ON LIN-NI'S CONJECTURE IN DIMENSIONS FOUR AND SIX

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ABSTRACT. We give negative answers to Lin-Ni's conjecture for any four and six dimensional domains.

1. INTRODUCTION

We start with the following nonlinear Neumann elliptic problem:

$$\begin{cases} \Delta u - \mu u + u^q = 0, \ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < q < +\infty$, $\mu > 0$ and Ω is a smooth and bounded domain in $\mathbb{R}^n$ $(n \geq 2)$.

Equation (1.1) arises in many branches of the applied sciences. For example, it can be viewed as a steady-state equation for the shadow system of the Gierer-Meinhardt system in mathematical biology [11], [17], or for parabolic equation in chemotaxis, e.g. Keller-Segel model [15].

Equation (1.1) has at least one solution, namely the constant solution $u \equiv \mu^{\frac{1}{q-1}}$. It turns out that this is the only solution, provided that μ is small and $q < \frac{n+2}{n-2}$. This was first proved by Lin-Ni-Takagi [15], via blow up analysis and compactness argument. Based on this, Lin and Ni [14] made the following conjecture:

Lin-Ni's Conjecture [14]**:** For μ small and $q = \frac{n+2}{n-2}$, problem (1.1) admits only the constant solution.

In recent years, many progress have been made towards the understanding of Lin-Ni's conjecture.

The first result was due to Adimurthi-Yadava [1]-[2] (and independently Budd-Knapp-Peletier [5]). They considered the following problem

$$\begin{cases} \Delta u - \mu u + u^{\frac{n+2}{n-2}} = 0 & \text{in } B_R(0), \\ u = u(|x|), \quad u > 0 & \text{in } B_R(0), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial B_R(0) \end{cases} \tag{1.2}$$

and the following results were proved

Theorem A. ([1]-[5]) For μ sufficiently small

(1) if $n = 3$ or $n \geq 7$, problem (1.2) admits only the constant solution;

(2) if $n = 4, 56$, problem (1.2) admits a nonconstant solution.

The proof of Theorem A relies on the radial symmetry of the domain and the solution. In the asymmetric case, the complete answer is not known yet, but there are a few results. In the general three dimensional domain case, Zhu [28] and Wei-Xu [26] proved

Theorem B. ([26],[28]) The conjecture is true if $n=3$ $(q=5)$ and Ω is convex.

Zhu's proof relies on blowing up analysis and a priori estimates, while Wei-Xu [26] gave a direct proof of Theorem B by using only integration by parts.

Part (1) of Theorem A is generalized by Druet-Robert-Wei [10] to mean convex domains with bounded energy.

Theorem C. ([10]) Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n=3$ or $n\geq 7$. Assume that $H(x)>0$ for all $x\in\partial\Omega$, where $H(x)$ is the mean curvature of $\partial\Omega$ at $x\in\partial\Omega$. Then for all $\mu>0$, there exists $\mu_0(\Omega,\Lambda)>0$ such that for all $\mu\in(0,\mu_0(\Omega,\Lambda))$ and for any $u\in C^2(\overline{\Omega})$, we have that

$$\left.\begin{cases}\Delta u+\mu u=u^{2^*-1} & \text{in } \Omega\\ u>0 & \text{in } \Omega\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega\\ \int_\Omega u^{2^*}dx\leq\Lambda & \end{cases}\right\}\Rightarrow u\equiv\mu^{\frac{n-2}{4}}.$$

It should be mentioned that the assumption of bounded energy in Theorem C is necessary. Without this technical assumption, it was proved that solutions to (1.1) may accumulate with infinite energy when the mean curvature is negative somewhere (see Wang-Wei-Yan [22]). More precisely, Wang-Wei-Yan gave a negative answer to Lin-Ni's conjecture in all dimensions $(n\geq 3)$ for non-convex domain by assuming that Ω is a smooth and bounded domain satisfying the following conditions:

(H_1) $y\in\Omega$ if and only if $(y_1,y_2,y_3,\cdots,-y_i,\cdots,y_n)\in\Omega,\ \forall i=3,\cdots,n.$
(H_2) If $(r,0,y'')\in\Omega$, then $(r\cos\theta,r\sin\theta,y'')\in\Omega,\ \forall\theta\in(0,2\pi)$, where $y''=(y_3,\cdots,y_n)$.
(H_3) Let $T:=\partial\Omega\cap\{y_3=\cdots=y_n=0\}$. There exists a connected component Γ of T such that $H(x)\equiv\gamma<0,\ \forall x\in\Gamma$.

Theorem D. ([22]) Suppose $n\geq 3,\ q=\frac{n+2}{n-2}$ and Ω is a bounded smooth domain satisfying (H_1)-(H_3). Let μ be any fixed positive number. Then problem (1.1) has infinitely many positive solutions, whose energy can be made arbitrarily large.

Wang-Wei-Yan [23] also gave a negative answer to Lin-Ni's conjecture in some convex domain including the balls for $n\geq 4$.

Theorem E. ([23]) Suppose $n\geq 4,\ q=\frac{n+2}{n-2}$ and Ω satisfies (H_1)-(H_2). Let μ be a any fixed positive number. Then problem (1.1) has infinitely many positive solutions, whose energy can be made arbitrarily large.

Theorems A-E reveal that Lin-Ni's conjecture depends very sensitively not only on the dimensions, but also on the shape of the domain (convexity). A natural question is: what about the general domains?

So far the only result for general domains is given by Rey-Wei [20] in which they disproved the conjecture in the five-dimensional case by constructing an nontrivial solution which blows up at K interior points in Ω provided μ is sufficiently small. In view of results of Theorem A, we expect to have an negative answer in the case $n = 4, 5, 6$. This is exactly what we shall achieve in this paper.

The purpose of this paper is to establish a result similar to (2) of Theorem A in general four, and six-dimensional domains by constructing a nontrivial solution which blows up at a single point in Ω provided μ is sufficiently small. From now on, we consider the problem

$$\Delta u - \mu u + u^{\frac{n+2}{n-2}} = 0 \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where $n = 4, 6$ and Ω is a smooth bounded domain in $\mathbb{R}^n$ and $\mu > 0$ very small. Our main result is stated as follows

Main Theorem. For problem (1.3) in $n = 4, 6$, there exists $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$, equation (1.3) possesses a nontrivial solution which blows up at an interior point of Ω.

In order to make this statement more precise, we introduce the following notation. Let $G(x, Q)$ be the Green's function defined as

$$\Delta_x G(x, Q) + \delta_Q - \frac{1}{|\Omega|} = 0 \text{ in } \Omega, \ \frac{\partial G}{\partial \nu} = 0 \text{ on } \partial\Omega, \ \int_\Omega G(x, Q)dx = 0. \tag{1.4}$$

We decompose

$$G(x, Q) = K(|x - Q|) - H(x, Q),$$

where

$$K(r) = \frac{1}{c_n r^{n-2}}, \ c_n = (n-2)|S^{n-1}|, \tag{1.5}$$

is the fundamental solution of the Laplacian operator in $\mathbb{R}^n$($|S^{n-1}|$ denotes the area of the unit sphere), $n = 4, 6$.

For the reason of normalization, we consider throughout the paper the following equation:

$$\Delta u - \mu u + n(n-2)u^{\frac{n+2}{n-2}} = 0, \ u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.6}$$

We recall that, according to [6], the functions

$$U_{\Lambda,Q} = (\frac{\Lambda}{\Lambda^2 + |x - Q|^2})^{\frac{n-2}{2}}, \ \Lambda > 0, \ Q \in \mathbb{R}^n, \tag{1.7}$$

are the only solutions to the problem

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}}, \ u > 0 \text{ in } \mathbb{R}^n. \tag{1.8}$$

Our main result can be stated precisely as follows:

Theorem 1.1. *Let Ω be any smooth bounded domain in $\mathbb{R}^n$.*

(1). *For $n = 4$, there exists $\mu_1 > 0$ such that for $0 < \mu < \mu_1$, problem (1.6) has a nontrivial solution*

$$u_\mu = U_{e^{-\frac{c_1}{\mu^2}}\Lambda, Q^\mu} + O(\mu^{-1} e^{-\frac{c_1}{\mu^2}}),$$

where c_1 is some constant depending on the domain, to be determined later, Λ will be some generic constant. The blow up point Q depends on the domain and parameter Λ.

(2). *For $n=6$, there exists $\mu_2>0$ such that for $0<\mu<\mu_2$, problem (1.6) has a nontrivial solution*

$$u_\mu = U_{\mu\Lambda, Q^\mu} + O(\mu),$$

where $\Lambda \to \Lambda_0$, and $\Lambda_0>0$ is some generic constant. The blow up point Q depends on the domain and parameter Λ.

We introduce several notations for late use. Set

$$\Omega_\varepsilon := \Omega/\varepsilon = \{z|\varepsilon z \in \Omega\}, \tag{1.9}$$

and

$$\mu = \begin{cases} (\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}, & n=4, \\ \varepsilon\ , & n=6. \end{cases} \tag{1.10}$$

Through the transformation $u(x) \longmapsto \varepsilon^{-\frac{n-2}{2}} u(x/\varepsilon)$, (1.6) becomes

$$\Delta u - \mu\varepsilon^2 u + n(n-2)u^{\frac{n+2}{n-2}} = 0,\ u>0 \text{ in } \Omega_\varepsilon,\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Omega_\varepsilon. \tag{1.11}$$

We set

$$S_\varepsilon[u] := -\Delta u + \mu\varepsilon^2 u - n(n-2)u_+^{\frac{n+2}{n-2}},\ u_+ = \max(u,0), \tag{1.12}$$

and introduce the following functional

$$J_\varepsilon[u] := \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{1}{2}\mu\varepsilon^2 \int_{\Omega_\varepsilon} u^2 - \frac{(n-2)^2}{2}\int_{\Omega_\varepsilon} |u|^{\frac{2n}{n-2}},\ u \in H^1(\Omega_\varepsilon). \tag{1.13}$$

Depending on the dimensions, we have to overcome different difficulties. In dimension four, the main problem is that the relation between μ and ϵ is only implicit. Dimension six is the *borderline* case, since in the linearized operator the constant term $-\mu u$ disappears. To remedy this problem, we have to introduce an artificial parameter η (see (2.14)).

The paper is organized as follows: In Section 2, we construct suitable approximated bubble solution W, and list their properties. In Section 3, we solve the linearized problem at W up to a finite-dimensional space. Then, in Section 4, we are able to solve the nonlinear problem in that space. In section 5, we study the remaining finite-dimensional problem and solve it in Section 6, finding critical points of the reduced energy functional. Some numerical results may be found in the last Section.

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2. Approximate bubble solutions

In this section, we construct suitable approximate solution, in the neighborhood of which solutions in Theorem 1.1 will be found. Depending on the dimensions, we shall make different ansatz.

Let ε be as defined at (1.10). For any $Q \in \Omega_\varepsilon$ with $d(Q, \partial\Omega_\varepsilon)$ large, $U_{\Lambda,Q}$ in (1.7) provides an approximate solution of (1.11). Because of the appearance of the additional linear term $\mu\varepsilon^2 u$ in the equation (1.11), we need to add an extra term to obtain a better approximation. Now we describe the next order terms in different dimensions.

When $n = 4$, we consider the following linear equation

$$\Delta\bar{\Psi} + U_{1,0} = 0 \quad \text{in } \mathbb{R}^4, \ \bar{\Psi}(0) = 1 \tag{2.1}$$

which has a unique radial solution with the following asymptotic behavior

$$\bar{\Psi}(|y|) = -\frac{1}{2}\ln|y| + I + O\Big(\frac{1}{|y|}\Big), \ \bar{\Psi}' = -\frac{1}{2|y|}\Big(1 + O\Big(\frac{\ln(1+|y|)}{|y|^2}\Big)\Big) \text{ as } |y| \to \infty, \tag{2.2}$$

where I is a generic constant. For $Q \in \Omega_\epsilon$, set

$$\Psi_{\Lambda,Q} = \frac{\Lambda}{2}\ln\frac{1}{\Lambda\varepsilon} + \Lambda\bar{\Psi}(\frac{y-Q}{\Lambda}) \tag{2.3}$$

which satisfies

$$\Delta\Psi_{\Lambda,Q} + U_{\Lambda,Q} = 0 \quad \text{in } \mathbb{R}^4.$$

From (2.2) we derive that

$$|\Psi_{\Lambda,Q}(y)|, \ |\partial_\Lambda\Psi_{\Lambda,Q}(y)| \le C\Big|\ln\frac{1}{\varepsilon(1+|y-Q|)}\Big|, \ |\partial_{Q_i}\Psi_{\Lambda,Q}(y)| \le \frac{C}{1+|y-Q|}. \tag{2.4}$$

Now we turn to the case of $n = 6$. Let $\Psi(|y|)$ be the radial solution of

$$\Delta\Psi + U_{1,0} = 0 \text{ in } \mathbb{R}^6, \quad \Psi \to 0 \text{ as } |y| \to +\infty. \tag{2.5}$$

Then, it is easy to check that

$$\Psi(y) = \frac{1}{4|y|^2}(1 + O(\frac{1}{|y|^2})) \text{ as } |y| \to +\infty. \tag{2.6}$$

For $Q \in \Omega_\varepsilon$, we set

$$\Psi_{\Lambda,Q}(y) = \Lambda^{-2}\Psi(\frac{y-Q}{\Lambda}).$$

Then it satsfies

$$\Delta\Psi_{\Lambda,Q}(y) + U_{\Lambda,Q} = 0 \text{ in } \mathbb{R}^6.$$

It is easy to check that

$$|\Psi_{\Lambda,Q}(y)|, \ |\partial_\Lambda\Psi_{\Lambda,Q}(y)| \le \frac{C}{(1+|y-Q|)^2}, \ |\partial_{Q_i}\Psi_{\Lambda,Q}(y)| \le \frac{C}{(1+|y-Q|)^3}. \tag{2.7}$$

The above considerations take care of the linear term $\mu\epsilon^2 u$ in the equation but we still need to obtain approximate solutions which satisfy the boundary boundary condition. To this end, we need an extra correction term. For this purpose, we define

$$\hat{U}_{\Lambda,\frac{Q}{\varepsilon}}(z) = -\Psi_{\Lambda,Q/\varepsilon}(z) - c_n\mu^{-1}\varepsilon^{n-4}\Lambda^{\frac{n-2}{2}}H(\varepsilon z, Q) + R_{\varepsilon,\Lambda,Q}(z)\chi(\varepsilon z), \tag{2.8}$$

where $R_{\varepsilon,\Lambda,Q}$ is the unique solution satisfying the following boundary value problem

$$\begin{cases} \Delta R_{\varepsilon,\Lambda,Q} - \varepsilon^2 R_{\varepsilon,\Lambda,Q} = 0 \text{ in } \Omega_\varepsilon \\ \mu\varepsilon^2 \frac{\partial R_{\varepsilon,\Lambda,Q}}{\partial\nu} = -\frac{\partial}{\partial\nu}\Big[U_{\Lambda,Q/\varepsilon} - \mu\varepsilon^2\Psi_{\Lambda,Q/\varepsilon} - c_n\varepsilon^{n-2}\Lambda^{\frac{n-2}{2}}H(\varepsilon z, Q)\Big] \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{2.9}$$

Here $\chi(x)$ is a smooth cut-off function in Ω such that $\chi(x) = 1$ for $d(x,\partial\Omega) < \delta/4$ and $\chi(x) = 0$ for $d(x,\partial\Omega) > \delta/2$.

Observe that from (2.2) and (2.6), an expansion of $U_{\Lambda,Q/\varepsilon}$ and the definition of H imply that the normal derivative of $R_{\varepsilon,Q}$ is of order ε^{n-3} on the boundary of Ω_ε, from which we deduce that

$$|R_{\varepsilon,\Lambda,Q}| + |\varepsilon^{-1}\nabla_z R_{\varepsilon,\Lambda,Q}| + |\varepsilon^{-2}\nabla_z^2 R_{\varepsilon,\Lambda,Q}| \le \begin{cases} C, & n = 4, \\ C\varepsilon^2, & n = 6. \end{cases} \tag{2.10}$$

Such an estimate also holds for the derivatives of $R_{\varepsilon,\Lambda,Q}$ with respect to $\Lambda,\ Q$.

Finally we are able to define the approximate bubble solutions. Depending on the dimensions we shall use different ansatz. For $n = 4$, let

$$\Lambda_{4,1} \le \Lambda \le \Lambda_{4,2},\ Q \in \mathcal{M}_{\delta_4} := \{x \in \Omega|\ d(x,\partial\Omega) > \delta_4\}, \tag{2.11}$$

where $\Lambda_{4,1}$ and $\Lambda_{4,2}$ are constants may depending on the domain and δ_4 is a small constant, to be determined later. In viewing of the rescaling, we write

$$\bar{Q} = \frac{1}{\varepsilon}Q,$$

and we define our approximate solutions as

$$W_{\varepsilon,\Lambda,Q} = U_{\Lambda,Q/\varepsilon} + \mu\varepsilon^2\hat{U}_{\Lambda,Q/\varepsilon} + \frac{c_4\Lambda}{|\Omega|}\mu^{-1}\varepsilon^2. \tag{2.12}$$

For $n = 6$, let (Λ, Q, η) satisfy

$$\begin{aligned} \sqrt{\frac{|\Omega|}{c_6}\Big(\frac{1}{96} - \Lambda_6\varepsilon^{\frac{2}{3}}\Big)} \le \Lambda &\le \sqrt{\frac{|\Omega|}{c_6}\Big(\frac{1}{96} + \Lambda_6\varepsilon^{\frac{2}{3}}\Big)}, \\ Q \in \mathcal{M}_{\delta_6} &:= \{x \in \Omega|\ d(x,\partial\Omega) > \delta_6\}, \\ \frac{1}{48} - \eta_6\varepsilon^{\frac{1}{3}} \le \eta &\le \frac{1}{48} + \eta_6\varepsilon^{\frac{1}{3}}, \end{aligned} \tag{2.13}$$

where Λ_6 and η_6 are constants may depending on the domain, δ_6 is a small constant, to be determined later. Our approximate solution for $n = 6$ is the following

$$W_{\varepsilon,\Lambda,Q,\eta} = U_{\Lambda,Q/\varepsilon} + \mu\varepsilon^2\hat{U}_{\Lambda,Q/\varepsilon} + \eta\mu^{-1}\varepsilon^4. \tag{2.14}$$

We remark that unlike the case of $n=4$, in the case of $n=6$, *an extra parameter* η is introduced. The main reason is that when $n=6$ the linear term $-\mu\epsilon^2$ is lost in linearized outer problem. Actually this is one of the main difficulties. This seems to be quite *new* in the Neumann boundary value problems.

For convenience, in the following, we write W, U, $\hat{U}$, R, and Ψ instead of $W_{\varepsilon,\Lambda,Q}$, $U_{\varepsilon,Q/\varepsilon}$, $\hat{U}_{\Lambda,Q/\varepsilon}$, $R_{\varepsilon,\Lambda,Q}$ and $\Psi_{\Lambda,Q/\varepsilon}$ respectively in the following. By construction, the normal derivative of W vanishes on the boundary of Ω_ε, and W satisfies

$$-\Delta W+\mu\varepsilon^2 W=\begin{cases}8U^3+\mu^2\varepsilon^4\hat{U}-\mu\varepsilon^2\Delta(R_{\varepsilon,\Lambda,Q}\chi), & n=4,\\ 24U^2+\mu^2\varepsilon^4\hat{U}-\mu\varepsilon^2\Delta(R_{\varepsilon,\Lambda,Q}\chi)+\varepsilon^6(\eta-\frac{c_6\Lambda^2}{|\Omega|}), & n=6.\end{cases} \tag{2.15}$$

We note that W depends smoothly on Λ and $\bar{Q}$. Setting, for $z\in\Omega_\varepsilon$,

$$\langle z-\bar{Q}\rangle=(1+|z-\bar{Q}|^2)^{\frac12},$$

a simple computation yields

$$|W(z)|\le\begin{cases}C(\varepsilon^2(-\ln\varepsilon)^{\frac12}+\langle z-\bar{Q}\rangle^{-2}), & n=4,\\ C(\varepsilon^3+\langle z-\bar{Q}\rangle^{-4}), & n=6,\end{cases} \tag{2.16}$$

$$|D_\Lambda W(z)|\le\begin{cases}C(\varepsilon^2(-\ln\varepsilon)^{\frac12}+\langle z-\bar{Q}\rangle^{-2}), & n=4,\\ C\langle z-\bar{Q}\rangle^{-4}, & n=6,\end{cases} \tag{2.17}$$

and

$$|D_{\bar{Q}}W(z)|\le\begin{cases}C(\langle z-\bar{Q}\rangle^{-3}), & n=4,\\ C(\langle z-\bar{Q}\rangle^{-5}), & n=6.\end{cases} \tag{2.18}$$

According to the choice of W, we have the following error and energy estimates, whose proof will be given in Section 7.

Lemma 2.1. *For $n=4$, we have*

$$\begin{aligned}|S_\varepsilon[W](z)|\le C\Big(&\langle z-\bar{Q}\rangle^{-4}\varepsilon^2(-\ln\varepsilon)^{\frac12}+\langle z-\bar{Q}\rangle^{-2}\varepsilon^4(-\ln\varepsilon)\\&+\frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac12}}+\frac{\varepsilon^4}{(-\ln\varepsilon)}|\ln(\frac{1}{\varepsilon(1+|z-\bar{Q}|)})|\Big),\end{aligned} \tag{2.19}$$

$$\begin{aligned}|D_\Lambda S_\varepsilon[W](z)|\le C\Big(&\langle z-\bar{Q}\rangle^{-4}\varepsilon^2(-\ln\varepsilon)^{\frac12}+\langle z-\bar{Q}\rangle^{-2}\varepsilon^4(-\ln\varepsilon)\\&+\frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac12}}+\frac{\varepsilon^4}{(-\ln\varepsilon)}|\ln(\frac{1}{\varepsilon(1+|z-\bar{Q}|)})|\Big),\end{aligned} \tag{2.20}$$

$$\begin{aligned}|D_{\bar{Q}}S_\varepsilon[W](z)|\le C\Big(&\langle z-\bar{Q}\rangle^{-5}\varepsilon^2(-\ln\varepsilon)^{\frac12}+\langle z-\bar{Q}\rangle^{-3}\varepsilon^4(-\ln\varepsilon)\\&+\langle z-\bar{Q}\rangle^{-1}\frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac12}}\Big),\end{aligned} \tag{2.21}$$

and

$$\begin{aligned} J_\varepsilon[W] = 2\int_{\mathbb{R}^4} U_{1,0}^4 + \frac{\Lambda^2}{4}c_4 \ln\frac{1}{\Lambda\varepsilon}\varepsilon^2\left(\frac{c_1}{-\ln\varepsilon}\right)^{\frac{1}{2}} - \frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2\left(\frac{c_1}{-\ln\varepsilon}\right)^{-\frac{1}{2}} \\ + \frac{1}{2}c_4^2\Lambda^2\varepsilon^2 H(Q,Q) + o(\varepsilon^2). \end{aligned} \tag{2.22}$$

For $n = 6$, *we have*

$$S_\varepsilon[W](z) = -\varepsilon^6(24\eta^2 - \eta + \frac{c_6\Lambda^2}{|\Omega|}) + O(1)\langle z - \bar{Q}\rangle^{-4}\varepsilon^3, \tag{2.23}$$

$$|D_\Lambda S_\varepsilon[W](z)|, |D_\eta S_\varepsilon[W](z)| \le C\langle z - \bar{Q}\rangle^{-3\frac{2}{3}}\varepsilon^3, \tag{2.24}$$

$$|D_{\bar{Q}} S_\varepsilon[W](z)| \le C\langle z - \bar{Q}\rangle^{-4\frac{2}{3}}\varepsilon^3, \tag{2.25}$$

and

$$\begin{aligned} J_\varepsilon[W] =& 4\int_{\mathbb{R}^6} U_{1,0}^3 + (\frac{1}{2}\eta^2|\Omega| - c_6\Lambda^2\eta + \frac{1}{48}c_6\Lambda^2 - 8\eta^3|\Omega|)\varepsilon^3 \\ &+ \frac{1}{2}c_6^2\Lambda^4 H(Q,Q)\varepsilon^4 + o(\varepsilon^4). \end{aligned} \tag{2.26}$$

3. Finite-Dimensional Reduction

We now apply finite-dimensional reduction procedure for critical exponent problems. The original finite dimensional Liapunov-Schmidt reduction method was first introduced in a seminal paper by Floer and Weinstein [24] in their construction of single bump solutions to one dimensional nonlinear Schrodinger equations. Subsequently this method has been modified and adapted to critical exponent problems. For critical exponents problems, we refer to Bahri-Li-Rey [4], Del Pino-Felmer-Musso [7], Rey-Wei [20, 21] and Wei-Yan [27] and the references therein. For the most updated references and optimal treatments of finite dimensional reduction for critical problems, we refer to Li-Wei-Xu [13].

The general strategy of this method is as follows: the nonlinear equation (1.11) is solved in two steps. In the first step, we solve it up to finite dimensional approximate kernels. In the second step, we reduce the problem to finding critical points of a finite dimensional problems in a suitable sets.

The new element in our proof is in the case of $n = 6$: an extra space (corresponding to η) is introduced. Unlike the traditional critical exponent problems, in which the dimensional of approximate kernels is $n + 1$, we now have $n + 2 = 8$ dimensions.

Equipping $H^1(\Omega_\varepsilon)$ with the scalar product

$$(u, v)_\varepsilon = \int_{\Omega_\varepsilon} (\nabla u \cdot \nabla v + \mu\varepsilon^2 uv). \tag{3.1}$$

For the case $n=4$, orthogonality to the functions

$$Y_0=\frac{\partial W}{\partial \Lambda},\ Y_i=\frac{\partial W}{\partial \bar{Q}_i}, 1\leq i\leq 4, \tag{3.2}$$

in that space is equivalent to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle\cdot,\cdot\rangle$, to the functions $Z_i, 0\leq i\leq 4$, defined as

$$\begin{cases} Z_0=-\Delta\frac{\partial W}{\partial \Lambda}+\mu\varepsilon^2\frac{\partial W}{\partial \Lambda},\\ Z_i=-\Delta\frac{\partial W}{\partial \bar{Q}_i}+\mu\varepsilon^2\frac{\partial W}{\partial \bar{Q}_i},\ 1\leq i\leq 4. \end{cases} \tag{3.3}$$

Straightforward computations provide us with the estimate:

$$|Z_i(z)|\leq C(\varepsilon^4(\frac{1}{-\ln\varepsilon})^{\frac{1}{2}}+\langle z-\bar{Q}\rangle^{-6}). \tag{3.4}$$

Then, we consider the following problem: given h, find a solution ϕ which satisfies

$$\begin{cases} -\Delta\phi+\mu\varepsilon^2\phi-24W^2\phi=h+\Sigma_{i=0}^4 c_iZ_i & \text{in } \Omega_\varepsilon,\\ \frac{\partial\phi}{\partial\nu}=0 & \text{on } \partial\Omega_\varepsilon,\\ \langle Z_i,\phi\rangle=0, & 0\leq i\leq 4, \end{cases} \tag{3.5}$$

for some numbers c_i.

While for the case $n=6$, orthogonality to the functions

$$Y_0=\frac{\partial W}{\partial \Lambda},\ Y_i=\frac{\partial W}{\partial \bar{Q}_i}, 1\leq i\leq 6,\ Y_7=\frac{\partial W}{\partial \eta}, \tag{3.6}$$

in that space is equivalent to the orthogonality in $L^2(\Omega_\varepsilon)$, equipped with the usual scalar product $\langle\cdot,\cdot\rangle$, to the functions $Z_i, 0\leq i\leq 7$, defined as

$$\begin{cases} Z_0=-\Delta\frac{\partial W}{\partial \Lambda}+\mu\varepsilon^2\frac{\partial W}{\partial \Lambda},\\ Z_i=-\Delta\frac{\partial W}{\partial \bar{Q}_i}+\mu\varepsilon^2\frac{\partial W}{\partial \bar{Q}_i},\ 1\leq i\leq 6,\\ Z_7=-\Delta\frac{\partial W}{\partial \eta}+\mu\varepsilon^2\frac{\partial W}{\partial \eta}. \end{cases} \tag{3.7}$$

Direct computations provide us the following estimate:

$$|Z_i(z)|\leq C(\varepsilon^7+\langle z-\bar{Q}\rangle^{-8}),\ 0\leq i\leq 6,\ Z_7(z)=O(\varepsilon^6). \tag{3.8}$$

Then, we consider the following problem: given h, find a solution ϕ which satisfies

$$\begin{cases} -\Delta\phi+\mu\varepsilon^2\phi-48W\phi=h+\Sigma_{i=0}^7 d_iZ_i & \text{in } \Omega_\varepsilon,\\ \frac{\partial\phi}{\partial\nu}=0 & \text{on } \partial\Omega_\varepsilon,\\ \langle Z_i,\phi\rangle=0, & 0\leq i\leq 7, \end{cases} \tag{3.9}$$

for some numbers d_i.

Existence and uniqueness of ϕ will follow from an inversion procedure in suitable weighted function space. To this end, we define

$$\begin{cases}\|\phi\|_* = \|\langle z-\bar{Q}\rangle\phi(z)\|_\infty,\ \|f\|_{**} = \varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}|\overline{f}| + \|\langle z-\bar{Q}\rangle^3 f(z)\|_\infty, n=4,\\ \|\phi\|_{***} = \|\langle z-\bar{Q}\rangle^2\phi(z)\|_\infty,\ \|f\|_{****} = \|\langle z-\bar{Q}\rangle^4 f(z)\|_\infty, n=6,\end{cases} \tag{3.10}$$

where $\|f\|_\infty = \max_{z\in\Omega_\varepsilon}|f(z)|$ and $\overline{f} = |\Omega_\varepsilon|^{-1}\int_{\Omega_\varepsilon} f(z)\mathrm{d}z$ denotes the average of f in Ω_ε.

Before stating an existence result for ϕ in (3.5) and (3.9), we need the following lemma:

Lemma 3.1. *Let u and f satisfy*

$$-\Delta u = f,\ \frac{\partial u}{\partial\nu} = 0,\ \bar{u} = \bar{f} = 0.$$

Then

$$|u(x)| \le C\int_{\Omega_\varepsilon}\frac{|f(y)|}{|x-y|^{n-2}}\mathrm{d}y. \tag{3.11}$$

Proof. The proof is similar to Lemma 3.1 in [20], we omit it here. □

As a consequence, we have

Corollary 3.2. *For $n=4$, suppose u and f satisfy*

$$-\Delta u + \mu\varepsilon^2 u = f \text{ in } \Omega_\varepsilon,\ \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega_\varepsilon.$$

Then

$$\|u\|_* \le C\|f\|_{**}. \tag{3.12}$$

For $n=6$, suppose u and f satisfy

$$-\Delta u + c\mu\varepsilon^2 u = f \text{ in } \Omega_\varepsilon,\ \frac{\partial u}{\partial\nu} = 0 \text{ on } \partial\Omega_\varepsilon,\ \overline{u} = \overline{f} = 0,$$

where c is an arbitrary constant. Then

$$\|u\|_{***} \le C\|f\|_{****}. \tag{3.13}$$

Proof. For $n=4$, integrating the equation yields $\bar{f} = \mu\varepsilon^2\bar{u}$. We may rewrite the original equation as

$$\Delta(u-\bar{u}) = \mu\varepsilon^2(u-\bar{u}) - (f-\bar{f}).$$

With the help of Lemma 3.1, we get

$$|u(y)-\bar{u}| \le C\mu\varepsilon^2\int_{\Omega_\varepsilon}\frac{|u(x)-\bar{u}|}{|x-y|^2}\mathrm{d}x + C\int_{\Omega_\varepsilon}\frac{|f(x)-\bar{f}|}{|x-y|^2}\mathrm{d}x.$$

Since

$$\langle y-\bar{Q}\rangle\int_{\mathbb{R}^4}\frac{1}{|x-y|^2}\langle x-\bar{Q}\rangle^{-3}\mathrm{d}x < \infty,$$

we obtain

$$\begin{aligned}\|\langle y-\bar{Q}\rangle|u-\bar{u}|\|_\infty &\le C\mu\varepsilon^2\|\langle y-\bar{Q}\rangle^3|u-\bar{u}|\|_\infty + C\|\langle y-\bar{Q}\rangle^3|f-\bar{f}|\|_\infty\\ &\le C\mu\|\langle y-\bar{Q}\rangle|u-\bar{u}|\|_\infty + C\|\langle y-\bar{Q}\rangle^3|f-\bar{f}|\|_\infty,\end{aligned}$$

which gives

$$\|\langle y-\bar{Q}\rangle|u-\bar{u}|\|_\infty \le C\|\langle y-\bar{Q}\rangle^3|f-\bar{f}|\|_\infty,$$

whence

$$\begin{aligned}\|\langle y-\bar{Q}\rangle u\|_\infty &\le C\|\langle y-\bar{Q}\rangle\|_\infty|\bar{u}| + C\varepsilon^{-3}|\bar{f}| + \|\langle y-\bar{Q}\rangle^3 f\|_\infty\\ &\le C\|f\|_{**}.\end{aligned}$$

Hence we finish the proof of the case $n=4$.

For $n=6$, by the help of Lemma 3.1,

$$\begin{aligned}|\langle y-\bar{Q}\rangle^2 u| \le & C\int_{\Omega_\varepsilon}\frac{\langle y-\bar{Q}\rangle^2(|\mu\varepsilon^2 u|+|f|)}{|x-y|^4}\mathrm{d}x\\ \le & C(|\mu\ln\varepsilon|\|u\|_{***}+\|f\|_{****}),\end{aligned}$$

where we used some similar estimates appeared in $n=4$. From the above inequality, we obtain $\|u\|_{***}\le\|f\|_{****}$. Hence we finish the proof. □

We now state the main result of this section

Proposition 3.3. *There exists $\varepsilon_0>0$ and a constant $C>0$, independent of $\varepsilon,\ \Lambda,\ \bar{Q}$ satisfying (2.11) and independent of $\varepsilon,\ \eta,\ \Lambda,\ \bar{Q}$, such that for all $0<\varepsilon<\varepsilon_0$ and all $h\in L^\infty(\Omega_\varepsilon)$, problem (3.5) and (3.9) has a unique solution $\phi=L_\varepsilon(h)$. Furthermore*

$$\begin{aligned}&\|L_\varepsilon(h)\|_*\le C\|h\|_{**},\ |c_i|\le C\|h\|_{**}\text{ for }0\le i\le 4,\\ &\|L_\varepsilon(h)\|_{***}\le C\|h\|_{****},\ |d_i|\le C\|h\|_{****}\text{ for }0\le i\le 6,\end{aligned}\tag{3.14}$$

Moreover, the map $L_\varepsilon(h)$ is C^1 with respect to $\Lambda,\ \bar{Q}$ of the L^∞_*-norm in $n=4$ and with respect to $\Lambda,\ \bar{Q},\ \eta$ of the L^∞_{***}-norm in $n=6$, i.e.,

$$\|D_{(\Lambda,\bar{Q})}L_\varepsilon(h)\|_*\le C\|h\|_{**}\text{ in }n=4,\quad \|D_{(\eta,\Lambda,\bar{Q})}L_\varepsilon(h)\|_{***}\le C\|h\|_{****}\text{ in }n=6.\tag{3.15}$$

The argument goes the same as the Proposition 3.1 in [20], for the convenience of the reader, we sketch the proof here. First, we need the following Lemma

Lemma 3.4. *For $n=4$, assume that ϕ_ε solves (3.5) for $h=h_\varepsilon$. If $\|h_\varepsilon\|_{**}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_*$. While for $n=6$, assume that ϕ_ε solves (3.9) for $h=h_\varepsilon$. If $\|h_\varepsilon\|_{****}$ goes to zero as ε goes to zero, so does $\|\phi_\varepsilon\|_{***}$.*

Proof. We prove this lemma by contradiction and first consider $n=4$. Assuming $\|\phi_\varepsilon\|_*=1$. Multiplying the first equation in (3.5) by Y_j and integrating in Ω_ε we find

$$\sum_i c_i\langle Z_i,Y_j\rangle = \langle -\Delta Y_j+\mu\varepsilon^2 Y_j-24W^2Y_j,\phi_\varepsilon\rangle - \langle h_\varepsilon,Y_j\rangle.$$

We can easily get the following equalities from the definition of Z_i, Y_j

$$\begin{aligned}&\langle Z_0,Y_0\rangle=\|Y_0\|_\varepsilon^2=\gamma_0+o(1),\\ &\langle Z_i,Y_i\rangle=\|Y_i\|_\varepsilon^2=\gamma_1+o(1),\ 1\le i\le 4,\end{aligned}\tag{3.16}$$

where γ_0,γ_1 are strictly positive constants, and

$$\langle Z_i,Y_j\rangle=o(1),\ i\ne j.\tag{3.17}$$

On the other hand, in view of the definition of Y_j and W, straightforward computations yield

$$\langle -\Delta Y_j + \mu\varepsilon^2 Y_j - 24W^2 Y_j, \phi_\varepsilon\rangle = o(\|\phi_\varepsilon\|_*)$$

and

$$\langle h_\varepsilon, Y_j\rangle = O(\|h_\varepsilon\|_{**}).$$

Consequently, inverting the quasi diagonal linear system solved by the c_i's we find

$$c_i = O(\|h_\varepsilon\|_{**}) + o(\|\phi_\varepsilon\|_*). \tag{3.18}$$

In particular, $c_i = o(1)$ as ε goes to zero.

Since $\|\phi_\varepsilon\|_* = 1$, elliptic theory shows that along some subsequence, the functions $\phi_{\varepsilon,0} = \phi_\varepsilon(y - \bar{Q})$ converge uniformly in any compact subset of $\mathbb{R}^4$ to a nontrivial solution of

$$-\Delta\phi_0 = 24U_{\Lambda,0}^2\phi_0.$$

A bootstrap argument (see e.g.Proposition 2.2 of [24]) implies $|\phi_0(y)| \le C(1+|y|)^{-2}$. As consequence, ϕ_0 can be written as

$$\phi_0 = \alpha_0\frac{\partial U_{\Lambda,0}}{\partial\Lambda} + \sum_i \alpha_i\frac{\partial U_{\Lambda,0}}{\partial y_i}$$

(see [19]). On the other hand, equalities $\langle Z_i, \phi_\varepsilon\rangle = 0$ yield

$$\begin{aligned}
&\int_{\mathbb{R}^4} -\Delta\frac{\partial U_{\Lambda,0}}{\partial\Lambda}\phi_0 = \int_{\mathbb{R}^4} U_{\Lambda,0}^2\frac{\partial U_{\Lambda,0}}{\partial\Lambda}\phi_0 = 0,\\
&\int_{\mathbb{R}^4} -\Delta\frac{\partial U_{\Lambda,0}}{\partial y_i}\phi_0 = \int_{\mathbb{R}^4} U_{\Lambda,0}^2\frac{\partial U_{\Lambda,0}}{\partial y_i}\phi_0 = 0,\ 1\le i\le 4.
\end{aligned}$$

As we also have

$$\int_{\mathbb{R}^4} |\nabla\frac{\partial U_{\Lambda,0}}{\partial\Lambda}|^2 = \gamma_0 > 0,\ \int_{\mathbb{R}^4} |\nabla\frac{\partial U_{\Lambda,0}}{\partial y_i}|^2 = \gamma_1 > 0,\ 1\le i\le 4,$$

and

$$\int_{\mathbb{R}^4} \nabla\frac{\partial U_{\Lambda,0}}{\partial\Lambda}\nabla\frac{\partial U_{\Lambda,0}}{\partial y_i} = \int_{\mathbb{R}^4} \nabla\frac{\partial U_{\Lambda,0}}{\partial y_i}\nabla\frac{\partial U_{\Lambda,0}}{\partial y_j} = 0,\ i\ne j,$$

the $\alpha_i's$ solve a homogeneous quasi diagonal linear system, yielding $\alpha_i = 0, 0\le i\le 4$, and $\phi_0 = 0$. So $\phi_\varepsilon(z-\bar{Q}) \to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Next, we will show $\|\phi_\varepsilon\|_* = o(1)$ by using the equation (3.5).

Using (3.5) and Corollary 3.2, we have

$$\|\phi_\varepsilon\|_* \le C(\|W^2\phi_\varepsilon\|_{**} + \|h\|_{**} + \sum_i |c_i|\|Z_i\|_{**}). \tag{3.19}$$

Then we estimate the right hand side of (3.19) term by term. By the help of (2.16), we deduce that

$$|\langle z-\bar{Q}\rangle^3 W^2\phi_\varepsilon| \le C\varepsilon^4(-\ln\varepsilon)\langle z-\bar{Q}\rangle^2\|\phi_\varepsilon\|_* + \langle z-\bar{Q}\rangle^{-1}|\phi_\varepsilon|. \tag{3.20}$$

Since $\|\phi_\varepsilon\|_* = 1$, the first term on the right hand side of (3.20) is dominated by $\varepsilon^2(-\ln\varepsilon)$. The last term goes uniformly to zero in any ball $B_R(\bar{Q})$, and is dominated by $\langle z-\bar{Q}\rangle^{-2}\|\phi_\varepsilon\|_* =$

$\langle z-\bar{Q}\rangle^{-2}$, which, through the choice of R, can be made as small as possible in $\Omega_\varepsilon\backslash B_R(\bar{Q})$. Consequently,

$$|\langle z-\bar{Q}\rangle^3 W^2\phi_\varepsilon| = o(1) \tag{3.21}$$

as ε goes to zero, uniformly in Ω_ε. On the other hand, we can also get

$$\begin{aligned}\varepsilon^{-3}(-\ln\varepsilon)^{\frac12}\overline{W^2\phi_\varepsilon} &\le C\varepsilon(-\ln\varepsilon)^{\frac12}\int_{\Omega_\varepsilon}(\langle z-\bar{Q}\rangle^{-4}+\varepsilon^4(-\ln\varepsilon))|\phi_\varepsilon|\\ &\le C\varepsilon(-\ln\varepsilon)^{-\frac12}\int_{\Omega_\varepsilon}(\langle z-\bar{Q}\rangle^{-5}+\varepsilon^4(-\ln\varepsilon)\langle z-\bar{Q}\rangle^{-1})\|\phi_\varepsilon\|_*\\ &= o(1).\end{aligned}$$

Finally, we obtain

$$\|W^2\phi_\varepsilon\|_{**} = o(1).$$

In view of the formula (3.4), we have

$$\langle z-\bar{Q}\rangle^3|Z_i| \le C(\langle z-\bar{Q}\rangle^3\varepsilon^4(\frac{1}{-\ln\varepsilon})+\langle z-\bar{Q}\rangle^{-3}) = O(1).$$

and

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac12}\overline{Z_i} \le C\varepsilon(-\ln\varepsilon)^{\frac12}\int_{\Omega_\varepsilon}|\langle z-\bar{Q}\rangle^{-6}+\varepsilon^4(\frac{1}{-\ln\varepsilon})^{\frac12}|\mathrm{d}x = o(1).$$

Hence, $\|Z_i\|_{**}=O(1)$. Therefore, we have

$$\|\phi_\varepsilon\|_* \le C(\|W^2\phi_\varepsilon\|_{**}+\|h\|_{**}+\sum_i|c_i|\|Z_i\|_{**}) = o(1), \tag{3.22}$$

which contradicts our assumption that $\|\phi_\varepsilon\|_*=1$.

Finally we turn to case of $n=6$. We still assume that $\|\phi_\varepsilon\|_{***}=1$. Using the similar arguments in previous case, we obtain the following

$$d_i = O(\|h\|_{****})+o(\|\phi\|_{***}) \text{ for } 0\le i\le 6,\ d_7=\varepsilon^{-2}O(\|h\|_{****})+(-\ln\varepsilon)o(\|\phi\|_{***}) \tag{3.23}$$

and $\phi_\varepsilon(z-\bar{Q})\to 0$ in $C^1_{loc}(\Omega_\varepsilon)$. Then, we will show $\|\phi\|_{***}=o(1)$ by using the equation (3.9). At first, we write the equation (3.9) into the following

$$-\Delta\phi+\mu\varepsilon^2(1-48\eta)\phi = h+\sum_i d_iZ_i+48U\phi+48\varepsilon^3\hat{U}\phi. \tag{3.24}$$

Using Corollary 3.2 again, we have

$$\|\phi_\varepsilon\|_{***} \le C(\|(U+\varepsilon^3\hat{U})\phi\|_{****}+\|h\|_{****}+\sum_i|d_i|\|Z_i\|_{****}). \tag{3.25}$$

From the formula of U and $\hat{U}$, it is not difficult to show

$$U+\varepsilon^3\hat{U} \le C\langle z-\bar{Q}\rangle^{-4}.$$

Similar to the case $n=4$, we could show $\|\langle z-\bar{Q}\rangle^{-4}\phi\|_{****}=o(1)$, $\|Z_i\|_{****}=O(1)$ for $0\le i\le 6$ and $\|Z_7\|_{****}=O(\varepsilon^2)$. Therefore, by the above facts and (3.23), we conclude

$$\|\phi\|_{***} \le o(1)+C\|h\|_{****}+o(1)\|\phi\|_{***} = o(1)$$

which contradicts the previous assumption that $\|\phi_\varepsilon\|_{***} = 1$. Hence, we finish the proof. $\square$

Proof of Proposition 3.3. Since the proof of the case $n = 4$ and $n = 6$ are almost the same, we only give the proof for the former one. We set

$$H = \{\phi \in H^1(\Omega_\varepsilon) \mid \langle Z_i, \phi\rangle = 0,\ 0 \le i \le 4\},$$

equipped with the scalar product $(\cdot,\cdot)_\varepsilon$. Problem (3.5) is equivalent to find $\phi \in H$ such that

$$(\phi, \theta)_\varepsilon = \langle 24W^2\phi + h, \theta\rangle, \quad \forall \theta \in H,$$

that is

$$\phi = T_\varepsilon(\phi) + \tilde{h}, \tag{3.26}$$

where $\tilde{h}$ depends on h linearly, and T_ε is a compact operator in H. Fredholm's alternative ensures the existence of a unique solution, provided that the kernel of $Id - T_\varepsilon$ is reduced to 0. We notice that any $\phi_\varepsilon \in \mathrm{Ker}(Id - T_\varepsilon)$ solves (3.5) with $h = 0$. Thus, we deduce from Lemma 3.4 that $\|\phi_\varepsilon\|_* = o(1)$ as ε goes to zero. As $\mathrm{Ker}(Id - T_\varepsilon)$ is a vector space and is $\{0\}$. The inequalities (3.14) follows from Lemma 3.4 and (3.18). This completes the proof of the first part of Proposition 3.3.

The smoothness of L_ε with respect to Λ and $\bar{Q}$ is a consequence of the smoothness of T_ε and $\tilde{h}$, which occur in the implicit definition (3.26) of $\phi \equiv L_\varepsilon(h)$, with respect to these variables. Inequality (3.15) is obtained by differentiating (3.5), writing the derivatives of ϕ with respect Λ and $\bar{Q}$ as linear combinations of the Z_i's and an orthogonal part, and estimating each term using the first part of the proposition see [7],[12] for detailed computations. $\square$

4. Finite-dimensional reduction:a nonlinear problem

In this section, we turn our attention to the nonlinear problem, which we solve in the finite-dimensional subspace orthogonal to the Z_i. Let $S_\varepsilon[u]$ be as defined at (1.12). Then (1.11) is equivalent to

$$S_\varepsilon[u] = 0 \text{ in } \Omega_\varepsilon, \quad u_+ \neq 0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon. \tag{4.1}$$

Indeed, if u satisfies (4.1) the Maximal Principle ensures that $u > 0$ in Ω_ε and (1.12) is satisfied. Observing that

$$S_\varepsilon[W+\phi] = -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - n(n-2)(W+\phi)^{\frac{n+2}{n-2}}$$

may be written as

$$S_\varepsilon[W+\phi] = -\Delta\phi + \mu\varepsilon^2\phi - n(n+2)W^{\frac{4}{n-2}}\phi + R^\varepsilon - n(n-2)N_\varepsilon(\phi) \tag{4.2}$$

with

$$N_\varepsilon(\phi) = (W+\phi)^{\frac{n+2}{n-2}} - W^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}W^{\frac{4}{n-2}}\phi \tag{4.3}$$

and

$$R^\varepsilon = S_\varepsilon[W] = -\Delta W + \mu\varepsilon^2 W - n(n-2)W^{\frac{n+2}{n-2}}. \tag{4.4}$$

From Lemma 2.1 we get

$$\begin{cases} \|R^\varepsilon\|_{**} + \|D_{(\Lambda,\bar{Q})} R^\varepsilon\|_{**} \le C\varepsilon, & n=4, \\ \|R^\varepsilon\|_{****} + \|D_{(\Lambda,\bar{Q},\eta)} R^\varepsilon\|_{****} \le C\varepsilon^{2\frac{2}{3}}, & n=6. \end{cases} \tag{4.5}$$

We now consider the following nonlinear problem: find ϕ such that, for some numbers c_i,

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 8(W+\phi)^3 = \sum_i c_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi \rangle = 0, & 0 \le i \le 4. \end{cases} \tag{4.6}$$

for $n = 4$, and find ϕ such that, for some numbers d_i,

$$\begin{cases} -\Delta(W+\phi) + \mu\varepsilon^2(W+\phi) - 24(W+\phi)^2 = \sum_i d_i Z_i & \text{in } \Omega_\varepsilon, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \langle Z_i, \phi \rangle = 0, & 0 \le i \le 7. \end{cases} \tag{4.7}$$

for $n = 6$. The first equation in (4.6) and (4.7) reads

$$\begin{aligned} -\Delta\phi + \mu\varepsilon^2\phi - 24W^2\phi &= 8N_\varepsilon(\phi) + R^\varepsilon + \sum_i c_i Z_i, \\ -\Delta\phi + \mu\varepsilon^2\phi - 48W\phi &= 24N_\varepsilon(\phi) + R^\varepsilon + \sum_i d_i Z_i. \end{aligned} \tag{4.8}$$

In order to employ the contraction mapping theorem to prove that (4.6) and (4.7) are uniquely solvable in the set where $\|\phi\|_*$ and $\|\phi\|_{***}$ are small respectively, we need to estimate N_ε in the following lemma.

Lemma 4.1. *There exists $\varepsilon_1 > 0$, independent of $\Lambda, \bar{Q}$, and C independent of $\varepsilon, \Lambda, \bar{Q}$ such that for $\varepsilon \le \varepsilon_1$ and*

$$\|\phi\|_* \le \varepsilon \text{ for } n=4, \quad \|\phi\|_{***} \le \varepsilon^{2\frac{2}{3}} \text{ for } n=6.$$

Then,

$$\|N_\varepsilon(\phi)\|_{**} \le C\varepsilon\|\phi\|_* \text{ for } n=4, \quad \|N_\varepsilon(\phi)\|_{****} \le C\varepsilon\|\phi\|_{***} \text{ for } n=6. \tag{4.9}$$

For

$$\|\phi_i\|_* \le \varepsilon \text{ for } n=4, \quad \|\phi_i\|_{***} \le \varepsilon^{2\frac{2}{3}} \text{ for } n=6, \qquad i=1,2.$$

Then,

$$\begin{aligned} &\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \le C\varepsilon\|\phi_1-\phi_2\|_* \text{ for } n=4, \\ &\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{****} \le C\varepsilon\|\phi_1-\phi_2\|_{***} \text{ for } n=6. \end{aligned} \tag{4.10}$$

Proof. Since the proof of these two cases are similar, we only consider $n = 4$ here. From (4.3), we see

$$|N_\varepsilon(\phi)| \le C(W\phi^2 + |\phi|^3). \tag{4.11}$$

Using (2.15), we gain

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}\overline{W\phi^2 + |\phi|^3} = \varepsilon(-\ln\varepsilon)^{\frac{1}{2}} \int_{\Omega_\varepsilon} (W\phi^2 + |\phi|^3),$$

where the integration term on the right hand side of the above equality can be estimated as

$$\begin{aligned}|W\phi^2+|\phi|^3| \le & C((\langle z-\bar{Q}\rangle^{-2}+\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}})|\phi|^2+|\phi|^3)\\ \le & C(\langle z-\bar{Q}\rangle^{-4}+\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}\langle z-\bar{Q}\rangle^{-2})\|\phi\|_*^2+\langle z-\bar{Q}\rangle^{-3}\|\phi\|_*^3\\ \le & C(\varepsilon\langle z-\bar{Q}\rangle^{-4}+\varepsilon^3(-\ln\varepsilon)^{\frac{1}{2}}\langle z-\bar{Q}\rangle^{-2})\|\phi\|_*.\end{aligned}$$

As a consequence,

$$\varepsilon^{-3}(-\ln\varepsilon)^{\frac{1}{2}}\overline{W\phi^2+|\phi|^3}\le C\varepsilon^2(-\ln\varepsilon)^{\frac{3}{2}}\|\phi\|_*\le C\varepsilon\|\phi\|_*.$$

On the other hand,

$$\|\langle z-\bar{Q}\rangle^3(W\phi^2+|\phi|^3)\|_\infty\le C\varepsilon\|\phi\|_*.$$

and (4.9) follows. Concerning (4.10), we write

$$N_\varepsilon(\phi_1)-N_\varepsilon(\phi_2)=\partial_\eta N_\varepsilon(\eta)(\phi_1-\phi_2)$$

for some $\eta=x\phi_1+(1-x)\phi_2,\ x\in[0,1]$. From

$$\partial_\eta N_\varepsilon(\eta)=3[(W+\eta)^2-W^2]$$

we deduce that

$$\partial_\eta N_\varepsilon(\eta)\le C(|W||\eta|+\eta^2) \tag{4.12}$$

and the proof (4.10) is similar to the previous one. □

Proposition 4.2. *For the case $n=4$, there exists C, independent of ε and Λ, $\bar{Q}$ satisfying (2.11), such that for small ε problem (4.6) has a unique solution $\phi=\phi(\Lambda,\bar{Q},\varepsilon)$ with*

$$\|\phi\|_*\le C\varepsilon. \tag{4.13}$$

Moreover, $(\Lambda,\bar{Q})\to\phi(\Lambda,\bar{Q},\varepsilon)$ is C^1 with respect to the $*$-norm, and

$$\|D_{\Lambda,\bar{Q}}\phi\|_*\le C\varepsilon. \tag{4.14}$$

For the case $n=6$, there exists C, independent of ε and Λ, η, $\bar{Q}$ satisfying (2.13), such that for small ε problem (4.7) has a unique solution $\phi=\phi(\Lambda,\eta,\bar{Q},\varepsilon)$ with

$$\|\phi\|_{***}\le C\varepsilon^{2\frac{2}{3}}. \tag{4.15}$$

Moreover, $(\Lambda,\bar{Q})\to\phi(\Lambda,\eta,\bar{Q},\varepsilon)$ is C^1 with respect to the $***$-norm, and

$$\|D_{\Lambda,\eta,\bar{Q}}\phi\|_{***}\le C\varepsilon^{2\frac{2}{3}}. \tag{4.16}$$

Proof. We only give the proof of $n=4$, the other case can be argued similarly. In the same spirit of [7], we consider the map A_ε from $\mathcal{F}$=$\{\phi\in H^1(\Omega_\varepsilon)|\|\phi\|_*\le C'\varepsilon\}$ to $H^1(\Omega_\varepsilon)$ defined as

$$A_\varepsilon(\phi)=L_\varepsilon(8N_\varepsilon(\phi)+R^\varepsilon).$$

Here C' is a large number, to be determined later, and L_ε is given by Proposition 3.3. We remark that finding a solution ϕ to problem (4.6) is equivalent to finding a fixed point of A_ε. On the one hand, we have for $\phi\in\mathcal{F}$, using (4.5), Proposition 3.3 and Lemma 4.1,

$$\begin{aligned}\|A_\varepsilon(\phi)\|_*&\le\|L_\varepsilon(N_\varepsilon(\phi))\|_*+\|L_\varepsilon(R^\varepsilon)\|_*\le C_1(\|N_\varepsilon(\phi)\|_{**}+\varepsilon)\\&\le C_2C'\varepsilon^2+C_1\varepsilon\le C'\varepsilon\end{aligned}$$

for $C^{'} = 2C_1$ and ε small enough, implying that A_ε sends $\mathcal{F}$ into itself. On the other hand, A_ε is a contraction. Indeed, for ϕ_1 and ϕ_2 in $\mathcal{F}$, we write

$$\|A_\varepsilon(\phi_1) - A_\varepsilon(\phi_2)\|_* \leq C\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_{**} \leq C\varepsilon\|\phi_1 - \phi_2\|_* \leq \frac{1}{2}\|\phi_1 - \phi_2\|_*$$

for ε small enough. The contraction Mapping Theorem implies that A_ε has a unique fixed point in $\mathcal{F}$, that is, problem (4.6) has a unique solution ϕ such that $\|\phi\|_* \leq C^{'}\varepsilon$.

In order to prove that $(\Lambda, \bar{Q}) \to \phi(\Lambda, \bar{Q})$ is C^1, we remark that if we set for $\psi \in F$,

$$B(\Lambda, \bar{Q}, \psi) \equiv \psi - L_\varepsilon(8N_\varepsilon(\psi) + R^\varepsilon)$$

then ϕ is defined as

$$B(\Lambda, \bar{Q}, \phi) = 0. \tag{4.17}$$

We have

$$\partial_\psi B(\Lambda, \bar{Q}, \psi)[\theta] = \theta - 8L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi)).$$

Using Proposition 3.3 and (4.12) we write

$$\begin{aligned}\|L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi))\|_* &\leq C\|\theta(\partial_\psi N_\varepsilon)(\psi)\|_{**} \leq \|\langle z - \bar{Q}\rangle^{-1}(\partial_\psi N_\varepsilon)(\psi)\|_{**}\|\theta\|_* \\ &\leq C\|\langle z - \bar{Q}\rangle^{-1}(W_+|\psi| + |\psi|^2)\|_{**}\|\theta\|_*.\end{aligned}$$

Using (2.16), (3.10) and $\psi \in \mathcal{F}$, we obtain

$$\|L_\varepsilon(\theta(\partial_\psi N_\varepsilon)(\psi))\|_* \leq \varepsilon\|\theta\|_*.$$

Consequently, $\partial_\psi B(\Lambda, \bar{Q}, \phi)$ is invertible with uniformly bounded inverse. Then the fact that $(\Lambda, \bar{Q}) \mapsto \phi(\Lambda, \bar{Q})$ is C^1 follows from the fact that $(\Lambda, \bar{Q}, \psi) \mapsto L_\varepsilon(N_\varepsilon(\psi))$ is C^1 and the implicit function theorem.

Finally, let's consider (4.14). Differentiating (4.17) with respect to Λ, we find

$$\partial_\Lambda\phi = (\partial_\psi B(\Lambda, \xi, \phi))^{-1}((\partial_\Lambda L_\varepsilon)(N_\varepsilon(\phi)) + L_\varepsilon((\partial_\Lambda N_\varepsilon)(\phi)) + L_\varepsilon(\partial_\Lambda R^\varepsilon)),$$

Then by Proposition 3.3,

$$\|\partial_\Lambda\phi\|_* \leq C(\|N_\varepsilon(\phi)\|_{**} + \|(\partial_\Lambda N_\varepsilon)(\phi)\|_{**} + \|\partial_\Lambda R^\varepsilon\|_{**}).$$

From Lemma 4.1 and (4.13), we know that $\|N_\varepsilon(\phi)\|_{**} \leq C\varepsilon^2$. Concerning the next term, we notice that according to the definition of N_ε,

$$|\partial_\Lambda N_\varepsilon(\phi)| = 3\phi^2|\partial_\Lambda W|.$$

Note that

$$|D_\Lambda W(z)| \leq C(\langle z - \bar{Q}\rangle^{-2} + \varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}}),$$

we have

$$\|\partial_\Lambda N_\varepsilon(\phi)\|_{**} \leq C\varepsilon.$$

Finally, using (4.5), we obtain

$$\|\partial_\Lambda\phi\|_* \leq C\varepsilon.$$

The derivative of ϕ with respect to $\bar{Q}$ may be estimated in the same way. This concludes the proof. $\square$

5. Finite-dimensional reduction:reduction energy

Let us define a reduced energy functional as

$$I_\varepsilon(\Lambda,\bar{Q})\equiv J_\varepsilon[W_{\Lambda,\bar{Q}}+\phi_{\varepsilon,\Lambda,\bar{Q}}] \tag{5.1}$$

for $n=4$ and

$$I_\varepsilon(\Lambda,\eta,\bar{Q})\equiv J_\varepsilon[W_{\Lambda,\eta,\bar{Q}}+\phi_{\varepsilon,\Lambda,\eta,\bar{Q}}] \tag{5.2}$$

for $n=6$. We have

Proposition 5.1. *The function $u=W_{\Lambda,\bar{Q}}+\phi_{\varepsilon,\Lambda,\bar{Q}}$ is a solution to problem (1.6) for $n=4$ if and only if $(\Lambda,\bar{Q})$ is a critical point of I_ε. The function $u=W_{\Lambda,\eta,\bar{Q}}+\phi_{\varepsilon,\Lambda,\eta,\bar{Q}}$ is a solution to problem (1.6) for $n=6$ if and only if $(\Lambda,\eta,\bar{Q})$ is a critical point of I_ε.*

Proof. Here we only give the proof for the case $n=6$, the other case can be proved in the same way. We notice that $u=W+\phi$ being a solution of (1.6) is equivalent to being a critical point of J_ε, which is also equivalent to the vanish of the d_i's in (4.7) or, in view of

$$\begin{aligned}
&\langle Z_0,Y_0\rangle=\|Y_0\|_\varepsilon^2=\gamma_0+o(1),\\
&\langle Z_i,Y_i\rangle=\|Y_i\|_\varepsilon^2=\gamma_1+o(1),\ 1\le i\le 6,\\
&\langle Z_7,Y_7\rangle=\|Y_7\|_\varepsilon^2=\gamma_2\varepsilon^3,
\end{aligned} \tag{5.3}$$

where $\gamma_0,\gamma_1,\gamma_2$ are strictly positive constants, and

$$\langle Z_i,Y_j\rangle=o(1),i\neq j,0\le i,j\le 6,\quad \langle Z_i,Y_j\rangle=o(\varepsilon^3),i\neq j,i=7\text{ or }j=7. \tag{5.4}$$

We have

$$J_\varepsilon'[W+\phi][Y_i]=0,\quad 0\le i\le 7. \tag{5.5}$$

On the other hand, we deduce from (5.2) that $I_\varepsilon'(\Lambda,\eta,Q)=0$ is equivalent to the cancelation of $J_\varepsilon'(W+\phi)$ applied to the derivative of $W+\phi$ with respect to Λ, η and $\bar{Q}$. By the definition of Y_i's and Proposition 4.2, we have

$$\frac{\partial(W+\phi)}{\partial\Lambda}=Y_0+y_0,\quad \frac{\partial(W+\phi)}{\partial\bar{Q}_i}=Y_i+y_i,\quad 1\le i\le 6,\quad \frac{\partial(W+\phi)}{\partial\eta}=Y_7+y_7$$

with $\|y_i\|_{***}=o(\varepsilon^2),\ 0\le i\le 7$. We write

$$y_i=y_i'+\sum_j a_{ij}Y_j,\quad \langle y_i',Z_j\rangle=(y_i',Y_j)_\varepsilon=0,\quad 0\le i,j\le 7,$$

and

$$J_\varepsilon'[W+\phi][Y_i]=\alpha_i,$$

where $a_{ij}=\langle y_i,Z_j\rangle$. It turns out that $I_\varepsilon'(\Lambda,\eta,\bar{Q})=0$ is equivalent, since $J_\varepsilon'[W+\phi][\theta]=0$ for $\langle\theta,Z_i\rangle=(\theta,Y_i)_\varepsilon=0,\ 0\le i\le 7$, to

$$([b_{ij}]+[a_{ij}])[\alpha_j]=0,$$

where $b_{ij} = \langle Y_i, Z_j \rangle$. Using the estimate $\|y_i\|_{***} = o(\varepsilon^2)$ and the expression of $Z_i, Y_i, 0 \le i \le 7$, we directly obtain

$$\begin{aligned}
&b_{00} = \gamma_0 + o(1), \quad b_{ii} = \gamma_1 + o(1) \text{ for } 1 \le i \le 6, \quad b_{77} = \gamma_2 \varepsilon^3, \\
&b_{ij} = o(1) \text{ for } 0 \le i \ne j \le 6, \quad b_{ij} = o(\varepsilon^3) \text{ for } i = 7 \text{ or } j = 7, i \ne j, \\
&a_{ij} = o(\varepsilon^2) \text{ for } 0 \le i \le 7, 0 \le j \le 6, \quad a_{i7} = o(\varepsilon^4) \text{ for } 0 \le i \le 7.
\end{aligned}$$

Then it is easy to see the matrix $[b_{ij}+a_{ij}]$ is invertible by the above estimates of each components, hence $\alpha_i = 0$. We see that $I'_\varepsilon(\Lambda, \eta, \bar{Q}) = 0$ means exactly that (5.5) is satisfied. $\square$

With Proposition 5.1, it remains to find critical points of I_ε. First, we establish an expansion of I_ε.

Proposition 5.2. *In the case $n = 4$, for ε sufficiently small, we have*

$$I_\varepsilon(\Lambda, \eta, Q) = J_\varepsilon[W] + \varepsilon^2 \sigma_\varepsilon(\Lambda, Q) \tag{5.6}$$

where $\sigma_\varepsilon = o(1)$ and $D_\Lambda(\sigma_\varepsilon) = o(1)$ as ε goes to 0, uniformly with respect to Λ, Q satisfying (2.11).

In the case $n = 6$, for ε sufficiently small, we have

$$I_\varepsilon(\Lambda, \eta, Q) = J_\varepsilon[W] + \varepsilon^4 \sigma_\varepsilon(\Lambda, \eta, Q) \tag{5.7}$$

where $\sigma_\varepsilon = o(1)$ and $D_\Lambda(\sigma_\varepsilon) = o(1)$ as ε goes to 0, uniformly with respect to Λ, η, Q satisfying (2.13).

Proof. We only consider the case $n = 6$ here, the left case can be argued similarly with minor changes. We first prove

$$I_\varepsilon(\Lambda, Q) - J_\varepsilon[W] = o(\varepsilon^4). \tag{5.8}$$

Actually, in view of (5.2), a Taylor expansion and the fact that $J'_\varepsilon[W + \phi][\phi] = 0$ yield

$$\begin{aligned}
I_\varepsilon(\Lambda, \eta, Q) - J_\varepsilon[W] =& J_\varepsilon[W + \phi] - J_\varepsilon[W] = -\int_0^1 J''_\varepsilon(W + t\phi)[\phi, \phi](t) \mathrm{d}t \\
=& -\int_0^1 \Big(\int_{\Omega_\varepsilon} (|\nabla \phi|^2 + \mu \varepsilon^2 \phi^2 - 48(W + t\phi)\phi^2)\Big) t \mathrm{d}t,
\end{aligned}$$

whence

$$\begin{aligned}
&I_\varepsilon(\Lambda, \eta, Q) - J_\varepsilon[W] \\
&= -\int_0^1 \Big(24 \int_{\Omega_\varepsilon} (N_\varepsilon(\phi)\phi + 2[W - (W + t\phi)]\phi^2)\Big) t \mathrm{d}t - \int_{\Omega_\varepsilon} R^\varepsilon \phi.
\end{aligned} \tag{5.9}$$

The first term on the right hand side of (5.9) can be estimated as

$$\Big| \int_{\Omega_\varepsilon} N_\varepsilon(\phi)\phi \Big| \le C \int_{\Omega_\varepsilon} |\phi|^3 = o(\varepsilon^5).$$

Similarly, for the second term on the right hand side of (5.9), we obtain

$$\Big| \int_{\Omega_\varepsilon} [W - (W + t\phi)]\phi^2 \Big| \le C \int_{\Omega_\varepsilon} |\phi|^3 = o(\varepsilon^5).$$

Concerning the last one, recalling

$$|R^\varepsilon| = |S_\varepsilon[W]| \leq \ C\varepsilon^{3\frac{2}{3}}\langle z-\bar{Q}\rangle^{-3},$$

uniformly in Ω_ε. A simple computation shows that

$$|\int_{\Omega_\varepsilon} R^\varepsilon\phi| = o(\varepsilon^4).$$

This concludes the proof of the first part of Proposition (5.7).

An estimate for the derivatives with respect to Λ is established exactly in the same way, differentiating the right side in (5.9) and estimating each term separately, using (4.3), (4.5) and Lemma 2.1. $\square$

6. Proof of Theorem 1.1

In this section, we prove the existence of a critical point of $I_\varepsilon(\Lambda, Q)$ and $I_\varepsilon(\Lambda,\eta,Q)$, thereby prove Theorem 1.1 by Proposition 5.1. According Proposition 5.2 and Lemma 2.1. Setting

$$K_\varepsilon(\Lambda,Q) = \frac{I_\varepsilon(\Lambda,Q) - 2\int_{\mathbb{R}^n} U^4}{(-\frac{\ln\varepsilon}{c_1})^{\frac{1}{2}}\varepsilon^2} \tag{6.1}$$

and

$$K_\varepsilon(\Lambda,\eta,Q) = \frac{I_\varepsilon(\Lambda,\eta,Q) - 4\int_{\mathbb{R}^n} U^3}{\varepsilon^3} \tag{6.2}$$

Then, we have when $n=4$,

$$\begin{aligned} K_\varepsilon(\Lambda,Q) =& \frac{1}{4}c_4\Lambda^2 \ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon}) - \frac{c_4^2\Lambda^2}{2|\Omega|} + \frac{1}{2}c_4^2\Lambda^2 H(Q,Q)(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}} \\ &+ O(\frac{c_1}{-\ln\varepsilon}), \end{aligned} \tag{6.3}$$

and when $n=6$,

$$K_\varepsilon(\Lambda,\eta,Q) =(\frac{1}{2}\eta^2|\Omega| - c_6\Lambda^2\eta + \frac{1}{48}c_6\Lambda^2 - 8\eta^3|\Omega|) + \frac{1}{2}c_6^2\Lambda^4 H(Q,Q)\varepsilon + O(\varepsilon^2). \tag{6.4}$$

Then we begin to consider $K_\varepsilon(\Lambda,Q)$, find its critical points with respect to Λ, Q, and $K_\varepsilon(\Lambda,\eta,Q)$ with its critical points with respect to the parameters Λ,η,Q.

First, we consider $K_\varepsilon(\Lambda,Q)$ for $n=4$. For the setting of the parameters Λ, Q, we see that Λ, Q are located on a compact set, we can obtain a maximal value of $K_\varepsilon(\Lambda,Q)$. We claim that:

Claim: The maximal point of $K_\varepsilon(\Lambda,Q)$ with respect to Λ, Q can not happen on the boundary of the parameters.

If we can prove this claim, then we could obtain an interior critical point of $K_\varepsilon(\Lambda, Q)$. Before proving the claim, we first consider

$$F_\varepsilon(\Lambda) = \frac{1}{4}c_4\Lambda^2 \ln \frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon}) - \frac{c_4^2\Lambda^2}{2|\Omega|}.$$

Note that

$$\frac{\partial}{\partial\Lambda}[F_\varepsilon\Lambda] = \frac{1}{2}c_4\Lambda \ln \frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon}) - \frac{1}{4}c_4\Lambda(\frac{c_1}{-\ln\varepsilon}) - \frac{c_4^2\Lambda}{|\Omega|},$$

Choosing $c_1 = \frac{2c_4}{|\Omega|}$, we could obtain that there exists

$$\Lambda^* = 1 + o(1) \in (\varepsilon^\beta, \varepsilon^{-\beta})$$

with some proper fixed constant $\beta \in (0, \frac{1}{3})$, such that

$$\frac{\partial}{\partial\Lambda}[F_\varepsilon\Lambda^*] = 0.$$

It can be also found that such Λ^* provides the maximal value of $F_\varepsilon(\Lambda)$ in $[\Lambda_{4,1}, \Lambda_{4,2}]$, where $\Lambda_{4,1} = \varepsilon^\beta, \Lambda_{4,2} = \varepsilon^{-\beta}$. In order to prove the above claim, we need a more accurate formula of the energy for $\Lambda \in [\varepsilon^\beta, \varepsilon^{-\beta}]$, . In other words, here we need to take Λ into consideration of the formula, go through the first part of the Appendix, we have

$$\begin{aligned} K_\varepsilon(\Lambda, Q) =& \frac{1}{4}c_4\Lambda^2 \ln \frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon}) - \frac{c_4^2\Lambda^2}{2|\Omega|} + \frac{1}{2}c_4^2\Lambda^2 H(Q,Q)(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}} \\ &+ O(\Lambda^2\frac{c_1}{-\ln\varepsilon}). \end{aligned}$$

Now, we come back to prove the claim, choosing $\Lambda = \Lambda^*$ and $Q = p$. (Here p refers to the point where $H(Q,Q)$ obtain its maximal value, it is possible to find such a point. Indeed, we notice a fact $H(Q,Q) \to -\infty$ as $d(Q, \partial\Omega_\varepsilon) \to 0$ see [20] and references therein for a proof of this fact. Therefore we could find such p.)

First, we prove that the maximal value can not happen $\partial\mathcal{M}_{\delta_4}$. We choose δ_4 such that $d_2 < \max_{\partial\mathcal{M}_{\delta_4}} H < d_1$ for some proper constant d_2, d_1 sufficiently negative, then we fixed $\mathcal{M}_{\delta_4}$. It is easy to see that $K_\varepsilon(\Lambda, Q) < K_\varepsilon(\Lambda, p)$, where Q lies on the boundary of $\mathcal{M}_{\delta_4}$ and $\Lambda \in (\Lambda_{4,1}, \Lambda_{4,2})$. For $\Lambda = \Lambda_{4,1}$ or $\Lambda_{4,2}$, we go to the arguments stated below. Therefore, we prove that the maximal point can not lie on the boundary of $\mathcal{M}_{\delta_4}$.

Next, we show $K_\varepsilon(\Lambda^*, p) > K_\varepsilon(\Lambda_{4,2}, Q)$. It is easy to see that

$$F_\varepsilon[\Lambda_{4,2}] \leq c\varepsilon^{-2\beta},$$

where $c < 0$. Then we can find $c_1 < 0$ such that $K_\varepsilon(\Lambda_{4,2}, Q) \leq c_1\varepsilon^{-2\beta}$ for the other terms compared to $\varepsilon^{-2\beta}$ are higher order term. On the other hand, for the choice of Λ^*, p, we see that $K_\varepsilon(\Lambda^*, p) \sim 1$. Therefore, we prove that $K_\varepsilon(\Lambda^*, p) > K_\varepsilon(\Lambda_{4,2}, Q)$.

It remains to prove that the maximal value can not happen at $\Lambda = \Lambda_{4,1}$. We choose $\Lambda = \varepsilon^{\beta/2}, Q = p$, and we show $K_\varepsilon(\varepsilon^{\beta/2}, p) > K_\varepsilon(\Lambda_{4,1}, Q)$. Direct computation yields.

$$K_\varepsilon(\varepsilon^{\beta/2}, p) = \frac{\beta c_4^2\varepsilon^\beta}{4|\Omega|}(1 + o(1)), \quad K_\varepsilon(\Lambda_{4,1}, Q) = \frac{\beta c_4^2\varepsilon^{2\beta}}{2|\Omega|}(1 + o(1)).$$

Hence, we finish the proof of the claim. In other words, we could obtain an interior maximal point in $[\Lambda_{4,1}, \Lambda 4, 2] \times \mathcal{M}_{\delta_4}$. Therefore, we show the existence of the critical points of $K_\varepsilon(\Lambda, Q)$ with respect to Λ, Q.

For $n = 6$. We set $\eta = \frac{1}{48} + a\varepsilon^{\frac{1}{3}}$, $\frac{c_6\Lambda^2}{|\Omega|} = \frac{1}{96} + b\varepsilon^{\frac{2}{3}}$, then

$$K_\varepsilon(a, b, Q) := K_\varepsilon(\Lambda, \eta, Q) = \Big[|\Omega|(\frac{1}{6912} - 8a^3 - ab) + \frac{1}{18432}|\Omega|^2 H(Q, Q)\Big]\varepsilon + o(\varepsilon), \tag{6.5}$$

where $-\eta_6 \le a \le \eta_6, -\Lambda_6 \le b \le \Lambda_6$. Setting $C_0 = H(p, p)$, p refers to the point where $H(p, p)$ obtains its maximal value. Let us introduce another five constants $C_i, i = 1, 2, 3, 4, 5$, with $C_2 < C_1 < C_0$, $0 < C_3 < C_4 < \eta_6$ and $0 < C_3 < C_5 < \Lambda_6$, the value of these five constants will be determined later.

We set

$$\Sigma_0 = \Big\{ - C_4 \le a \le C_4,\ -C_5 \le b \le C_5,\ Q \in \mathcal{N}_{C_2} \Big\}, \tag{6.6}$$

where $\mathcal{N}_{C_2} = \{q : H(q, q) > C_2\}$

We also define

$$B = \{(a, b, Q) \mid (a, b) \in B_{C_3}(0),\ Q \in \mathcal{N}_{C_1}\},\ B_0 = \{(a, b) \mid (a, b) \in B_{C_3}(0)\} \times \partial\mathcal{N}_{C_1}, \tag{6.7}$$

where $B_c(0) := \{0 \le a^2 + b^2 \le c\}$.

It is trivial to see that $B_0 \subset B \subset \Sigma_0$, B are compact and B_0 is connected. Let Γ be the class of continuous functions $\varphi : B \to \Sigma_0$ with the property that $\varphi(y) = y, y = (a, b, Q)$ for all $y \in B_0$. Define the min-max value c as

$$c = \min_{\varphi\in\Gamma} \max_{y\in B} K_\varepsilon(\varphi(y)).$$

We now show that c defines a critical value. To this end, we just have to verify the following conditions

- T1 $\max_{y\in B_0} K_\varepsilon(\varphi(y)) < c,\ \forall\varphi \in \Gamma$,
- T2 For all $y \in \partial\Sigma_0$ such that $K_\varepsilon(y) = c$, there exists a vector τ_y tangent to $\partial\Sigma_0$ at y such that

$$\partial_{\tau_y} K_\varepsilon(y) \ne 0.$$

Suppose T1 and T2 hold. Then standard deformation argument ensures that the min-max value c is a (topologically nontrivial) critical value for $K_\varepsilon(\Lambda, \eta, Q)$ in Σ_0. (Similar notion has been introduced in [8]) for degenerate critical points of mean curvature.)

To check T1 and T2, we define $\varphi(y) = \varphi(a, b, Q) = (\varphi_a, \varphi_b, \varphi_Q)$ where $(\varphi_a, \varphi_b) \in [-C_4, C_4] \times [-C_5, C_5]$ and $\varphi_Q \in \mathcal{N}_{C_2}$.

For any $\varphi \in \Gamma$ and $Q \in \mathcal{N}_{C_2}$, the map $Q \to \varphi_Q(a, b, Q)$ is a continuous function from $\mathcal{N}_{C_1}$ to $\mathcal{N}_{C_2}$ such that $\varphi_Q(a, b, Q) = Q$ for $Q \in \partial\mathcal{N}_{C_1}$. Let $\mathcal{D}$ be the smallest ball which contain $\mathcal{N}_{C_1}$, we extend φ_Q to a continuous function $\tilde{\varphi}_Q$ from $\mathcal{D}$ to $\mathcal{D}$ where $\tilde{\varphi}(Q)$ is defined as follows:

$$\tilde{\varphi}_Q(x) = \varphi(x),\ x \in \mathcal{N}_{C_1}, \quad \tilde{\varphi}_Q(x) = Id,\ x \in \mathcal{D} \setminus \mathcal{N}_{C_1}.$$

Then we claim there exists $Q' \in \mathcal{D}$ such that $\tilde{\varphi}(Q') = p$. (p stands for the point where $H(p, p)$ obtain its maximal value). Otherwise $\frac{\tilde{\varphi}(Q)-p}{|\tilde{\varphi}Q-p|}$ provides a continuous map from $\mathcal{D}$ to S^5, which is impossible in algebraic topology. Hence, there exists $Q' \in \mathcal{D}$ such that $\tilde{\varphi}(Q') = p$. By the definition of $\tilde{\varphi}$, we can further conclude $Q' \in \mathcal{N}_{C_1}$. Whence

$$\begin{aligned}\max_{y\in B} K_\varepsilon(\varphi(y)) \geq & K_\varepsilon(\varphi_a(a,b,Q'), \varphi_b(a,b,Q'), p) \\ \geq & (\frac{1}{18432}|\Omega|^2 C_0 + \frac{1}{6912}|\Omega| - C_6|\Omega|)\varepsilon + o(\varepsilon),\end{aligned} \tag{6.8}$$

where $C_6 = 8C_4^3 + C_4C_5$ which stands for the maximal value of $8a^3 + ab$ in $[-C_4, C_4] \times [-C_5, C_5]$. As a consequence

$$c \geq (\frac{1}{18432}|\Omega|^2 C_0 + \frac{1}{6912}|\Omega| - C_6|\Omega|)\varepsilon + o(\varepsilon). \tag{6.9}$$

For $(a, b, Q) \in B_0$, we have $H(\varphi_Q(a, b, Q), \varphi_Q(a, b, Q)) = C_1$. So we have

$$K_\varepsilon(a,b,Q) \leq (\frac{1}{18432}|\Omega|^2 C_1 + \frac{1}{6912}|\Omega| + C_7|\Omega|)\varepsilon + o(\varepsilon), \tag{6.10}$$

where $C_7 = \max_{(a,b)\in B_{C_3}(0)} 8a^3 + ab < 8C_3^3 + C_3^2$.

If we choose $\frac{1}{18432}|\Omega|(C_0 - C_1) > C_6 + C_7 = 8C_4^3 + C_4C_5 + 8C_3^3 + C_3^2$. Then $\max_{y\in B_0} K_\varepsilon(\varphi(y)) < c$ holds. So T1 is verified.

To verify T2, we observe that

$$\partial\Sigma_0 =: \{a, b, Q \mid a = -C_4 \text{ or } a = C_4 \text{ or } b = -C_5 \text{ or } b = C_5 \text{ or } Q \in \partial\mathcal{N}_{C_2}\}.$$

Since C_4, C_5 are arbitrary, we choose $0 < 24C_4^2 < C_5$. Then on $a = -C_4$ or $a = C_4$, we choose $\tau_y = \frac{\partial}{\partial b}$, on $b = -C_5$ or $b = C_5$, we choose $\tau_y = \frac{\partial}{\partial a}$. By our setting, we could show $\partial_{\tau_y} K_\varepsilon(y) \neq 0$. It only remains to consider the case $Q \in \partial\mathcal{N}_{C_2}$. If $Q \in \partial\mathcal{N}_{C_2}$, then

$$K_\varepsilon(a,b,Q) \leq (\frac{1}{18432}|\Omega|^2 C_2 + \frac{1}{6912}|\Omega| + C_7|\Omega|)\varepsilon + o(\varepsilon), \tag{6.11}$$

which is obviously less than c for $C_2 < C_1$. So T2 is also verified.

In conclusion, we proved that for ε sufficiently small, c is a critical value, i.e., a critical point $(a, b, Q) \in \Sigma_0$ of K_ε exists. Which means K_ε indeed has critical points respect to Λ, η, Q in (2.13).

Proof of Theorem 1.1 completed. For $n = 4$, we proved that for ε small enough, I_ε has a critical point $(\Lambda^\varepsilon, Q^\varepsilon)$. Let $u_\varepsilon = W_{\Lambda^\varepsilon, \bar{Q}^\varepsilon, \varepsilon}$. Then u_ε is a nontrivial solution to problem (1.12). The strong maximal principle shows $u_\varepsilon > 0$ in Ω_ε. Let $u_\mu = \varepsilon^{-1}u_\varepsilon(x/\varepsilon)$. By our construction, u_μ has all the properties stated in Theorem 1.1.

For $n = 6$, we proved that for ε small enough, I_ε has a critical point $(\Lambda^\varepsilon, \eta^\varepsilon, Q^\varepsilon)$. Let $u_\varepsilon = W_{\Lambda^\varepsilon, \eta^\varepsilon, \bar{Q}^\varepsilon, \varepsilon}$. Then u_ε is a nontrivial solution to problem (1.12). The strong maximal principle shows $u_\varepsilon > 0$ in Ω_ε. Let $u_\mu = \varepsilon^{-2}u_\varepsilon(x/\varepsilon)$. By our construction, u_μ has all the properties stated in Theorem 1.1. □

7. Appendix A: Proof of Lemma 2.1

We divide the proof into two parts.

First, we study the case $n = 4$. From the definition (2.12) of W, (2.10) and (2.15), we know that

$$\begin{aligned}
S_\varepsilon[W] =& -\Delta W + \mu\varepsilon^2 - 8W^3 \\
=& 8U^3 + \varepsilon^4(\frac{c_1}{-\ln\varepsilon})\hat{U} - \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\Delta(R_{\varepsilon,\Lambda,Q}\chi) - 8W^3 \\
=& O\Big(\langle z-\bar{Q}\rangle^{-2}\varepsilon^4(-\ln\varepsilon) + \langle z-\bar{Q}\rangle^{-4}\varepsilon^2(-\ln\varepsilon)^{\frac{1}{2}} + \frac{\varepsilon^4}{-\ln\varepsilon}|\ln\frac{1}{\varepsilon\langle z-\bar{Q}\rangle}| \\
&+ \frac{\varepsilon^4}{(-\ln\varepsilon)^{\frac{1}{2}}}\Big).
\end{aligned}$$

Estimates for $D_\Lambda S_\varepsilon[W]$ and $D_{\bar{Q}}S_\varepsilon[W]$ are obtained in the same way.

We now turn to the proof of the energy estimate (2.22). From (2.15) and (2.16) we deduce that

$$\begin{aligned}
\int_{\Omega_\varepsilon}|\nabla W|^2 + \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\int_{\Omega_\varepsilon}W^2 =& 8\int_{\Omega_\varepsilon}U^3W + \varepsilon^4(\frac{c_1}{-\ln\varepsilon})\int_{\Omega_\varepsilon}\hat{U}W \\
&- \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\Delta(R\chi)W.
\end{aligned} \tag{7.1}$$

Concerning the first term on the right hand side of (7.1), we have

$$\int_{\Omega_\varepsilon}U^3W = \int_{\Omega_\varepsilon}U^4 + \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}U^3 + \eta\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\int_{\Omega_\varepsilon}U^3. \tag{7.2}$$

We note that

$$\int_{\Omega_\varepsilon}U^4 = \int_{\mathbb{R}^4}U_{1,0}^4 + o(\varepsilon^2), \quad \int_{\Omega_\varepsilon}U^3 = \frac{c_4\Lambda}{8} + O(\varepsilon^2).$$

Then, we get

$$\int_{\Omega_\varepsilon}U^3W = \int_{\mathbb{R}^4}U_{1,0}^4 + \frac{c_4^2\Lambda^2}{8|\Omega|}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} + \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\int_{\Omega_\varepsilon}\hat{U}U^3 + o(\varepsilon^2),$$

where for the third term on the right hand side, we have

$$\begin{aligned}
\int_{\Omega_\varepsilon}\hat{U}U^3 =& -\int_{\Omega_\varepsilon}\Psi U^3 - c_4(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\Lambda\int_{\Omega_\varepsilon}H(x,Q)U^3 + O(1) \\
=& -\frac{\Lambda^2}{16}\ln\frac{1}{\Lambda\varepsilon}c_4 - \frac{c_4^2\Lambda^2}{8}(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}H(Q,Q) + O(1).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\int_{\Omega_\varepsilon}U^3W =& \int_{\mathbb{R}^4}U_{1,0}^4 + \frac{c_4^2\Lambda^2}{8|\Omega|}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} - \frac{c_4\Lambda^2}{16}\ln\frac{1}{\Lambda\varepsilon}\varepsilon^2(\frac{c_1}{-\ln\varepsilon}) - \frac{c_4^2\Lambda^2}{8|\Omega|}\varepsilon^2H(Q,Q) \\
&+ O(\varepsilon^2(\frac{1}{-\ln\varepsilon})).
\end{aligned} \tag{7.3}$$

For the second term on the right hand side of (7.1)

$$\int_{\Omega_\varepsilon} \hat{U}W = \int_{\Omega_\varepsilon} \hat{U}U + \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}} \int_{\Omega_\varepsilon} \hat{U}^2 + \eta\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} \int_{\Omega_\varepsilon} \hat{U}.$$

By noting that

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \hat{U}U = O(\varepsilon^{-2}), \ \int_{\Omega_\varepsilon} \hat{U}^2 = O(\varepsilon^{-4}(\frac{c_1}{-\ln\varepsilon})^{-1}),\\
&\int_{\Omega_\varepsilon} \hat{U} = -\varepsilon^{-4}(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} \int_{\Omega} \frac{\Lambda}{|x-Q|^2} + O(\varepsilon^{-4}),
\end{aligned}$$

we obtain

$$\varepsilon^4(\frac{c_1}{-\ln\varepsilon}) \int_{\Omega_\varepsilon} \hat{U}W = -\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} \int_{\Omega} \frac{c_4\Lambda^2}{|x-Q|^2|\Omega|}. \tag{7.4}$$

For the last term on the right hand side of (7.1),

$$\begin{aligned}
\int_{\Omega_\varepsilon} \Delta(R\chi)W =&\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\eta \int_{\Omega_\varepsilon} \Delta(R\chi) + O(1)\\
=&\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\eta \int_{\partial\Omega_\varepsilon} \frac{\partial(R\chi)}{\partial\nu}\\
=&\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\eta \int_{\partial\Omega_\varepsilon} \frac{\partial(U-\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\Psi - c_4\Lambda\varepsilon^2 H)}{\partial\nu}\\
=&\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\eta \int_{\Omega_\varepsilon} \Delta(U-\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\Psi - c_4\Lambda\varepsilon^2 H)\\
=&\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\eta \int_{\Omega_\varepsilon} (-8U^3 + \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}U + c_4\Lambda\varepsilon^4\frac{1}{|\Omega|})\\
=&(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} \int_{\Omega} \frac{c_4\Lambda^2}{(\varepsilon^2\Lambda^2+|x-Q|^2)|\Omega|}.
\end{aligned} \tag{7.5}$$

(7.3)-(7.5) implies

$$\begin{aligned}
\frac{1}{2}\int_{\Omega_\varepsilon} \Big(|\nabla W|^2 + \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}W^2\Big) =&4\int_{\mathbb{R}^4} U_{1,0}^4 + \varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\frac{c_4^2\Lambda^2}{2|\Omega|} - \frac{c_4^2\Lambda^2}{2}H(p,p)\varepsilon^2\\
&- \frac{c_4\Lambda^2}{4}\ln\frac{1}{\Lambda\varepsilon}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}} + O(\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}).
\end{aligned} \tag{7.6}$$

At last, we compute the term $\int_{\Omega_\varepsilon} W^4$.

$$\begin{aligned}
\int_{\Omega_\varepsilon} W^4 =& 4\int_{\Omega_\varepsilon} U^4 + 4\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}\int_{\Omega_\varepsilon} U^3\hat{U} + 4\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}}\eta\int_{\Omega_\varepsilon} U^3 \\
&+ O(\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}) \\
=& \int_{\mathbb{R}^4} U_{1,0}^4 - \frac{c_4\Lambda^2}{4}\varepsilon^2\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}} - \frac{c_4^2\Lambda^2}{2}\varepsilon^2 H(Q,Q) \\
&+ \frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} + O(\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}).
\end{aligned}\tag{7.7}$$

Combining (7.6) and (7.7), we obtain

$$\begin{aligned}
J_\varepsilon[W] =& \frac{1}{2}\int_{\Omega_\varepsilon}|\nabla W|^2 + \frac{\mu\varepsilon^2}{2}\int_{\Omega_\varepsilon} W^2 - 2\int_{\Omega_\varepsilon} W^4 \\
=& 2\int_{\mathbb{R}^4} U_{1,0}^4 + \frac{c_4\Lambda^2}{4}\varepsilon^2\ln\frac{1}{\Lambda\varepsilon}(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}} - \frac{c_4^2\Lambda^2}{2|\Omega|}\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{-\frac{1}{2}} \\
&+ \frac{1}{2}c_4^2\Lambda^2\varepsilon^2 H(Q,Q) + O(\varepsilon^2(\frac{c_1}{-\ln\varepsilon})^{\frac{1}{2}}).
\end{aligned}\tag{7.8}$$

In the end of this section, we prove (2.23)-(2.26). From the definition (2.14), of W, (2.10) and (2.15), we know that

$$\begin{aligned}
S_\varepsilon[W] =& -\Delta W + \varepsilon^3 W - 24W^2 \\
=& 24U^2 + \varepsilon^6\hat{U} - \varepsilon^3\Delta(R\chi) + \varepsilon^6(\eta - \frac{c_6\Lambda^2}{|\Omega|}) - 24U^2 - 24\eta\varepsilon^2 + O(\varepsilon^3\langle z-\bar{Q}\rangle^{-4}) \\
=& -\varepsilon^6(24\eta^2 - \eta + \frac{c_6\Lambda^2}{|\Omega|}) + O(\varepsilon^3\langle z-\bar{Q}\rangle^{-4}) \\
=& O(\langle z-\bar{Q}\rangle^{-3\frac{2}{3}}\varepsilon^3).
\end{aligned}$$

Estimates for $D_\Lambda S_\varepsilon[W]$, $D_{\bar{Q}} S_\varepsilon[W]$ and $D_\eta S_\varepsilon[W]$ are derived in the same way. Now we are in the position to compute the energy. From (2.15) and (2.16), we deduce that

$$\begin{aligned}
\int_{\Omega_\varepsilon}|\nabla W|^2 + \int_{\Omega_\varepsilon}\varepsilon^3 W^2 =& \int_{\Omega_\varepsilon}(-\Delta W + \varepsilon^3 W)W \\
=& \int_{\Omega_\varepsilon}\Big(24U^2 + \varepsilon^6\hat{U} - \varepsilon^3\Delta(R\chi) + \varepsilon^6(\eta - \frac{c_6\Lambda^2}{|\Omega|})\Big)W.
\end{aligned}\tag{7.9}$$

Concerning the first term on the right hand side of (7.9), we have

$$\begin{aligned}\int_{\Omega_\varepsilon} U^2 W &= \int_{\Omega_\varepsilon} U^3 + \varepsilon^3 \int_{\Omega_\varepsilon} \hat{U} U^2 + \eta\varepsilon^3 \int_{\Omega_\varepsilon} U^2 \\ &= \int_{\mathbb{R}^6} U_{1,0}^3 + \frac{1}{24} c_6 \eta \Lambda^2 \varepsilon^3 - \varepsilon^3 \int_{\Omega_\varepsilon} U^2 \Psi - c_6 \Lambda^2 \varepsilon^4 \int_{\Omega_\varepsilon} U^2 H + O(\varepsilon^5) \\ &= \int_{\mathbb{R}^6} U_{1,0}^3 + \frac{1}{24} c_6 \eta \Lambda^2 \varepsilon^3 - \frac{1}{24} c_6^2 \Lambda^4 \varepsilon^4 H(Q,Q) - \frac{1}{576} c_6 \Lambda^2 \varepsilon^3 + O(\varepsilon^5). \end{aligned} \tag{7.10}$$

For the second, third and fourth term on the right hand side of (7.9), following a similar as we did in case $n = 4$.

$$\varepsilon^6 \int_{\Omega_\varepsilon} \hat{U} W = \varepsilon^6 \int_{\Omega_\varepsilon} \hat{U}(U + \varepsilon^3 \hat{U} + \eta \varepsilon^3) = -\eta \Lambda^2 \varepsilon^4 \int_{\Omega} \frac{1}{|x-Q|^4} + O(\varepsilon^5), \tag{7.11}$$

$$-\varepsilon^3 \int_{\Omega_\varepsilon} \Delta(R\chi) W = \varepsilon^3 \eta \int_{\Omega_\varepsilon} \Delta(U - \varepsilon^3 \Psi - c_6 \varepsilon^4 \Lambda^2 H) + O(\varepsilon^5) = \varepsilon^6 \eta \int_{\Omega} U + O(\varepsilon^5), \tag{7.12}$$

and

$$\varepsilon^6 \int_{\Omega_\varepsilon} W = (\eta^2 |\Omega| - c_6 \Lambda^2 \eta) \varepsilon^3 + O(\varepsilon^5). \tag{7.13}$$

(7.10)-(7.13) implies

$$\begin{aligned}\frac{1}{2} \int_{\Omega_\varepsilon} |\nabla W|^2 + \frac{\varepsilon^3}{2} \int_{\Omega_\varepsilon} W^2 =& 12 \int_{\mathbb{R}^6} U_{1,0}^3 + (\frac{1}{2} \eta^2 |\Omega| - \frac{1}{48} c_6 \Lambda^2) \varepsilon^3 - \frac{c_6^2 \Lambda^4}{2} H(Q,Q) \varepsilon^4 \\ &+ O(\varepsilon^5). \end{aligned} \tag{7.14}$$

Then,

$$\begin{aligned}\int_{\Omega_\varepsilon} W^3 =& \int_{\mathbb{R}^6} U_{1,0}^3 + 3\varepsilon^3 \int_{\Omega_\varepsilon} U^2 \hat{U} + 3\varepsilon^3 \int_{\Omega_\varepsilon} U^2 \eta + 3\varepsilon^6 \int_{\Omega_\varepsilon} U \eta^2 + 3\varepsilon^9 \int_{\Omega_\varepsilon} \hat{U} \eta^2 \\ &+ \varepsilon^9 \int_{\Omega_\varepsilon} \eta^3 + O(\varepsilon^5) \\ =& \int_{\mathbb{R}^6} U_{1,0}^3 + \frac{1}{8} c_6 \eta \Lambda^2 \varepsilon^3 - \frac{1}{192} c_6 \Lambda^2 \varepsilon^3 + \eta^3 |\Omega| \varepsilon^3 - \frac{1}{8} c_6^2 \Lambda^4 H(Q,Q) \varepsilon^4 + O(\varepsilon^5). \end{aligned} \tag{7.15}$$

Combining (7.14)-(7.15), we gain the energy

$$\begin{aligned}J_\varepsilon[W] =& 4 \int_{\mathbb{R}^6} U_{1,0}^3 + \Big(\frac{1}{2} \eta^2 |\Omega| - c_6 \Lambda^2 \eta + \frac{1}{48} c_6 \Lambda^2 - 8\eta^3 |\Omega|\Big) \varepsilon^3 \\ &+ \frac{1}{2} c_6^2 \Lambda^4 H(Q,Q) + O(\varepsilon^5). \end{aligned} \tag{7.16}$$

Hence, we finish the whole proof of Lemma 2.1. □

References

[1] Adimurthi, Yadava, S. L.: Existence and nonexistence of positive radial solutions of Neumann problems with critical Sobolev exponents. *Arch. Rat. Mech. Anal.* **115** (1991), 275-296.

[2] Adimurthi, Yadava, S. L.: On a conjecture of Lin-Ni for a semilinear Neumann problem. *Trans. Amer. Math. Soc.* **336** (1993), 631-637.

[3] Adimurthi, Yadava, S. L.: Nonexistence of positive radial solutions of a quasilinear Neumann problem with a critical Sobolev exponent. *Arch. Rat. Mech. Anal.* **139** (1997), 239-253.

[4] A. Bahri, Y.Y. Li, O. Rey: On a variational problem with lack of compactness: the topological effect of the critical points at infinity. *Calc. Var. Partial Differential Equations* **3** (1995), 67 - 93.

[5] Budd, C., Knapp, M., Peletier, L.: Asymptotic behavior of solutions of elliptic equations with critical exponent and Neumann boundary conditions. *Proc. Roy. Soc. Edinburgh.* **117** (1991), 225-250.

[6] Caffarelli, L., Gidas, B., Spruck, J.: Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.* **42** (1989), 271-297.

[7] del Pino, M., Felmer, P., Musso, M.: Two-bubble solutions in the super-critical Bahri-Coron's problem. *Calc. Var. Partial Differential Equations.* **16** (2003), 113-145.

[8] del Pino, M., Felmer, P., Wei, J.C.: On the role of mean curvature in some singularly perturbed Neumann problems. *SIAM J. Math. Anal.* 31(1999), no. 1, 63-79.

[9] Druet, O., Hebey, E., Robert, F.: Blow-up theory for elliptic PDEs in Riemannian geometry. Mathematical Notes, 45. Princeton University Press, Princeton, NJ, 2004. viii+218 pp.

[10] Druet, O., Robert, F., Wei, J.C.: The Lin-Ni's problem for mean convex domains. *Mem. Amer. Math. Soc,* 218 (2012), no. 1027, vi+105 pp.

[11] A. Gierer, H. Meinhardt, A theory of biological pattern formation. Kybernetik (Berlin) 12 (1972), 30-39.

[12] Khenissy, S., Rey, O.: A criterion for existence of solutions to the supercritical Bahri-Coron's problem. *Houston J. Math.* **30** (2004), 587-613.

[13] Y.Y. Li, J. Wei, H. Xu: Multi-bump solutions of $-\Delta u = K(x)u^{\frac{n+2}{n-2}}$ on lattices in R^n, preprint 2013.

[14] Lin, C. S., Ni, W. M.: *On the diffusion coefficient of a semilinear Neumann problem*, Lecture Notes in Mathematics 1340 (Springer, Berlin, 1986) 160-174

[15] Lin, C. S., Ni, W. M., Takagi, I.: Large amplitude stationary solutions to a chemotaxis system. *J. Differential Equations.* **72** (1988), 1-27.

[16] Lin, F.H., Ni, W. M., Wei, J. C.: On the number of solutions for some singularly perturbed Neumann problems, *Comm. Pure Appl. Math.* **60** (2007), no. 2, 252-281.

[17] Ni, W.M.: Qualitative properties of solutions to elliptic problems. Stationary partial differential equations. Vol. I, 157-233, Handb. Diker. Equ., North-Holland, Amsterdam, 2004.

[18] Ni, W. M., Takagi, I.: On the shape of least-energy solutions to a semilinear Neumann problem. *Comm. Pure Appl. Math.* **44** (1991), 819-851.

[19] Rey, O.: The role of the Green's function in a nonlinear elliptic problem involving the critical Sobolev exponent. *J. Funct. Anal.* **89** (1990), 1-52.

[20] Rey, O., Wei, J.: Arbitrary number of positive solutions for an elliptic problem with critical nonlinearity. *J. Eur. Math. Soc.* **7** (2005), 449-476.

[21] Rey, O., Wei, J.: Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity. I. N=3. *J. Funct. Anal.* **212** (2004), no. 2, 472-499.

[22] Wang, L., Wei, J., Yan, S.: A Neumann Problem with Critical Exponent in Non-convex Domains and Lin-Ni's Conjecture. *Transaction of Amer. Math. Soc.* **361** (2009), 1189-1208.

[23] Wang, L., Wei, J., Yan, S.: On Lin-Ni's conjecture in convex domains. *Proc. Lond. Math. Soc.* **(3) 102** (2011), no. 6, 1099-1126.

[24] Wang, X., Wei, J.: On the equation $\Delta u + K(x)u^{\frac{n+2}{n-2}\pm\varepsilon^2}$ in $\mathbb{R}^n$. *Rend. Circ. Mat. Palermo.* **44** (1995) , 365-400.

[25] Wei, J.: Existence and stability of spikes for the Gierer-Meinhardt system. Handbook of differential equations: stationary partial differential equations. Vol. V, 487-585, Handb. Diker. Equ., Elsevier/North-Holland, Amsterdam, 2008.

[26] Wei, J., Xu, X.: Uniqueness and a priori estimates for some nonlinear elliptic Neumann equations in $\mathbb{R}^3$. *Pacific J. Math.* **221** (2005), no. 1, 159-165.

[27] J. Wei, S. Yan, Infinitely many solution for prescribed curvature problem on S^N. *J. Func. Anal.* **258** (2010), 3048 - 3081.

[28] Zhu, M.: Uniqueness results through a priori estimates, I. A three dimensional Neumann problem. *J. Differential Equations.* **154** (1999), 284-317.

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