. Inst. H. Poincar Anal. Non Linaire 22 (2005), 459–484.
[29] O. Rey, J. Wei Arbitrary number of positive solutions for an elliptic problem with critical nonlinearity. J. Eur. Math. Soc. (JEMS) 7 (2005), 449–476.
[30] R. Schoen and D. Zhang, Prescribed scalar curvature problem on the $n-$sphere, Calc. Var. PDE, 4(1996), 1-25.
[31] S. Yan, Concentration of solutions for the scalar curvature equation on R^N. J. Diff. Equats. 163(2000), 239–264.

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: wei@math.cuhk.edu.hk

School of Mathematics, Statistics and Computer Science, The University of New England, Armidale, NSW 2351, Australia
E-mail address: syan@turing.une.edu.au