CLUSTERED SOLUTIONS AROUND HARMONIC CENTERS TO A COUPLED ELLIPTIC SYSTEM

SOLUTIONS EN GRAPPE AUTOUR DES CENTRES HARMONIQUES D'UN SYSTÈME ELLIPTIQUE COUPLÉ

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ABSTRACT. We study the following system of Schrödinger-Maxwell equations

$$\varepsilon^2\Delta v - v - \omega v\phi + f(v) = 0, \quad \Delta\phi + \gamma v^2 = 0 \text{ in } \Omega,$$
$$v,\ \phi > 0 \text{ in } \Omega, \quad v,\ \phi = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth and bounded domain of $\mathbb{R}^3$. We prove that for any integer k the system has a family of solutions $(v_\varepsilon, \phi_\varepsilon)$ such that the form of v_ε consists of k spikes concentrating at a harmonic center of Ω as $\varepsilon \to 0^+$. Furthermore we show that the spikes approach the vertexes of a configuration which maximizes a suitable geometrical problem.

RÉSUMÉ. On étudie le système d'équations de Schrödinger-Maxwell suivant:

$$\varepsilon^2\Delta v - v - \omega v\phi + f(v) = 0, \quad \Delta\phi + \gamma v^2 = 0 \text{ dans } \Omega,$$
$$v,\ \phi > 0 \text{ dans } \Omega, \quad v,\ \phi = 0 \text{ sur } \partial\Omega,$$

où Ω est un ouvert borné régulier. On montre que pour tout entier k le système a une famille de solutions $(v_\varepsilon, \phi_\varepsilon)$ telle que la forme de v_ε consiste en k pointes qui se concentrent sur un centre harmonique de Ω lorsque $\varepsilon \to 0^+$. On montre, en plus, que les pointes approchent les sommets d'une configuration qui maximise un problème géométrique.

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1. Introduction

In this paper we study the stationary waves for a system of Schrödinger-Maxwell equations in the electrostatic case. After suitable rescalation, such system takes the form:

$$\begin{cases} \varepsilon^2\Delta v - v - \omega\phi v + f(v) = 0 \quad \text{in } \Omega, \\ \Delta\phi + \gamma v^2 = 0 \quad \text{in } \Omega, \\ v,\ \phi > 0 \text{ in } \Omega, \ \ v = \phi = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth domain, $\varepsilon, \omega, \gamma > 0$, $v, \phi : \Omega \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. Problem (1.1) was first proposed by Benci-Fortunato (see [6]): it describes a charged quantum particle constrained to move in the 3-dimensional region Ω interacting with its own electrostatic field. The unknowns $v = v(x)$ and $\phi = \phi(x)$ represent the wave function associated to the particle and the scalar electric potential respectively.

The system (1.1) with $f \equiv 0$ has been studied in [6] (in the case of a bounded space region Ω) and in [10] (in the case $\Omega = \mathbb{R}^3$ and under the action of an external nonzero potential). In both papers, for fixed $\varepsilon > 0$, the authors prove the existence of infinitely many solutions. Furthermore existence results for (1.1) in $\mathbb{R}^3$ have been established in [14] for power-like nonlinearities f.

This paper deals with the *semiclassical limit* of the system (1.1), i.e. it is concerned with the problem of finding nontrivial solutions and studying their asymptotic behaviour when $\varepsilon \to 0^+$; hence such solutions are usually referred to as *semiclassical* ones. The analysis of the Schrödinger-Maxwell equations in the limit $\varepsilon \to 0^+$ is not only a challenging mathematical task, but also of some relevance for the understanding of a wide class of quantum phenomena. Indeed, according to the *correspondence principle*, letting ε go to zero in the Schrödinger equation formally describes the transition from Quantum Mechanics to Classical Mechanics.

While there is a wide literature concerning semiclassical states for the single nonlinear Schrödinger equation in an assigned potential ϕ (we recall, among many others, [1], [2], [3], [4], [12], [17], [18], [19], [20], [23], [26], [27], [32], [33], [35], [37], [38], [39], [41], [42]), there are few papers dealing with the case of an unknown potential. The first time the semiclassical limit for a Schrödinger-Maxwell system has been considered seems to be in [15], [16], [40]. In such papers problem (1.1) is studied and it is proved that the solutions exhibit some kind of notable concentration behaviour: their form consists of very sharp peaks which become highly concentrated when ε is small. More precisely in [15] and [40] the authors construct a family of radially symmetric waves concentrating around a sphere when $\Omega = \mathbb{R}^3$. In [16] a new kind of solutions is found for the system (1.1) in $\mathbb{R}^N$ ($N \geq 3$), the so called *clusters*, i.e. a combination of several interacting peaks concentrating at the same point as $\varepsilon \to 0^+$. The object of this paper is to construct *clusters* for (1.1) when $\Omega \subset \mathbb{R}^3$ is a bounded and smooth domain. The problem is more complicated than in all $\mathbb{R}^3$: indeed the loss of the translation invariance gives rise to the natural question on the location of the concentration point. The analysis reveals that the configuration of the limiting clustered peaks is determined by two crucial aspects: the interaction of the spikes and the shape of Ω.

In order to state our main result we first enumerate the assumptions on the function f that will be steadily assumed:

(f1) $f \in C^{1+\sigma}_{loc}(\mathbb{R}) \cap C^2(0, +\infty)$ with $\frac{1}{2} < \sigma \leq 1$; $f(t) = 0$ for $t \leq 0$.

(f2) The problem in the whole space

$$\begin{cases} \Delta w - w + f(w) = 0, \quad w > 0 \text{ in } \mathbb{R}^3, \\ w(0) = \max_{x \in \mathbb{R}^3} w(x), \quad \lim_{|x| \to +\infty} w(x) = 0, \end{cases} \tag{1.2}$$

has a unique solution w, which is nondegenerate, i.e., denoting by L the linearized operator

$$L : H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad L[u] := \Delta u - u + f'(w)u,$$

then

$$\text{Kernel}(L) = \text{span}\left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_3} \right\}. \tag{1.3}$$

By the well-known result of Gidas, Ni and Nirenberg ([24]) w is radially symmetric and strictly decreasing in $r=|x|$. Moreover, by classical regularity results, $w\in C^4(\mathbb{R}^3)$ and the following asymptotic behavior holds:

$$w(r),\ w''(r),\ w'''' = \frac{A}{r}e^{-r}\Big(1+O\Big(\frac{1}{r}\Big)\Big),\quad w'(r),\ w'''(r) = -\frac{A}{r}e^{-r}\Big(1+O\Big(\frac{1}{r}\Big)\Big), \tag{1.4}$$

where $A_N>0$ is a suitable positive constant.

Typical examples of f satisfying (f1)-(f2) include $f(t)=t_+^p-at_+^q$ where $a\geq 0$ and $1<p<q<5$, or $f(t)=t_+(t-a)(1-t)$ where $0<a<\frac{1}{2}$. Other nonlinearities can be found in [11]. The uniqueness of w is proved in [34] for the case of power-like f; for a general nonlinearity, see [9]. The nondegeneracy condition can be derived from the uniqueness argument (see [36]).

Then we will prove that, roughly speaking, up to a suitable rescaling in the coordinates, the limit profile of each peak resembles the function w, while the rescaled cluster (by making the minimum distance between two vertexes equal to 1) approaches an optimal configuration for the following geometric problem:

(*) *Given k points $P_1,\dots,P_k\in\mathbb{R}^3$ with $|P_i-P_j|\geq 1$ for $i\neq j$, find the configuration which maximizes $\sum_{i\neq j}\frac{1}{|P_i-P_j|}$.*

A final question arises on the location in Ω of the asymptotic peaks: what we will show is that the concentration point can be identified in terms of the Robin's function of Ω, i.e. the diagonal of the regular part of the Green's function. Let us briefly introduce some notation. It is well known that for a smooth domain there exists a unique Green's function G of the Laplace operator with Dirichlet boundary condition and it can be decomposed as

$$G(x,y)=\frac{1}{4\pi|x-y|}-H(x,y) \tag{1.5}$$

(see, for example, [5]) where $\frac{1}{4\pi|x-y|}$ (the singular part) is the fundamental solution of the negative Laplace operator in $\mathbb{R}^3$, and H (the regular part) is harmonic in both variables. The restriction of the regular part to the diagonal $H(x):=H(x,x)$ is called the Robin's function of Ω. Finally the points where the Robin's function attains its minimum

$$H_0:=\inf_{x\in\Omega}H(x)$$

are called harmonic centers of Ω. Then concentration of the clustered solutions of (1.1) occurs at the harmonic centers of Ω. Now we proceed to provide the exact formulation of the main result of this paper.

Theorem 1.1. *Assume that $\Omega\subset\mathbb{R}^3$ is a smooth and bounded domain and that hypotheses (f1)-(f2) hold. Then, for any given integer $k\geq 1$, there exists $\varepsilon_k>0$ such that for every $\varepsilon\in(0,\varepsilon_k)$ the system (1.1) has a solution $(v_\varepsilon,\phi_\varepsilon)$ such that*

(1) *$v_\varepsilon,\ \phi_\varepsilon\in H_0^1(\Omega)$;*

furthermore there exist $P_1^\varepsilon,\dots,P_k^\varepsilon\in\Omega$ such that, as $\varepsilon\to 0^+$,

(2) *$v_\varepsilon(x)=\sum_{i=1}^k w\big(\frac{x-P_i^\varepsilon}{\varepsilon}\big)+o(\varepsilon^{3/2})$ uniformly for $x\in\overline{\Omega}$;*

(3) $|P_i^\varepsilon - P_j^\varepsilon| = O(\varepsilon \log \frac{1}{\varepsilon^2})$ *and* $\frac{1}{\varepsilon|\log \varepsilon^2|}(P_1^\varepsilon, \dots, P_k^\varepsilon)$ *approaches an optimal configuration in (*).*

Finally, for every sequence $\varepsilon_n \to 0^+$*, up to a subsequence,*

(4) $P_1^{\varepsilon_n}, \dots, P_k^{\varepsilon_n} \to P_0$*, where* $P_0 \in \Omega$ *is a harmonic center (i.e.* $H(P_0) = H_0$*);*

(5) $\phi_{\varepsilon_n}(x) = \varepsilon_n^3 \big(G(x, P_0) + o(1)\big) k \int_{\mathbb{R}^3} w^2 dx$ *uniformly for any compact subset of* $\overline{\Omega} \setminus \{P_0\}$*;*

Remark 1.2. *In order to provide a more precise description of the behaviour of the solutions* $(v_\varepsilon, \phi_\varepsilon)$*, the question of the number of the harmonic centers is crucial. In section 2 we will show that the Robin's function for a smooth bounded domain is a continuous function and tends to infinity at the boundary. Then the set of the harmonic radii is nonempty. Furthermore in* [7] *the authors prove that in the case of a convex bounded domain the Robin's function is strictly convex and, consequently, there exists a unique harmonic center* P_0 *(for example, the harmonic center of a ball is its geometric center); then, if, in addition, we assume the convexity of* Ω*, the parts (4)-(5) of Theorem 1.1 hold for all the families* v_ε*,* ϕ_ε*,* P_i^ε*, without need to pass to sequences and all the waves* v_ε *concentrate at that point* P_0 *as* $\varepsilon \to 0^+$*. In general we have* $H(P_i^\varepsilon) \to H_0$ *as* $\varepsilon \to 0^+$ *for every* $i = 1, \dots, k$*.*

Theorem 1.1 is proved by using an approach relied upon a finite dimensional reduction which is related to the procedure introduced in [28] and [29], and also developed in [15]-[16]. This approach is based on a combination of a Lyapunov-Schmidt reduction procedure together with a variational method. The object is to discover the solutions around a small neighborhood of a well chosen first approximation. First we construct an approximated solution obtained as the sum of suitable truncations and rescalations of w; then we find a solution of (1.1) in the normal direction of the approximated solution surface as fixed point of a suitable map. Next we study the remaining finite dimensional equation. After this reduction process, by using the implicit function theorem, we prove that, in a small neighborhood of the first approximation, solving (1.1) is equivalent to solving some finite dimensional maximization problem

$$\max M_\varepsilon(Q_1, \dots, Q_k), \text{ where } M_\varepsilon : \Omega^k \to \mathbb{R},$$

being $Q_1, \dots, Q_k$ the centers of the approximating bumps. The solution of such reduced problem also provides the location of the clusters.

We point out that multi-peak solutions concentrating on a single point have been proved to exist for the Gierer-Meinhardt system on the real line ([8]) and in a bounded interval ([43]). Similar results in $\mathbb{R}^2$ were obtained in [21], where spikes are located at regular polygons or at concentric regular polygons. The existence of clusters is known for the single Schrödinger equation in all $\mathbb{R}^N$ ([33]) or in a bounded domain with Neumann conditions ([13], [29]). In [44] the authors show that interior clusters occur for the coupled FitxHugh-Nagumo system in a domain of $\mathbb{R}^2$ with Dirichlet boundary conditions. However in the case $N \geq 3$ we are unaware of clusters for coupled elliptic systems in bounded domains. This paper seems to be the first result in this line.

Let us now briefly outline the organization of the contents of this paper. In Section 2 we recall some basic properties of the Robin's function and of the set of the harmonic centers. Section 3 is devoted to the study of the geometrical problem (*). In Section 4 we derive some key energy estimates which will play a key role in the rest of the arguments. In Section 5 and 6 we reduce the problem to finite dimension by the Liapunov-Schmidt reduction method. In Section 7 we compute the reduced energy M_ε and show that its critical points give rise to a solution of (1.1). Finally the proof of Theorem 1.1 is completed in Section 8.

NOTATION

- Given $A \subset \mathbb{R}^3$ an open subset, $C(A)$ denotes the space of the continuous function $u : A \to \mathbb{R}$. $L^p(A)$ is the usual Lebesgue space endowed with the norm
$$\|u\|_p^p := \int_A |u|^p \, dx \text{ for } 1 \le p < +\infty, \quad \|u\|_\infty = \sup_{x \in A} |u(x)|.$$
Furthermore $H_0^1(A)$ is the usual Sobolev space endowed with the norm
$$\|u\|_{H^1}^2 = \int_A \left(|\nabla u|^2 + |u|^2\right) dx.$$
- We will often use the symbol C for denoting a positive constant independent on ε. The value of C is allowed to vary from line to line (and also in the same formula).
- $o(1)$ denotes a vanishing quantity as $\varepsilon \to 0^+$.
- Given $\{a_\varepsilon\}_{\varepsilon>0}$ and $\{b_\varepsilon\}_{\varepsilon>0}$ two family of numbers, we write $a_\varepsilon = o(b_\varepsilon)$ (resp. $a_\varepsilon = O(b_\varepsilon)$) to mean that $\frac{a_\varepsilon}{b_\varepsilon} \to 0$ (resp. $|a_\varepsilon| \le C|b_\varepsilon|$) as $\varepsilon \to 0^+$.

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2. Robin's Function and Harmonic Centers

Let $\Omega \subset \mathbb{R}^3$ be a smooth and bounded domain. The Green's function G of the operator $-\Delta$ in Ω with Dirichlet boundary conditions is the solution (in the sense of the distributions) of the problem
$$\begin{cases} -\Delta_y G(x,y) = \delta_x & \text{in } \Omega, \\ G(x,y) = 0 & \text{on } \partial\Omega, \end{cases}$$
where δ_x denotes the Dirac measure at the point x. It is well known that for sufficiently smooth domains a unique Green's function exists and can be decomposed as in (1.5), where $H(x,\cdot)$ is defined as the unique harmonic function with the same boundary conditions as the singular part, i.e. it is the unique (classical) solution in $C^2(\Omega) \cap C(\overline{\Omega})$ of the following Dirichlet problem:
$$\begin{cases} \Delta_y H(x,y) = 0 & \text{in } \Omega, \\ H(x,y) = \dfrac{1}{4\pi|y-x|} & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

$H(x,\cdot)$ also coincides with the weak solution in $H^1(\Omega)$ of the system (2.6). The Green's function and, consequently, its regular part H are symmetric in x and y. Furthermore

$$G(x,y)\geq 0 \quad \forall x,y\in\Omega\times\Omega,\ x\neq y \tag{2.7}$$

(see [30]). In general an explicit calculation for the Green's function of a given domain is a difficult matter, except for domains with simple geometry. For example, in the case of a ball $B(x_0,r)$ the Green's function is given by

$$\frac{1}{4\pi}\bigg(\frac{1}{|y-x|}-\frac{|x-x_0|r}{\big||x-x_0|^2(y-x_0)-r^2(x-x_0)\big|}\bigg). \tag{2.8}$$

In the next two propositions we derive some basic properties of the Robin's function.

Proposition 2.1. *The function H is continuous in $\Omega\times\Omega$. As a corollary, the Robin's function $x\in\Omega\mapsto H(x):=H(x,x)$ is also continuous.*

Proof. If $(x_0,y_0)\in\Omega\times\Omega$, let U be a ball centered at x_0 with closures in Ω. Since $H(x,\cdot)$ is a harmonic function in Ω, according to the maximum principle we have

$$0\leq H(x,y)\leq \sup_{z\in\partial\Omega}\frac{1}{4\pi|z-x|}\leq \sup_{x\in U}\sup_{z\in\partial\Omega}\frac{1}{4\pi|z-x|}<+\infty,\quad \forall x\in U,\ y\in\Omega.$$

Therefore the family $\{H(x,\cdot)\,|\,x\in U\}$ is uniformly bounded on Ω, and then, according to theorem 2.18 of [30], is equiuniformly continuous on compact subsets of Ω. Hence we get

$$|H(x,y)-H(x_0,y_0)|\leq |H(x,y)-H(x,y_0)|+|H(x,y_0)-H(x_0,y_0)|.$$

The first term can be made arbitrarily small by the equicontinuity of the family $\{H(x,\cdot)\,|\,x\in U\}$ in y_0 and the last by the continuity of $H(\cdot,y_0)$ in x_0. □

Next lemma describes the boundary behaviour of the Robin's function.

Proposition 2.2. *$H(x)\to+\infty$ as $d(x)\to 0$, where $d(x)=dist(x,\partial\Omega)$.*

*P*roof. Fix $x_0\in\Omega$; then $B(x_0,d(x_0))\subset\Omega$. Denote by $\widetilde{H}$ the regular part of the Green's function in $B(x_0,d(x_0))$, which is given (2.8). Since $H(x_0,\cdot)\geq\tilde{H}(x_0,\cdot)$ on $\partial B(x_0,d(x_0))$, then, by using again the maximum principle, $H(x_0,\cdot)\geq\tilde{H}(x_0,\cdot)$ in $B(x_0,d(x_0))$; in particular $H(x_0)\geq\tilde{H}(x_0)=\frac{1}{4\pi d(x_0)}$. □

We recall that a harmonic center of Ω is a minimum point for the Robin's function. Combining Proposition 2.1 and Proposition 2.2 we deduce that a smooth bounded domain has at least one harmonic center. Since, by (2.8), the Robin's function of the ball $B(x_0,r)$ is $\frac{r}{r^2-|x-x_0|^2}$, the unique harmonic center of a ball is its geometric center. More generally, the Robin's function of a convex bounded domain is strictly convex; this implies, in particular, the existence of a unique harmonic center. This result has been established by Cardaliaguet and Rabah ([7]). However the problem of establishing the number of harmonic centers for non convex domains is open in general.

3. Optimal Configurations For Problem (*)

The object of this section is to study the geometrical problem (*), which plays an important role in the location of the clusters.

To begin with, fix $k \in \mathbb{N}$ and define the following set:

$$\Sigma_k := \left\{ (P_1, \ldots, P_k) \in \mathbb{R}^{3k} \,\middle|\, |P_i - P_j| \geq 1 \text{ for } \ i \neq j \right\}.$$

Hence problem (*) consists in determining the number

$$m(k) := \sup \left\{ \sum_{i \neq j} \frac{1}{|P_i - P_j|} \,\middle|\, (P_1, \ldots, P_k) \in \Sigma_k \right\}.$$

and in characterizing the configurations which achieve this optimal number. We remark that the functional $I_s(P_1, \ldots, P_k) := \sum_{i \neq j} \frac{1}{|P_i - P_j|^s}$ is called **Riesz s-energy**. For a study of I_s and related packing problems, we refer to a recent article [31].

Note that this problem has a physical meaning in $\mathbb{R}^3$: consider k rigid balls of radius 1 centered at $P_1, \ldots, P_k$. Assume that the attractive force between different balls is proportional to $\frac{1}{r}$, where r is the distance between the two centers of the balls. Then $-\sum_{i \neq j} \frac{1}{|P_i - P_j|}$ is the total potential energy and solving problem (*) becomes minimizing the total energy of the system. In the next lemma we study the maximization problem (*).

Lemma 3.1. *The value $m(k)$ is always attained by some configuration. Furthermore, if $(P_1, \ldots, P_k) \in \Sigma_k$ is an optimal configuration, then $\min_{i \neq j} |P_i - P_j| = 1$. Moreover, if $k_1, \ldots, k_l \in \mathbb{N}$ are such that $k_1 + \ldots + k_l = k$, then*

$$m(k) > \sum_{i=1}^{l} m(k_i) \tag{3.1}$$

Proof. The proof can be found in [16] in a more general framework. For sake of completeness we repeat it. First we prove (3.1) for $\ell = 2$. Take two configurations $(P_1, \ldots, P_{k_1}) \in \Sigma_{k_1}$, $(P_{k_1+1}, \ldots, P_k) \in \Sigma_{k_2}$. Then translate the convex hulls of each configuration in such a way that their mutual distance is equal to 1 and there are at least two vertexes belonging to different hulls at distance 1. Hence we obtain a configuration $(Q_1, \ldots, Q_k) \in \Sigma_k$ such that $\sum_{i \neq j} \frac{1}{|Q_i - Q_j|} > \sum_{i,j \leq k_1,\, i \neq j} \frac{1}{|P_i - P_j|} + \sum_{i,j > k_1,\, i \neq j} \frac{1}{|P_i - P_j|} + 2$, by which we get (3.1).

Let $\{(P_1^n, \ldots, P_k^n)\}_n \subset \Sigma_k$ be a maximizing sequence. By the translation invariance we may assume $P_1^n = 0$. We claim that the sequences $\{P_i^n\}$ are bounded in $\mathbb{R}^3$. Otherwise, up to a subsequence, we could find I, $J \subset \{1, \ldots, k\}$, I, $J \neq \emptyset$, such that $I \cup J = \{1, \ldots, k\}$, $I \cap J = \emptyset$, $|P_i^n| \leq C$ for $i \in I$, and $|P_j^n| \to +\infty$ for $j \in J$. Then

$$m(k) = \sum_{i \neq j} \frac{1}{|P_i^n - P_j^n|} + o(1) = \sum_{i \neq j, (i,j) \in I} \frac{1}{|P_i^n - P_j^n|} + \sum_{i \neq j, (i,j) \in J} \frac{1}{|P_i^n - P_j^n|} + o(1) \leq m(\#I) + m(\#J) + o(1),$$

in contradiction with (3.1). By compactness, we can obtain an optimal configuration $(P_1, \dots, P_k)$. If it was $l = \min_{i \neq j} |P_i - P_j| > 1$, then the new configuration $(\frac{1}{l}P_1, \dots, \frac{1}{l}P_k)$ would still belong to Σ_k and would contradict the optimality of $(P_1, \dots, P_k)$. $\square$

In general, it is difficult to find the number $m(k)$, except for some special cases. For example, it is obvious that $m(3) = 6$ and $m(4) = 12$, with the optimal configurations given by a regular triangle and a regular tetrahedron with edge 1 respectively. Note that in general $m(k) \leq k(k-1)$.

4. Key Energy Estimate

For every $\varepsilon > 0$ set

$$\Omega_\varepsilon = \varepsilon^{-1}\Omega = \{x \,|\, \varepsilon x \in \Omega\}.$$

It is convenient to make a change of variables in the system (1.1) to obtain the version

$$\begin{cases} \Delta u - u - \delta u\psi + f(u) = 0 \text{ in } \Omega_\varepsilon, \\ \Delta\psi + \varepsilon^2 u^2 = 0 \text{ in } \Omega_\varepsilon, \\ u,\, \psi > 0 \text{ in } \Omega_\varepsilon, \;\; u = \psi = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{4.1}$$

where

$$\delta = \omega\gamma, \;\; u(x) = v(\varepsilon x), \;\; \psi(x) = \frac{1}{\gamma}\phi(\varepsilon x) \tag{4.2}$$

We begin with the following proposition.

Proposition 4.1. *For every $g \in L^2(\Omega_\varepsilon)$ denote by $T_\varepsilon[g]$ the unique solution in $H_0^1(\Omega_\varepsilon)$ of*

$$-\Delta\psi = \varepsilon^2 g. \tag{4.3}$$

Then the following representation formula holds:

$$T_\varepsilon[g](x) = \varepsilon^3 \int_{\Omega_\varepsilon} G(\varepsilon x, \varepsilon y) g(y) dy, \tag{4.4}$$

where G is the Green's function defined in the section 2. Furthermore

a) $T_\varepsilon[g] \geq 0$ *for every* $g \in L^2(\Omega_\varepsilon)$ *such that* $g \geq 0$;
b) $\int_{\Omega_\varepsilon} T_\varepsilon[g] g dx \geq 0$ *for every* $g \in L^2(\Omega_\varepsilon)$;
c) $\|T_\varepsilon[g]\|_\infty \leq C\varepsilon\sqrt{\varepsilon}\|g\|_2$ *for every* $g \in L^2(\Omega_\varepsilon)$;
d) $\|T_\varepsilon[g]\|_\infty \leq C\varepsilon^2(\|g\|_1 + \|g\|_\infty)$ *for every* $g \in L^\infty(\Omega_\varepsilon)$;
e) *the functional* $J : u \in H_0^1(\Omega_\varepsilon) \mapsto \int_{\Omega_\varepsilon} u^2 T_\varepsilon[u^2] dx$ *is* C^1 *and*

$$J'(u)[v] = 4\int_{\Omega_\varepsilon} uv T_\varepsilon[u^2] dx \quad \forall u, v \in H_0^1(\Omega_\varepsilon).$$

Proof. By Lax-Milgram's Lemma we get the existence of a unique solution in $H^1_0(\Omega_\varepsilon)$ of (4.3). The representation formula (4.4) holds for $u \in C^\infty_0(\Omega_\varepsilon)$ (see, for example, [22, p. 23, Theorem 1]); by density (4.4) can be extended to any $g \in L^2(\Omega_\varepsilon)$. a) follows immediately from (2.7). Furthermore

$$\varepsilon^2 \int_{\Omega_\varepsilon} T_\varepsilon[g] g dx = \int_{\Omega_\varepsilon} |\nabla T_\varepsilon[g]|^2 dx \geq 0.$$

By (1.5) and (2.7), for every $g \in L^2(\Omega_\varepsilon)$, by using Hölder's inequality we have

$$|T_\varepsilon[g](x)| \leq \frac{\varepsilon^2}{4\pi} \int_{\Omega_\varepsilon} \frac{|g(y)|}{|y-x|} dy \leq \frac{\varepsilon^2}{4\pi} \|g\|_2 \Big(\int_{|y| \leq C/\varepsilon} \frac{1}{|y|^2} dy \Big)^{1/2} \leq C\varepsilon\sqrt{\varepsilon} \|g\|_2,$$

while, for $g \in L^\infty(\Omega_\varepsilon)$,

$$|T_\varepsilon[g](x)| \leq \frac{\varepsilon^2}{4\pi} \int_{|y-x| \leq 1} \frac{|g(y)|}{|y-x|} dy + \frac{\varepsilon^2}{4\pi} \int_{\Omega_\varepsilon} |g(y)| dy \leq C\varepsilon^2 (\|g\|_\infty + \|g\|_1)$$

and we obtain c)-d). Part e) is a direct computation. □

Associated with (4.1) is the following energy functional $E_\varepsilon \in C^1(H^1_0(\Omega_\varepsilon), \mathbb{R})$:

$$E_\varepsilon[u] := \frac{1}{2} \int_{\Omega_\varepsilon} \big(|\nabla u|^2 + |u|^2 \big) dx - \int_{\Omega_\varepsilon} F(u) dx + \frac{\delta}{4} \int_{\Omega_\varepsilon} u^2 T_\varepsilon[u^2] dx,$$

where $F(w) = \int_0^w f(s) ds$. By using Proposition 4.1 the energy functional can be rewritten as

$$E_\varepsilon[u] = \frac{1}{2} \int_{\Omega_\varepsilon} \big(|\nabla u|^2 + u^2 \big) dx - \int_{\Omega_\varepsilon} F(u) dx + \frac{\delta}{4} \varepsilon^3 \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} G(\varepsilon x, \varepsilon y) u^2(x) u^2(y) dx dy.$$

We denote by I the energy associated to (1.2):

$$I[w] = \frac{1}{2} \int_{\mathbb{R}^3} \big(|\nabla w|^2 + w^2 \big) dx - \int_{\mathbb{R}^3} F(w) dx.$$

Fix $k \in \mathbb{N}$ and for every $\mathbf{Q} \in \mathbb{R}^{3k}$ set

$$w_i = w(x - Q_i), \quad i = 1, \dots, k, \quad w_{\mathbf{Q}} = \sum_{i=1}^k w(x - Q_i).$$

According to Proposition 2.2 it makes sense to choose $\eta \in (0, \frac{2\sigma - 1}{10})$ sufficiently small such that

$$H(x) \geq 2H_0 \text{ for } x \in \Omega \text{ with } d(x) \leq \eta$$

where $H_0 := \min_\Omega H(x)$. Then define the configuration space:

$$\Gamma_\varepsilon = \left\{ \mathbf{Q} = (Q_1, \dots, Q_k) \in \Omega_\varepsilon^k \,\middle|\, H(\varepsilon Q_i) < 2H_0, \quad (1-\eta) \log \frac{1}{\varepsilon^2} < |Q_i - Q_j| < \Big(\log \frac{1}{\varepsilon^2} \Big)^2 \text{ for } i \neq j \right\}.$$

For every $\mathbf{Q} = (Q_1, \dots, Q_k) \in \overline{\Gamma}_\varepsilon$, we set

$$w_{\varepsilon,i}(x) = w(x - Q_i)\chi(\varepsilon x), \quad i = 1, \dots, k, \quad w_{\varepsilon,\mathbf{Q}} = \sum_{i=1}^k w_{\varepsilon,i}.$$

where $\chi \in C^\infty_0(\Omega)$ is a smooth cut-off function such that $\chi = 1$ if $d(x) \geq \frac{\eta}{2}$ and $\chi = 0$ if $d(x) \leq \frac{\eta}{4}$. Then we get $w_{\varepsilon,i} = w_i$ for $|x - Q_i| \leq \frac{\eta}{2\varepsilon}$; hence, by (1.4) we deduce

$$\big| w_{\varepsilon,i} - w_i \big|, \ \big| \nabla w_{\varepsilon,i} - \nabla w_i \big|, \ \big| \Delta w_{\varepsilon,i} - \Delta w_i \big| = O\big(e^{-\frac{\eta}{4\varepsilon}} \big) w_i^{1/2} = o(\varepsilon^3) w_i^{1/2} \tag{4.5}$$

and, by assumption (f1),

$$F(w_{\varepsilon,\mathbf{Q}})-F(w_{\mathbf{Q}}),\ f(w_{\varepsilon,\mathbf{Q}})-f(w_{\mathbf{Q}})=O(w_{\varepsilon,\mathbf{Q}}-w_{\mathbf{Q}})=o(\varepsilon^3)w_{\mathbf{Q}}^{1/2} \tag{4.6}$$

uniformly for $x\in\mathbb{R}^3$ and $\mathbf{Q}\in\overline{\Gamma}_\varepsilon$.

In order to provide the key result about the interaction of the w_i's we state two useful lemmas.

Lemma 4.1. *The following limits hold*

$$\frac{1}{w(Q_i-Q_j)}\int_{A_{\varepsilon,i}}f(w_i)w_j\,dx,\quad \frac{1}{w(Q_i-Q_j)}\int_{\mathbb{R}^3}f(w_i)w_j\,dx\to\int_{\mathbb{R}^3}f(w)e^{x_1}dx\quad \textit{as}\ \ \varepsilon\to 0^+,$$

uniformly for $\mathbf{Q}=(Q_1,\dots,Q_k)\in\overline{\Gamma}_\varepsilon$, *where*

$$A_{\varepsilon,i}=\Big\{x\in\mathbb{R}^3\ \Big|\ |x-Q_i|\le\frac{1-\eta}{2}\log\frac{1}{\varepsilon^2}\Big\}.$$

Proof. The proof is an easy consequence of Lebesgue's Dominated Convergence Theorem. First notice that

$$\int_{A_{i,j}}f(w_i)w_j=\int_{|x|\le\frac{1-\eta}{2}\log\frac{1}{\varepsilon^2}}f(w)w(x+Q_i-Q_j),\quad \int_{\mathbb{R}^3}f(w_i)w_j=\int_{\mathbb{R}^3}f(w)w(x+Q_i-Q_j).$$

According to (1.4) for every $x\in\mathbb{R}^3$ we have

$$\lim_{|Q_i-Q_j|\to+\infty}\frac{w(x+Q_i-Q_j)}{w(Q_i-Q_j)}-e^{-\frac{x(Q_i-Q_j)}{|Q_i-Q_j|}}=\lim_{|Q_i-Q_j|\to+\infty}e^{-|x+Q_i-Q_j|+|Q_i-Q_j|}-e^{-\frac{x(Q_i-Q_j)}{|Q_i-Q_j|}}=0. \tag{4.7}$$

Observe that, if $|x|\le\frac{2}{2+\sigma}|Q_i-Q_j|$ (with σ given by assumption (f1)), then $|x+Q_i-Q_j|\ge\frac{\sigma}{2+\sigma}|Q_i-Q_j|$; hence, by using (1.4), for $|Q_i-Q_j|$ sufficiently large we get

$$f(w)\frac{w(x+Q_i-Q_j)}{w(Q_i-Q_j)}\le 2f(w)\frac{|Q_i-Q_j|}{|x+Q_i-Q_j|}e^{|x|}\le 2f(w)\frac{2+\sigma}{\sigma}e^{|x|}.$$

On the other hand, for $|x|\ge\frac{2}{2+\sigma}|Q_i-Q_j|$, by (1.4) and (f1) we obtain

$$f(w)\frac{w(x+Q_i-Q_j)}{w(Q_i-Q_j)}\le C\|w\|_\infty\frac{|Q_i-Q_j|}{|x|^{1+\sigma}}e^{-(1+\sigma)|x|+|Q_i-Q_j|}\le C\|w\|_\infty\frac{2+\sigma}{2}e^{-\frac{\sigma}{2}|x|}.$$

Since $f(w)e^{|x|}\in L^1(\mathbb{R}^3)$, the convergence (4.7) is dominated. Taking into account that w is radial, we deduce $\int_{\mathbb{R}^3}f(w)e^{-\frac{x(Q_i-Q_j)}{|Q_i-Q_j|}}dx=\int_{\mathbb{R}^3}f(w)e^{x_1}dx$ for every Q_i, Q_j, then we obtain the thesis. □

Lemma 4.2. *For every* $g:\mathbb{R}^3\to\mathbb{R}$ *such that* $(1+|y|^3)g\in L^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ *set*

$$\Psi_1[g](x)=\int_{\mathbb{R}^3}\frac{g(y)}{|x-y|}dy,\quad \Psi_2[g](x)=\int_{\mathbb{R}^3}\frac{g(y)}{|x-y|^2}dy.$$

Then there exist constants $C_1(g)$, $C_1'(g)$, $C_2(g)$, $C_2'(g)$ *such that*

$$\left|\Psi_1[g](x)-\frac{C_1(g)}{|x|}\right|\le\frac{C_1'(g)}{|x|^2},\qquad \left|\Psi_2[g](x)-\frac{C_2(g)}{|x|^2}\right|\le\frac{C_2'(g)}{|x|^3}\quad \forall x\neq 0.$$

Furthermore $C_1(g)=C_2(g)=\int_{\mathbb{R}^3}g(y)$ *and*

$$|\Psi_1[g]|,\ |\Psi_2[g]|\le 4\pi\|g\|_\infty+\|g\|_{L^1},\quad C_1'(g),\,C_2'(g)\le 2\sum_{p=1}^{3}(4\pi\|y^pg\|_\infty+\|y^pg\|_{L^1}). \tag{4.8}$$

Proof. First observe that if $h \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} \frac{|h(y)|}{|x-y|} dy, \quad \int_{\mathbb{R}^3} \frac{|h(y)|}{|x-y|^2} dy \leq \int_{B(x,1)} \frac{|h(y)|}{|x-y|^2} dy + \int_{\mathbb{R}^3} |h(y)| dy \leq 4\pi \|h\|_\infty + \|h\|_{L^1},$$

by which we deduce the first part of (4.8). Next fix $p = 1, 2$. From the inequality

$$\big| |x|^p - |x-y|^p \big| \leq 2(|x-y|^{p-1}|y| + |y|^p) \quad \forall x, y \in \mathbb{R}^3,$$

we get

$$\int_{\mathbb{R}^3} g(y) \Big| \frac{1}{|x-y|^p} - \frac{1}{|x|^p} \Big| dy \leq \frac{2}{|x|^p} \Big(\int_{\mathbb{R}^3} g(y) \frac{|y|}{|x-y|} dy + \int_{\mathbb{R}^3} g(y) \frac{|y|^p}{|x-y|^p} dy \Big).$$

The function $\int_{\mathbb{R}^3} g(y) \frac{|y|^p |x|}{|x-y|^p} dy$ is bounded in $\mathbb{R}^3$ for $p = 1, 2$: more precisely we have $\int_{\mathbb{R}^3} g(y) \frac{|y|^p |x|}{|x-y|^p} dy \leq \int_{\mathbb{R}^3} g(y) \frac{|y|^p}{|x-y|^{p-1}} dy + \int_{\mathbb{R}^3} g(y) \frac{|y|^{p+1}}{|x-y|^p} dy \leq 4\pi(\|y^p g\|_\infty + \|y^{p+1} g\|_\infty) + \|y^p g\|_{L^1} + \|y^{p+1} g\|_{L^1}$, and we deduce the thesis. □

With the help of Lemma 4.1 and Lemma 4.2 we derive the following key energy estimate.

Proposition 4.2. *The following estimate holds:*

$$E_\varepsilon[w_{\varepsilon,\mathbf{Q}}] = kI[w] + \alpha(\mathbf{Q}) + c_1 \varepsilon^2 - c_2 \varepsilon^3 H(\varepsilon Q_1) + o(\varepsilon^3) \ \text{as } \varepsilon \to 0^+, \tag{4.9}$$

uniformly for $\mathbf{Q} = (Q_1, \dots, Q_k) \in \overline{\Gamma}_\varepsilon$, *where* c_1, c_2 *are positive constants and* $\alpha : \mathbb{R}^{3k} \to \mathbb{R}$ *satisfies*

$$\alpha(\mathbf{Q} + P^k) = \alpha(\mathbf{Q}) \quad \forall \mathbf{Q} \in \mathbb{R}^{3k}, \forall P \in \mathbb{R}^3 \tag{4.10}$$

(where $P^k = (P, \dots, P)$*). Furthermore*

$$\alpha(\mathbf{Q}) = c_3 \varepsilon^2 \sum_{i \neq j} \frac{1 + o(1)}{|Q_i - Q_j|} - c_4 \sum_{i \neq j} (1 + o(1)) w(Q_i - Q_j) \ \text{as } \varepsilon \to 0^+$$

uniformly for $\mathbf{Q} = (Q_1, \dots, Q_k) \in \overline{\Gamma}_\varepsilon$, *for some positive constants* c_3, c_4.

Proof. We split the functional E_ε as follows:

$$E_\varepsilon[w_{\varepsilon,\mathbf{Q}}] = E_{\varepsilon,1}[w_{\varepsilon,\mathbf{Q}}] + \frac{\delta \varepsilon^2}{16\pi} E_{\varepsilon,2}[w_{\varepsilon,\mathbf{Q}}] - \frac{\delta \varepsilon^3}{4} E_{\varepsilon,3}[w_{\varepsilon,\mathbf{Q}}],$$

where

$$E_{\varepsilon,1}[w_{\varepsilon,\mathbf{Q}}] = \frac{1}{2} \int_{\Omega_\varepsilon} \big(|\nabla w_{\varepsilon,\mathbf{Q}}|^2 + |w_{\varepsilon,\mathbf{Q}}|^2 \big) dx, \quad E_{\varepsilon,2}[w_{\varepsilon,\mathbf{Q}}] = \int_{\Omega_\varepsilon} |w_{\varepsilon,\mathbf{Q}}|^2 dy \int_{\Omega_\varepsilon} \frac{1}{|y-x|} w_{\varepsilon,\mathbf{Q}}^2 \, dx,$$

$$E_{\varepsilon,3}[w_{\varepsilon,\mathbf{Q}}] = \int_{\Omega_\varepsilon} w_{\varepsilon,\mathbf{Q}}^2 dy \int_{\Omega_\varepsilon} H(\varepsilon x, \varepsilon y) w_{\varepsilon,\mathbf{Q}}^2 dx.$$

Using (4.5) and (4.6) we compute

$$\begin{aligned} E_{\varepsilon,1}[w_{\varepsilon,\mathbf{Q}}] &= kI[w] + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^3} (\nabla w_i \nabla w_j + w_i w_j) dx - \int_{\mathbb{R}^3} \Big(F(w_{\mathbf{Q}}) - \sum_{i=1}^k F(w_i) \Big) dx + o(\varepsilon^3) \\ &= kI[w] + \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^3} f(w_i) w_j dx - \int_{\mathbb{R}^3} \Big(F(w_{\mathbf{Q}}) - \sum_{i=1}^k F(w_i) \Big) dx + o(\varepsilon^3) \end{aligned} \tag{4.11}$$

uniformly for $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$. Set

$$\alpha_1(\mathbf{Q}) = \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^3} f(w_i)w_j dx - \int_{\mathbb{R}^3}\Big(F(w_{\mathbf{Q}}) - \sum_{i=1}^k F(w_i)\Big)dx, \quad \mathbf{Q} \in \mathbb{R}^{3k}.$$

Consider the sets $A_{\varepsilon,i}$ defined in Lemma 4.1. For every $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ we have: $|x - Q_j| \geq |x - Q_i|$ on $A_{\varepsilon,i}$, by which, since w is decreasing in $|x|$, $w_j \leq w_i$ on $A_{\varepsilon,i}$. Then, by using assumption (f1), we get

$$\Big|F(w_{\mathbf{Q}}) - F(w_i) - f(w_i)\sum_{j\neq i} w_j\Big|, \ \Big|\sum_{j\neq i} F(w_j)\Big| \leq C w_i^\sigma \sum_{j\neq i} w_j^2 \ \text{ in } A_{\varepsilon,i};$$

on the other hand $|x-Q_j| \geq \frac{1}{2}|Q_i-Q_j|$ on $A_{\varepsilon,i}$ for $j \neq i$, consequently, by (1.4), $w_j^2(x) \leq w_j^2\big(\frac{1}{2}(Q_i-Q_j)\big) = o(w(Q_i - Q_j))$ on $A_{\varepsilon,i}$ for $j \neq i$; hence we achieve

$$\int_{A_{\varepsilon,i}}\Big(F(w_{\mathbf{Q}}) - \sum_{j=1}^k F(w_j) - f(w_i)\sum_{j\neq i} w_j\Big)dx = \sum_{j\neq i} o(w(Q_i - Q_j)) \quad \forall i = 1, \ldots, k. \tag{4.12}$$

Notice that $|F(w_{\mathbf{Q}}) - \sum_{i=1}^k F(w_i)| \leq C\sum_{i=1}^k |w_i|^{2+\sigma}$ on $\mathbb{R}^3$. Since $|x-Q_i| \geq \frac{1}{2}|Q_i-Q_j|$ on $\mathbb{R}^3\setminus(\cup_{i=1}^k A_{\varepsilon,i})$, we obtain $w_i^2 = o(w(Q_i - Q_j))$ on $\mathbb{R}^3 \setminus (\cup_{i=1}^k A_{\varepsilon,i})$ for every $i \neq j$, by which

$$\int_{\mathbb{R}^3\setminus(\cup_{i=1}^k A_{\varepsilon,i})}\Big(F(w_{\mathbf{Q}}) - \sum_{i=1}^k F(w_i)\Big)dx = \sum_{i\neq j} o(w(Q_i - Q_j)). \tag{4.13}$$

Combining (4.12) and (4.13) and using Lemma 4.1 we arrive at

$$\begin{aligned}\alpha_1(\mathbf{Q}) &= \frac{1}{2}\sum_{i\neq j}\int_{\mathbb{R}^3} f(w_i)w_j dx - \sum_{i=1}^k\sum_{j\neq i}\int_{A_{\varepsilon,i}} f(w_i)w_j dx + \sum_{i\neq j} o(w(Q_i - Q_j))\\ &= -\frac{1}{2}\int_{\mathbb{R}^3} f(w)e^{x_1}dx\sum_{i\neq j}(1 + o(1))w(Q_i - Q_j) \text{ as } \varepsilon \to 0^+\end{aligned}$$

uniformly for $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$. As regards $E_{\varepsilon,2}$, by using again (4.5) we obtain

$$\begin{aligned}E_{\varepsilon,2}(w_{\varepsilon,\mathbf{Q}}^2) &= \int_{\mathbb{R}^3} w_{\mathbf{Q}}^2 dx\int_{\mathbb{R}^3}\frac{1}{|y-x|}w_{\mathbf{Q}}^2 dy + o(\varepsilon^3) = \int_{\mathbb{R}^3}\Big(\sum_{i=1}^k w_i\Big)^2\Psi_1\Big[\Big(\sum_{i=1}^k w_i\Big)^2\Big]dx + o(\varepsilon^3)\\ &= \sum_{i=1}^k\int_{\mathbb{R}^3} w_i^2\Psi_1[w_i^2]dx + \alpha_2(\mathbf{Q}) + o(\varepsilon^3)\end{aligned}$$

where

$$\alpha_2(\mathbf{Q}) = \sum_{i\neq j} w_i^2\Psi[w_j^2]dx + \sum_{(i,j,l,m),i\neq j}\int_{\mathbb{R}^3}\Big(w_iw_j\Psi_1[w_lw_m]dx + w_lw_m\Psi_1[w_iw_j]\Big)dx$$

and Ψ_1 has been defined in Lemma 4.2. We immediately obtain

$$\int_{\mathbb{R}^3} w_i^2\Psi_1[w_i^2]\,dx = \int_{\mathbb{R}^3} w^2\Psi_1[w^2]\,dx \tag{4.14}$$

and the last expression is a real constant independent on Q_i.

For $i \neq j$ Lemma 4.2 gives (setting $C_1 = C_1(w^2)$)

$$\begin{aligned}\int_{\mathbb{R}^3} w_i^2 \Psi_1[w_j^2] dx &= \int_{\mathbb{R}^3} w^2 \Psi_1[w^2](x + Q_i - Q_j) dx \\ &= C_1 \int_{\mathbb{R}^3} \frac{w^2}{|x + Q_i - Q_j|} dx + O(1) \int_{\mathbb{R}^3} \frac{w^2}{|x + Q_i - Q_j|^2} dx \\ &= C_1 \Psi_1[w^2](Q_i - Q_j) + O(1) \Psi_2[w^2](Q_i - Q_j) \\ &= \frac{C_1^2}{|Q_i - Q_j|} + \frac{O(1)}{|Q_i - Q_j|^2} = C_1^2 \frac{1 + o(1)}{|Q_i - Q_j|}.\end{aligned}$$

Since by (4.8) $\Psi_1[w_l w_m] \leq C$ uniformly for $\mathbf{Q} \in \mathbb{R}^3$, for $i \neq j$ and $l, m \in \{1, \ldots, k\}$ we have

$$\int_{\mathbb{R}^3} w_i w_j \Psi_1[w_l w_m]\, dx \leq C \int_{\mathbb{R}^3} w_i w_j \, dx,$$

and, since for every $x \in \mathbb{R}^3$ $|x - Q_i| \geq \frac{1}{2}|Q_i - Q_j|$ or $|x - Q_j| \geq \frac{1}{2}|Q_i - Q_j|$, i.e. $w_i(x) \leq w(\frac{1}{2}|Q_i - Q_j|)$ or $w_j(x) \leq w(\frac{1}{2}|Q_i - Q_j|)$, then we deduce

$$\int_{\mathbb{R}^3} w_i w_j \Psi_1[w_l w_m]\, dx = \frac{o(1)}{|Q_i - Q_j|},$$

and, consequently,

$$\int_{\mathbb{R}^3} w_l w_m \Psi_1[w_i w_j]\, dx = \int_{\mathbb{R}^3} w_i w_j \Psi_1[w_l w_m]\, dx = \frac{o(1)}{|Q_i - Q_j|}.$$

Hence we have proved

$$\alpha_2(\mathbf{Q}) = C_1^2 \sum_{i \neq j} \frac{1 + o(1)}{|Q_i - Q_j|} \text{ as } \varepsilon \to 0^+$$

uniformly for $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$.

It remains to estimate $E_{\varepsilon,2}(w_{\varepsilon,\mathbf{Q}})$:

$$\begin{aligned}E_{\varepsilon,2}(w_{\varepsilon,\mathbf{Q}}) &= \sum_{i,j} \int_{\mathbb{R}^3} w_{\varepsilon,i}^2 dy \int_{\mathbb{R}^3} H(\varepsilon x, \varepsilon y) w_{\varepsilon,j}^2 dx \\ &= \sum_{i,j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w^2(x) \chi(\varepsilon(x + Q_i)) H(\varepsilon(x + Q_i), \varepsilon(y + Q_j)) w^2(y) \chi(\varepsilon(y + Q_j)) dx dy.\end{aligned}$$

We want to apply the Lebesgue's dominated convergence theorem. By construction for all ε it results $\varepsilon\overline{\Gamma}_\varepsilon \subset K_0^k := \{z \in \Omega \,|\, d(z) \geq \eta\}^k$. Hence, since by Proposition 2.1 H is uniformly continuous on $K_0 \times K_0$, for every x, $y \in \mathbb{R}^3$ and (i,j) we obtain

$$\lim_{\varepsilon \to 0^+} \sup_{\mathbf{Q} \in \Gamma_\varepsilon} \Big(H(\varepsilon(x + Q_i), \varepsilon(y + Q_j)) - H(\varepsilon Q_i, \varepsilon Q_j) \Big) \leq \lim_{\varepsilon \to 0^+} \sup_{z, z' \in K_0} \Big(H(z + \varepsilon x, z' + \varepsilon y)) - H(z, z') \Big) = 0.$$

and, since by construction $\chi = 1$ in K_0,

$$\lim_{\varepsilon \to 0^+} \sup_{\mathbf{Q} \in \Gamma_\varepsilon} \Big(\chi(\varepsilon(x + Q_i)) - 1 \Big) \leq \lim_{\varepsilon \to 0^+} \sup_{z \in K_0} \Big(\chi(\varepsilon x + z) - 1 \Big) = 0.$$

Hence Lebesgue's theorem applies (since $\chi(z) = 0$ for $d(z) \geq \frac{\eta}{4}$, the dominating function is given by $\sup_{d(z),d(z')\geq\frac{\eta}{4}} H(z,z')w^2(x)w^2(y))$ for every (i,j) we obtain

$$\int_{\mathbb{R}^3} w_{\varepsilon,i}^2 dy \int_{\mathbb{R}^3} H(\varepsilon x, \varepsilon y) w_{\varepsilon,j}^2 dx = (1+o(1)) \left(\int_{\mathbb{R}^3} w^2 \right)^2 \sum_{i,j} H(\varepsilon Q_i, \varepsilon Q_j) \text{ as } \varepsilon \to 0^+$$

uniformly for $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$. On the other hand the definition of Γ_ε implies that $|H(\varepsilon Q_i, \varepsilon Q_j) - H(\varepsilon Q_1)| \to 0$ as $\varepsilon \to 0^+$ uniformly for $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$, by which

$$E_{\varepsilon,2}(w_{\varepsilon,\mathbf{Q}}) = (1+o(1)) \left(\int_{\mathbb{R}^3} w^2 \right)^2 k^2 H(\varepsilon Q_1) \text{ as } \varepsilon \to 0^+$$

uniformly for $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$. By setting $\alpha(\mathbf{Q}) = \alpha_1(\mathbf{Q}) + \varepsilon^2 \frac{\delta}{16\pi} \alpha_2(\mathbf{Q})$, it is obvious that $\alpha(\mathbf{Q} + P^k) = \alpha(\mathbf{Q})$ for every $\mathbf{Q} \in \mathbb{R}^{3k}$ and $P \in \mathbb{R}^3$. Then the thesis follows. $\square$

Finally we are in position to provide the following error estimates.

Lemma 4.3. *There exists a constant $C > 0$ such that for every $\varepsilon > 0$ and $\mathbf{Q} = (Q_1, \dots, Q_k) \in \overline{\Gamma}_\varepsilon$:*

$$\left| \Delta w_{\varepsilon,\mathbf{Q}} - w_{\varepsilon,\mathbf{Q}} + f(w_{\varepsilon,\mathbf{Q}}) - \delta T_\varepsilon[w_{\varepsilon,\mathbf{Q}}^2] w_{\varepsilon,\mathbf{Q}} \right| \leq C \varepsilon^{\frac{5+2\sigma}{4}(1-\eta)} w_{\mathbf{Q}}^{\frac{2\sigma-1}{4}}.$$

Proof. According to d) of Proposition 4.1

$$|T_\varepsilon[w_{\varepsilon,\mathbf{Q}}^2] w_{\varepsilon,\mathbf{Q}}| \leq C \varepsilon^2 w_{\mathbf{Q}}.$$

We just need to estimate the local term: by (4.5) and (4.6) we deduce

$$\Delta w_{\varepsilon,\mathbf{Q}} - w_{\varepsilon,\mathbf{Q}} + f(w_{\varepsilon,\mathbf{Q}}) = \Delta w_{\mathbf{Q}} - w_{\mathbf{Q}} + f(w_{\mathbf{Q}}) + o(\varepsilon^3) w_{\mathbf{Q}}^{1/2} = f(w_{\mathbf{Q}}) - \sum_{j=1}^k f(w_j) + o(\varepsilon^3) w_{\mathbf{Q}}^{1/2}$$

uniformly for $x \in \mathbb{R}^3$ and $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$. To this aim set $\sigma' = \frac{1+2\sigma}{4} \in (\frac{1}{2}, \sigma)$ and consider the sets $A_{\varepsilon,i}$ defined in Lemma 4.1. Fix $i = 1, \dots, k$ and for $x \in A_{\varepsilon,i}$ and $j \neq i$ we have $|x - Q_j| \geq \frac{1-\eta}{2} \log \frac{1}{\varepsilon^2}$ and $w_j(x) \leq w_i(x)$; then, using assumption (f2) and (1.4),

$$\begin{aligned} \left| f(w_{\mathbf{Q}}) - \sum_{j=1}^k f(w_j) \right| &\leq C |w_i|^\sigma \sum_{j\neq i} w_j \leq C w_i^{\sigma-\sigma'} e^{-\sigma'|x-Q_i|} \sum_{j \neq i} e^{-|x-Q_j|} \\ &\leq C w_i^{\sigma-\sigma'} \sum_{j\neq i} e^{-\sigma'|Q_i-Q_j|} e^{-(1-\sigma')|x-Q_j|} \leq C \varepsilon^{(1+\sigma')(1-\eta)} w_i^{\sigma-\sigma'} \quad \text{in } A_{\varepsilon,i}. \end{aligned}$$

For $x \in \mathbb{R}^3 \setminus (\cup_{i=1}^k A_{\varepsilon,i})$ we get $|x - Q_i| \geq \frac{1-\eta}{2} \log \frac{1}{\varepsilon^2}$ for every $i = 1, \dots, k$, by which

$$\left| f(w_{\mathbf{Q}}) - \sum_{j=1}^k f(w_j) \right| \leq C \sum_{j=1}^k |w_j|^{1+\sigma} \leq C \sum_{j=1}^k e^{-(1+\sigma')|x-Q_j|} w_j^{\sigma-\sigma'} \leq C \varepsilon^{(1+\sigma')(1-\eta)} \sum_{j=1}^k w_j^{\sigma-\sigma'}.$$

$\square$

5. The Linearized Equation

Let us equip $H_0^1(\Omega_\varepsilon)$ and $L^2(\Omega_\varepsilon)$ with the following scalar product respectively:

$$(u,v)_\varepsilon = \int_{\Omega_\varepsilon} \big(\nabla u \nabla v + uv\big) dx, \quad \langle u,v\rangle_\varepsilon = \int_{\Omega_\varepsilon} uv\, dx.$$

Taken $\mathbf{Q} = (Q_1, \dots, Q_k) \in \overline{\Gamma}_\varepsilon$, we introduce the following functions:

$$Z_{\varepsilon,i,j} = (1-\Delta)\frac{\partial w_{\varepsilon,i}}{\partial x_j}, \quad i \in \{1,\dots,k\},\ j \in \{1,2,3\}.$$

By using (4.5) we deduce

$$Z_{\varepsilon,i,j} = (1-\Delta)\frac{\partial w_i}{\partial x_j} + o(\varepsilon^3) w_i^{1/2} = f'(w_i)\frac{\partial w_i}{\partial x_j} + o(\varepsilon^3) w_i^{1/2} \tag{5.1}$$

uniformly for $x \in \mathbb{R}^3$ and $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$. After integration by parts it is immediate to prove that

$$\Big(\phi, \frac{\partial w_{\varepsilon,i}}{\partial x_j}\Big)_\varepsilon = \langle \phi, Z_{\varepsilon,i,j}\rangle_\varepsilon \quad \forall \phi \in H_0^1(\Omega_\varepsilon), \tag{5.2}$$

then orthogonality to the functions $\frac{\partial w_{\varepsilon,i}}{\partial x_j}$ in $H_0^1(\Omega_\varepsilon)$ is equivalent to orthogonality to $Z_{\varepsilon,i,j}$ in $L^2(\Omega_\varepsilon)$. It is easy to show that $(\frac{\partial w_i}{\partial x_j}, \frac{\partial w_i}{\partial x_\ell})_{H^1(\mathbb{R}^3)} = 0$ for $j \neq \ell$; furthermore for $i \neq m$, since $|Q_i - Q_m| \to +\infty$ as $\varepsilon \to 0^+$, we get $(\frac{\partial w_i}{\partial x_j}, \frac{\partial w_m}{\partial x_\ell})_{H^1(\mathbb{R}^3)} = o(1)$. Hence, using again (4.5), we can write

$$\Big\langle Z_{\varepsilon,i,j}, \frac{\partial w_{\varepsilon,m}}{\partial x_\ell}\Big\rangle_\varepsilon = \Big(\frac{\partial w_{\varepsilon,i}}{\partial x_j}, \frac{\partial w_{\varepsilon,m}}{\partial x_\ell}\Big)_\varepsilon = \Big(\frac{\partial w_i}{\partial x_j}, \frac{\partial w_m}{\partial x_\ell}\Big)_{H^1(\mathbb{R}^3)} + o(1) = \delta_{im}\delta_{j\ell}\Big\|\frac{\partial w}{\partial x_1}\Big\|^2_{H^1(\mathbb{R}^3)} + o(1) \tag{5.3}$$

as $\varepsilon \to 0^+$ uniformly for $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ (δ_{im} and δ_{jn} denoting the Kroneker's symbols).

We first consider a linear problem: taken $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ and given $h \in C(\overline{\Omega}_\varepsilon)$, find a function ϕ and constants $\beta_{i,j}$ satisfying

$$\begin{cases} L_{\mathbf{Q}}[\phi] = h + \displaystyle\sum_{i,j} \beta_{ij} Z_{\varepsilon,i,j}, \\ \phi \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon),\ \langle \phi, Z_{\varepsilon,i,j}\rangle_\varepsilon = 0 \text{ for } i = 1,\dots,k,\ j = 1,2,3, \end{cases} \tag{5.4}$$

where

$$L_{\mathbf{Q}}[\phi] := \Delta\phi - \phi + f'(w_{\varepsilon,\mathbf{Q}})\phi - \delta T_\varepsilon[w^2_{\varepsilon,\mathbf{Q}}]\phi - 2\delta T_\varepsilon[w_{\varepsilon,\mathbf{Q}}\phi] w_{\varepsilon,\mathbf{Q}}.$$

Now we prove the following a priori estimate for (5.4).

Lemma 5.1. *There exists a constant $C > 0$ such that, provided that ε is sufficiently small, if $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ and $(\phi, h, \beta_{i,j})$ satisfies (5.4), the following holds:*

$$\|\phi\|_\infty \le C(\|h\|_2 + \|h\|_\infty).$$

Proof. We argue by contradiction. Assume the existence of a sequence $\varepsilon_n \to 0^+$, $\mathbf{Q}_n = (Q_1^n, \dots, Q_k^n) \in \overline{\Gamma}_{\varepsilon_n}$, $(\tilde\phi_n, \tilde\beta^n_{i,j}) \in \big(H^2(\Omega_{\varepsilon_n}) \cap H_0^1(\Omega_{\varepsilon_n})\big) \times \mathbb{R}^{3k}$, $h_n \in C(\overline{\Omega}_{\varepsilon_n})$ satisfying (5.4) such that

$$\|\tilde\phi_n\|_\infty > n(\|\tilde h_n\|_2 + \|\tilde h_n\|_\infty).$$

Since $H^2(\Omega_\varepsilon) \subset C(\overline{\Omega}_\varepsilon)$, it makes sense to set $\phi_n = \frac{\tilde{\phi}_n}{\|\tilde{\phi}_n\|_\infty}$, $\beta^n_{i,j} = \frac{\tilde{\beta}^n_{i,j}}{\|\tilde{\phi}_n\|_\infty}$, $h_n = \frac{\tilde{h}_n}{\|\tilde{\phi}_n\|_\infty}$. We obtain that $(\phi_n, \beta^n_{i,j}, h_n)$ satisfies (5.4) and

$$\|\phi_n\|_\infty = 1, \ \ \|h_n\|_2 + \|h_n\|_\infty = o(1).$$

Choose $(m, \ell) \in \{1, \dots, k\} \times \{1, 2, 3\}$ such that, up to a subsequence, $|\beta^n_{m,\ell}| \geq |\beta^n_{i,j}|$ for every (i, j) and n. By multiplying the equation in (5.4) by $\frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell}$ and integrating over Ω_{ε_n}, we get

$$\sum_{i,j} \beta^n_{i,j} \int_{\Omega_{\varepsilon_n}} Z_{\varepsilon_n,i,j} \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} dx = -\int_{\Omega_{\varepsilon_n}} h_n \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} dx + \int_{\Omega_{\varepsilon_n}} L_{\mathbf{Q}_n}[\phi_n] \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} dx. \tag{5.5}$$

First examine the left hand side of (5.5). By using (5.3)

$$\sum_{i,j} \beta^n_{i,j} \int_{\Omega_{\varepsilon_n}} Z_{\varepsilon_n,i,j} \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} dx = \beta^n_{m,\ell} \Big(\Big\| \frac{\partial w}{\partial x_1} \Big\|^2_{H^1(\mathbb{R}^3)} + o(1) \Big). \tag{5.6}$$

The first term on the right hand side of (5.5) can be estimated as

$$\int_{\Omega_{\varepsilon_n}} \Big| h_n \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} \Big| dx \leq \|h_n\|_\infty \int_{\mathbb{R}^3} |\nabla w_{\varepsilon_n,m}| dx = o(1). \tag{5.7}$$

As regards the last term in (5.5), by d) of Proposition 4.1 we have

$$|T_{\varepsilon_n}[w^2_{\varepsilon_n,\mathbf{Q}_n}]|, \, |T_{\varepsilon_n}[w_{\varepsilon_n,\mathbf{Q}_n} \phi_n]| \leq C\varepsilon_n^2. \tag{5.8}$$

Furthermore, by (4.5) and (5.1) we deduce

$$\begin{aligned} \int_{\Omega_{\varepsilon_n}} \Big| Z_{\varepsilon_n,m,\ell} - f'(w_{\varepsilon_n,\mathbf{Q}_n}) \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} \Big| dx &= \int_{\Omega_{\varepsilon_n}} \Big| f'(w_m) - \sum_{j=1}^k f'(w_j) \Big| \Big| \frac{\partial w_m}{\partial x_\ell} \Big| dx + o(1) \\ &= \sum_{j \neq m} \int_{\Omega_{\varepsilon_n}} \Big| f'(w_j) \frac{\partial w_m}{\partial x_\ell} \Big| dx + o(1) = o(1) \end{aligned}$$

since $|Q^n_j - Q^n_m| \to +\infty$ for $j \neq m$. Then using (5.8) we obtain

$$\begin{aligned} \int_{\Omega_{\varepsilon_n}} L_{\mathbf{Q}_n}[\phi_n] \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} dx &= \int_{\Omega_{\varepsilon_n}} \phi_n \Big[-Z_{\varepsilon_n,m,\ell} + f'(w_{\varepsilon_n,\mathbf{Q}_n}) \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} \Big] \, dx \\ &\quad - \delta \int_{\Omega_{\varepsilon_n}} \frac{\partial w_{\varepsilon_n,m}}{\partial x_\ell} \Big[T_{\varepsilon_n}[w^2_{\varepsilon_n,\mathbf{Q}_n}] \phi_n + 2T_{\varepsilon_n}[w_{\varepsilon_n,\mathbf{Q}_n} \phi_n] w_{\varepsilon_n,\mathbf{Q}_n} \Big] dx = o(1). \end{aligned}$$

Combining this with (5.5), (5.6) and (5.7), we achieve $\beta^n_{i,j} = o(1)$ for every (i, j), by which $\|h_n + \sum_{i,j} \beta^n_{i,j} Z_{\varepsilon_n,i,j}\|_\infty = o(1)$. Hence, by (5.8), we get

$$\|\Delta\phi_n - \phi_n + f'(w_{\varepsilon_n,\mathbf{Q}_n})\phi_n\|_\infty = o(1). \tag{5.9}$$

Fix $R > 0$. We claim that

$$\|\phi_n\|_{L^\infty(\cup_{j=1}^k B_R(Q^n_j))} = o(1). \tag{5.10}$$

Otherwise, we may assume that $\|\phi_n\|_{L^\infty(B_R(Q^n_1))} \geq c > 0$. By multiplying the equation in (5.4) by ϕ_n and integrating by parts, using a) and b) of Proposition 4.1, we immediately get

$$\int_{\Omega_{\varepsilon_n}} \big(|\nabla \phi_n|^2 + \phi_n^2 \big) dx \leq \int_{\Omega_{\varepsilon_n}} |f'(w_{\varepsilon_n,\mathbf{Q}_n})| dx + \int_{\Omega_{\varepsilon_n}} |h_n \phi_n| dx \leq C + \|h_n\|_2 \|\phi_n\|_2 = C + o(\|\phi_n\|_2),$$

then the sequence ϕ_n is bounded in $H^1(\mathbb{R}^3)$, and hence, possibly passing to a subsequence, $\phi_n(x+Q_1^n) \to \phi_0$ weakly in $H^1(\mathbb{R}^3)$ and a.e. in $\mathbb{R}^3$, and ϕ_0 satisfies

$$\Delta\phi_0 - \phi_0 + f'(w)\phi_0 = 0, \quad \|\phi_0\|_\infty \leq 1.$$

According to elliptic regularity theory we may assume $\phi_n(\cdot + Q_1^n) \to \phi_0$ uniformly on compact sets (see, for example, [25, corollary 4.7]), then $\|\phi_0\|_\infty \geq c$. By assumption (f2) $\phi_0 = \sum_{j=1}^3 a_j \frac{\partial w}{\partial x_j}$. On the other hand for $\ell = 1, 2, 3$, using (5.1), $0 = \int_{\mathbb{R}^3} \phi_n(x+Q_1^n) Z_{\varepsilon_n,1,\ell}(x+Q_1^n) \to \sum_{j=1}^3 a_j \int_{\mathbb{R}^3} \frac{\partial w}{\partial x_j}(1-\Delta)\frac{\partial w}{\partial x_\ell} = a_\ell \|\frac{\partial w}{\partial x_1}\|_{H^1}^2$, which implies $a_\ell = 0$, that is $\phi_0 = 0$. The contradiction follows.

Hence we have proved (5.10), by which we immediately obtain

$$\left\| f'(w_{\varepsilon_n,\mathbf{Q}_n})\phi_n \right\|_\infty = o(1)$$

and, by (5.9),

$$\|\Delta\phi_n - \phi_n\|_\infty = o(1)$$

By standard regularity results $\phi_n \in C^2(\Omega_{\varepsilon_n}) \cap C(\overline{\Omega}_{\varepsilon_n})$. Let $\overline{x}_n$ be the maximum point for $|\phi_n|$ in Ω_{ε_n}. Then we get $|\phi_n|(\overline{x}_n) = 1$ and $\Delta|\phi_n|(\overline{x}_n) \leq 0$, by which $|\Delta\phi_n(\overline{x}_n) - \phi_n(\overline{x}_n)| = |\Delta|\phi_n|(\overline{x}_n) - |\phi_n|(\overline{x}_n)| > 1$, which is a contradiction. $\square$

Now we are in position to provide the existence of a solution for the system (5.4).

Lemma 5.2. *For $\varepsilon > 0$ sufficiently small, for every $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ and $h \in C(\overline{\Omega}_\varepsilon)$, there exists a unique pair $(\phi, \beta_{i,j}) \in \left(H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)\right) \times \mathbb{R}^{3k}$ solving (5.4). Furthermore*

$$\|\phi\|_{H^1} + \|\phi\|_\infty \leq C(\|h\|_2 + \|h\|_\infty).$$

Proof. The existence follows from Fredholm alternative. To this aim, for every $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ let us consider $\mathcal{H}_{\mathbf{Q}}$ the closed subset of $H_0^1(\Omega_\varepsilon)$ defined by

$$\mathcal{H}_{\mathbf{Q}} = \left\{ \phi \in H_0^1(\Omega_\varepsilon) \,\middle|\, \left(\phi, \frac{\partial w_{\varepsilon,i}}{\partial x_j}\right)_\varepsilon = 0 \ \forall i = 1, \ldots, k, \ \forall j = 1, 2, 3 \right\}.$$

Notice that, by (5.2), $\phi \in \mathcal{H}_{\mathbf{Q}}$ solves the equation $L_{\mathbf{Q}}[\phi] = h + \sum_{i,j} \beta_{i,j} Z_{\varepsilon,i,j}$ if and only if

$$(\phi, \psi)_\varepsilon - \langle f'(w_{\varepsilon,\mathbf{Q}})\phi, \psi\rangle_\varepsilon + \delta\langle T_\varepsilon[w_{\varepsilon,\mathbf{Q}}^2]\phi + 2T_\varepsilon[w_{\varepsilon,\mathbf{Q}}\phi]w_{\varepsilon,\mathbf{Q}}, \psi\rangle_\varepsilon = -\langle h, \psi\rangle_\varepsilon \ \ \forall \psi \in \mathcal{H}_{\mathbf{Q}}. \tag{5.11}$$

Indeed, once we know ϕ, we can determine the unique $\beta_{i,j}$ from the linear system of equations

$$\begin{aligned}-\left\langle f'(w_{\varepsilon,\mathbf{Q}})\phi, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\right\rangle_\varepsilon + \delta\left\langle T_\varepsilon[w_{\varepsilon,\mathbf{Q}}^2]\phi + 2T_\varepsilon[w_{\varepsilon,\mathbf{Q}}\phi]w_{\varepsilon,\mathbf{Q}}, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\right\rangle_\varepsilon \\ = -\left\langle h, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\right\rangle_\varepsilon - \sum_{i,j} \beta_{i,j} \left\langle Z_{\varepsilon,i,j}, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\right\rangle_\varepsilon, \ \ m = 1, \ldots, k, \ n = 1, 2, 3.\end{aligned}$$

According to (5.3), the coefficient matrix is nonsingular since it is dominated by its diagonal, that is $det\langle Z_{\varepsilon,i,j}, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\rangle_\varepsilon = \|\frac{\partial w}{\partial x_1}\|_{H^1}^{2(3+k)} + o(1)$. By standard elliptic regularity, $\phi \in H^2(\Omega_\varepsilon)$.

Thus it remains to solve (5.11). According to Riesz's representation theorem, take $\mathcal{K}_{\mathbf{Q}}(\phi)$, $\overline{h} \in \mathcal{H}_{\mathbf{Q}}$ such that

$$(\mathcal{K}_{\mathbf{Q}}(\phi), \psi)_\varepsilon = -\langle f'(w_{\varepsilon,\mathbf{Q}})\phi, \psi\rangle_\varepsilon + \delta\langle T_\varepsilon[w_{\varepsilon,\mathbf{Q}}^2]\phi + 2T_\varepsilon[w_{\varepsilon,\mathbf{Q}}\phi]w_{\mathbf{Q}}, \psi\rangle_\varepsilon \;\; \forall \psi \in \mathcal{H}_{\mathbf{Q}},$$

$$(\overline{h}, \psi)_\varepsilon = -\langle h, \psi\rangle_\varepsilon \;\; \forall \psi \in \mathcal{H}_{\mathbf{Q}}.$$

Then problem (5.11) consists in finding $\phi \in \mathcal{H}_{\mathbf{Q}}$ such that

$$\phi + \mathcal{K}_{\mathbf{Q}}(\phi) = \overline{h}. \tag{5.12}$$

It is easy to prove that $\mathcal{K}_{\mathbf{Q}}$ is a linear compact operator form $\mathcal{H}_{\mathbf{Q}}$ to $\mathcal{H}_{\mathbf{Q}}$. Using Fredholm's alternatives, (5.12) has a unique solution for each $\overline{h}$, if and only if (5.12) has a unique solution for $\overline{h} = 0$. Let $\phi \in \mathcal{H}_{\mathbf{Q}}$ be a solution of $\phi + \mathcal{K}_{\mathbf{Q}}(\phi) = 0$; then ϕ solves the system (5.4) with $h = 0$ for some $\beta_{i,j} \in \mathbb{R}$. Lemma 5.1 implies $\phi \equiv 0$.

Finally, by multiplying the equation in (5.4) by ϕ and integrating by parts, using a) and b) of Proposition 4.1, we immediately get

$$\int_{\Omega_\varepsilon} \big(|\nabla\phi|^2 + \phi^2\big)dx \le \int_{\Omega_\varepsilon} |f'(w_{\varepsilon,\mathbf{Q}})\phi^2|dx + \int_{\Omega_\varepsilon} |h\phi|dx \le C(\|\phi\|_\infty + \|h\|_2)\|\phi\|_2 \le C(\|\phi\|_\infty + \|h\|_2)\|\phi\|_{H^1},$$

by which

$$\|\phi\|_{H^1} \le C(\|\phi\|_\infty + \|h\|_2)$$

and we conclude by using Lemma 5.1 □

6. Liapunov-Schmidt Reduction

The object is now to solve the following nonlinear problem: given $\mathbf{Q} = (Q_1, \dots, Q_k) \in \overline{\Gamma}_\varepsilon$, find $(\phi, \beta_{i,j})$ solving

$$\begin{cases} S_\varepsilon[w_{\varepsilon,\mathbf{Q}} + \phi] = \displaystyle\sum_{i,j} \beta_{i,j} Z_{\varepsilon,i,j}, \\ \phi \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon), \;\; \langle \phi, Z_{\varepsilon,i,j}\rangle_\varepsilon = 0 \quad i = 1, \dots, k, \; j = 1, 2, 3, \end{cases} \tag{6.13}$$

where

$$S_\varepsilon[\psi] = \Delta\psi - \psi + f(\psi) - \delta\psi T_\varepsilon[\psi^2].$$

Lemma 6.1. *Fix $\tau \in (\frac{3}{2}, \frac{5+2\sigma}{4}(1-\eta))$. Provided that $\varepsilon > 0$ is sufficiently small, for every $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ there is a unique pair $(\phi_{\mathbf{Q}}, \beta_{i,j}(\mathbf{Q})) \in \big(H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)\big) \times \mathbb{R}^{3k}$ satisfying (6.13) and*

$$\|\phi_{\mathbf{Q}}\|_\infty < \varepsilon^\tau, \;\; \|\phi_{\mathbf{Q}}\|_{H^1} < \varepsilon^\tau. \tag{6.14}$$

Proof. We write the equation in (6.13) in the following form:

$$L_{\mathbf{Q}}[\phi] = -S_\varepsilon[w_{\varepsilon,\mathbf{Q}}] - N_{\mathbf{Q}}[\phi] + \sum_{i,j} \beta_{ij} Z_{\varepsilon,i,j} \tag{6.15}$$

and use contraction mapping theorem. Here

$$N_{\mathbf{Q}}[\phi] = f(w_{\varepsilon,\mathbf{Q}} + \phi) - f(w_{\varepsilon,\mathbf{Q}}) - f'(w_{\varepsilon,\mathbf{Q}})\phi - \delta(w_{\varepsilon,\mathbf{Q}} + \phi)T_\varepsilon[\phi^2] - 2\delta\phi T_\varepsilon[w_{\varepsilon,\mathbf{Q}}\phi].$$

Consider the matric space $\mathcal{B}_{\mathbf{Q}} = \{\phi \in C(\overline{\Omega}_\varepsilon) \,|\, \|\phi\|_2 \le \varepsilon^\tau,\, \|\phi\|_\infty \le \varepsilon^\tau\}$ endowed with the norm $\|\cdot\|_* = \|\cdot\|_2 + \|\cdot\|_\infty$. Taken $\phi_1,\, \phi_2 \in \mathcal{B}_{\mathbf{Q}}$ we compute

$$\|f(w_{\varepsilon,\mathbf{Q}} + \phi_1) - f'(w_{\varepsilon,\mathbf{Q}})\phi_1 - f(w_{\varepsilon,\mathbf{Q}} + \phi_2) - f'(w_{\varepsilon,\mathbf{Q}})\phi_2\|_* \le \sup_{\xi \in \mathcal{B}_{\mathbf{Q}}} \|f'(w_{\varepsilon,\mathbf{Q}} + \xi) - f'(w_{\varepsilon,\mathbf{Q}})\|_\infty \|\phi_1 - \phi_2\|_* \le C\varepsilon^{\sigma\tau}\|\phi_1 - \phi_2\|_*,$$

by assumption (f1). By c) of Proposition 4.1 we get

$$\begin{aligned}\|(w_{\varepsilon,\mathbf{Q}} + \phi_1)T_\varepsilon[\phi_1^2] - (w_{\varepsilon,\mathbf{Q}} + \phi_2)T_\varepsilon[\phi_2^2]\|_* &\le \|T_\varepsilon[\phi_1^2 - \phi_2^2](w_{\varepsilon,\mathbf{Q}} + \phi_1)\|_* + \|T_\varepsilon[\phi_2^2](\phi_1 - \phi_2)\|_* \\ &\le C\varepsilon\sqrt{\varepsilon}\|\phi_1 - \phi_2\|_*.\end{aligned}$$

In a similar way

$$\|\phi_1 T_\varepsilon[w_{\varepsilon,\mathbf{Q}}\phi_1] - \phi_2 T_\varepsilon[w_{\varepsilon,\mathbf{Q}}]\phi_2\|_* \le C\varepsilon\sqrt{\varepsilon}\|\phi_1 - \phi_2\|_*,$$

by which

$$\|N_{\mathbf{Q}}[\phi_1] - N_{\mathbf{Q}}[\phi_2]\|_* \le C(\varepsilon^{\sigma\tau} + \varepsilon\sqrt{\varepsilon})\|\phi_1 - \phi_2\|_* \;\; \forall \phi_1,\, \phi_2 \in \mathcal{B}_{\mathbf{Q}},\;\; \forall \mathbf{Q} \in \overline{\Gamma}_\varepsilon. \tag{6.16}$$

For every $\phi \in \mathcal{B}_{\mathbf{Q}}$ we define $\mathcal{A}_{\mathbf{Q}}[\phi] \in H^2(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ to be the unique solution to the system (5.4) given by Lemma 5.2 with $h = h_{\mathbf{Q}}[\phi] := -S_\varepsilon[w_{\varepsilon,\mathbf{Q}}] - N_{\mathbf{Q}}[\phi]$. By (6.16), Lemma 4.3 and Lemma 5.2 and the choice of τ

$$\|\mathcal{A}_{\mathbf{Q}}[\phi]\|_{H^1} + \|\mathcal{A}_{\mathbf{Q}}[\phi]\|_\infty \le \|h_{\mathbf{Q}}[\phi]\|_* \le C(\varepsilon^{\frac{5+2\sigma}{4}(1-\eta)} + \varepsilon^{(\sigma+1)\tau} + \varepsilon^{\frac{3}{2}+\tau}) < \varepsilon^\tau$$

at least for small ε, and hence $\mathcal{A}_{\mathbf{Q}}[\phi] \in \mathcal{B}_{\mathbf{Q}}$. Moreover, since $\mathcal{A}_{\mathbf{Q}}[\phi_1] - \mathcal{A}_{\mathbf{Q}}[\phi_2]$ solves the system (5.4) with $h = -N_{\mathbf{Q}}[\phi_1] + N_{\mathbf{Q}}[\phi_2]$, by (6.16) and Lemma 5.2 we also have that

$$\|\mathcal{A}_{\mathbf{Q}}[\phi_1] - \mathcal{A}_{\mathbf{Q}}[\phi_2]\|_* \le C\|N_{\mathbf{Q}}[\phi_1] - N_{\mathbf{Q}}[\phi_2]\|_* < \|\phi_1 - \phi_2\|_* \;\; \forall \phi_1,\, \phi_2 \in \mathcal{B}_{\mathbf{Q}}, \quad \forall \mathbf{Q} \in \overline{\Gamma}_\varepsilon,$$

i.e. the map $\mathcal{A}_{\mathbf{Q}}$ is a contraction map from $\mathcal{B}_{\mathbf{Q}}$ to $\mathcal{B}_{\mathbf{Q}}$. By the contraction mapping theorem, (6.13) has a unique solution $(\phi_{\mathbf{Q}}, \beta_{i,j}(\mathbf{Q})) \in \mathcal{B}_{\mathbf{Q}} \times \mathbb{R}^{3k}$. □

Lemma 6.2. *For $\varepsilon > 0$ sufficiently small the map $\mathbf{Q} \in \Gamma_\varepsilon \to \phi_{\mathbf{Q}} \in H_0^1(\Omega_\varepsilon)$ constructed in Lemma 6.1 is C^1.*

Proof. Consider the following map $K : \Gamma_\varepsilon \times H_0^1(\Omega_\varepsilon) \times \mathbb{R}^{3k} \to H_0^1(\Omega_\varepsilon) \times \mathbb{R}^{3k}$ of class C^1:

$$K(\mathbf{Q}, \phi, \beta_{i,j}) = \begin{pmatrix} (1-\Delta)^{-1}(S_\varepsilon[w_{\varepsilon,\mathbf{Q}} + \phi]) - \sum_{i,j} \beta_{ij} \frac{\partial w_{\varepsilon,i}}{\partial x_j} \\ (\phi, \frac{\partial w_{\varepsilon,i}}{\partial x_j})_\varepsilon \end{pmatrix}, \tag{6.17}$$

where $v = \big(1-\Delta\big)^{-1}(h)$ is defined as the unique solution in $H_0^1(\Omega_\varepsilon)$ of

$$v - \Delta v = h.$$

It is immediate that $(\phi, \beta_{i,j})$ solves the system (6.13) if and only if $K(\mathbf{Q}, \phi, \beta_{i,j}) = 0$. We are going to prove that, provided that ε is sufficiently small, for every $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ the linear operator

$$\frac{\partial K(\mathbf{Q}, \phi, \beta_{i,j})}{\partial(\phi, \beta_{i,j})}\Big|_{(\mathbf{Q}, \phi_{\mathbf{Q}}, \beta_{i,j}(\mathbf{Q}))} : H_0^1(\Omega_\varepsilon) \times \mathbb{R}^{3k} \to H_0^1(\Omega_\varepsilon) \times \mathbb{R}^{3k}$$

is invertible. But first notice how, assuming this, the thesis easily follows: indeed the uniqueness of the local solution $(\phi_{\mathbf{Q}}, \beta_{i,j}(\mathbf{Q}))$ provided by Lemma 6.1 implies that the map $\Phi_\varepsilon : \mathbf{Q} \in \Gamma_\varepsilon \to \phi_{\mathbf{Q}} \in H^1_0(\Omega_\varepsilon)$ actually coincides with the implicit function associated to K, hence the C^1-regularity will follow from the Implicit Function Theorem.

Now we compute

$$\frac{\partial K(\mathbf{Q}, \phi, \beta_{i,j})}{\partial(\phi, \beta_{i,j})}|_{(\mathbf{Q},\phi_{\mathbf{Q}},\beta_{i,j}(\mathbf{Q}))}[\hat{\phi}, \hat{\beta}_{i,j}] \quad = \quad \begin{pmatrix} (1-\Delta)^{-1}\big(S'_\varepsilon[w_{\mathbf{Q}} + \phi_{\mathbf{Q}}](\hat{\phi})\big) - \sum_{i,j} \hat{\beta}_{ij} \frac{\partial w_{\varepsilon,i}}{\partial x_j} \\ (\hat{\phi}, \frac{\partial w_{\varepsilon,i}}{\partial x_j})_\varepsilon \end{pmatrix}.$$

Proceeding as in the proof of Lemma 5.2, $(\hat{\phi}, \hat{\beta}_{ij})$ solves the system

$$\frac{\partial K(\mathbf{Q}, \phi, \beta_{ij})}{\partial(\phi, \beta_{i,j})}|_{(\mathbf{Q},\phi_{\mathbf{Q}},\beta_{i,j}(\mathbf{Q}))}[\hat{\phi}, \hat{\beta}_{ij}] = (\theta, \gamma_{ij})$$

if and only if (since $\big((1-\Delta)^{-1}(h), \psi\big)_\varepsilon = \langle h, \psi\rangle_\varepsilon$ and since (5.2) holds) $\hat{\phi}$ satisfies

$$\big\langle S'_\varepsilon[w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}}](\hat{\phi}), \psi\big\rangle_\varepsilon = (\theta, \psi)_\varepsilon \;\; \forall \psi \in \mathcal{H}_{\mathbf{Q}}, \tag{6.18}$$

being $\mathcal{H}_{\mathbf{Q}}$ the closed subset of $H^1_0(\Omega_\varepsilon)$ defined in Lemma 5.2, and

$$\Big(\hat{\phi}, \frac{\partial w_{\varepsilon,i}}{\partial x_j}\Big)_\varepsilon = \gamma_{ij}. \tag{6.19}$$

Indeed, once we know $\hat{\phi}$, the related $\hat{\beta}_{i,j}$ are given by the following system of equations:

$$\Big\langle S'_\varepsilon[w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}}](\hat{\phi}), \frac{\partial w_{\varepsilon,m}}{\partial x_n}\Big\rangle_\varepsilon = \sum_{i,j} \hat{\beta}_{i,j} \Big(\frac{\partial w_{\varepsilon,i}}{\partial x_j}, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\Big)_\varepsilon + \Big(\theta, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\Big)_\varepsilon$$

for $m \in \{1, \dots, k\}$, $n \in \{1, 2, 3\}$ (as we have already observed in Lemma 5.2, this system is uniquely solvable for small ε).
Thus it remains to solve (6.18) and (6.19). We can decompose $\hat{\phi} = \overline{\phi} + \sum_{i,j} c_{ij} \frac{\partial w_{\varepsilon,i}}{\partial x_j}$, where $\overline{\phi} \in \mathcal{H}_{\mathbf{Q}}$. According to (6.19), the coefficients c_{ij} are immediately determined by the following system:

$$\sum_{i,j} c_{i,j} \Big(\frac{\partial w_{\varepsilon,i}}{\partial x_j}, \frac{\partial w_{\varepsilon,m}}{\partial x_n}\Big)_\varepsilon = \gamma_{m,n} \;\; m \in \{1, \dots, k\}, \; n \in \{1, 2, 3\},$$

which is uniquely solvable for small ε. (6.18) may be rewritten as

$$\begin{aligned} &(\overline{\phi}, \psi)_\varepsilon - \big\langle f'(w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})\hat{\phi}, \psi\big\rangle_\varepsilon + \delta\big\langle \hat{\phi} T_\varepsilon[(w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})^2], \psi\big\rangle_\varepsilon \\ &\qquad + 2\delta\big\langle (w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}}) T_\varepsilon\big[(w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})\hat{\phi}\big], \psi\big\rangle_\varepsilon = -(\theta, \psi)_\varepsilon, \;\; \forall \psi \in \mathcal{H}_{\mathbf{Q}}. \end{aligned} \tag{6.20}$$

According to Riesz's representation theorem, for every $\overline{\phi} \in \mathcal{H}_{\mathbf{Q}}$ take $\mathcal{W}_{\mathbf{Q}}(\overline{\phi})$, $\overline{\theta} \in \mathcal{H}_{\mathbf{Q}}$ such that

$$\begin{aligned} (\mathcal{W}_{\mathbf{Q}}(\overline{\phi}), \psi)_\varepsilon = &- \big\langle f'(w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})\overline{\phi}, \psi\big\rangle_\varepsilon + \delta\big\langle \overline{\phi} T_\varepsilon[(w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})^2], \psi\big\rangle_\varepsilon \\ &+ 2\delta\big\langle (w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}}) T_\varepsilon[(w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})\overline{\phi}], \psi\big\rangle_\varepsilon \qquad \forall \psi \in \mathcal{H}_{\mathbf{Q}}, \end{aligned}$$

and

$$(\overline{\theta},\psi)_\varepsilon = -\sum_{i,j} c_{i,j}\Big\langle f'(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})\frac{\partial w_{\varepsilon,i}}{\partial x_j},\psi\Big\rangle_\varepsilon + \delta\sum_{i,j} c_{i,j}\Big\langle \frac{\partial w_{\varepsilon,i}}{\partial x_j} T_\varepsilon[(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})^2],\psi\Big\rangle_\varepsilon$$
$$+ 2\delta\sum_{i,j} c_{i,j}\Big\langle (w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})T_\varepsilon\Big[(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})\frac{\partial w_{\varepsilon,i}}{\partial x_j}\Big],\psi\Big\rangle_\varepsilon \quad \forall \psi\in\mathcal{H}_{\mathbf{Q}},$$

Then problem (6.20) consists in finding $\overline{\phi}\in\mathcal{H}_{\mathbf{Q}}$ such that

$$\overline{\phi}+\mathcal{W}_{\mathbf{Q}}(\overline{\phi}) = -\theta-\overline{\theta}.$$

By (6.14) and c)-d) of Proposition 4.1, comparing the definition of $\mathcal{W}_{\mathbf{Q}}$ and $\mathcal{K}_{\mathbf{Q}}$ (see Lemma 5.2), when $\varepsilon\to 0^+$ we have

$$\mathcal{W}_{\mathbf{Q}}-\mathcal{K}_{\mathbf{Q}}\to 0 \text{ uniformly for } \mathbf{Q}\in\Gamma_\varepsilon.$$

Since we have proved that $I+\mathcal{K}_{\mathbf{Q}}$ is invertible, then the theory of the linear operators assures the invertibility of $I+\mathcal{W}_{\mathbf{Q}}$ for small ε. This concludes the proof of the lemma. □

7. Reduced energy functional

For $\varepsilon>0$ sufficiently small consider the reduced functional

$$M_\varepsilon:\overline{\Gamma}_\varepsilon\to\mathbb{R},\ \ M_\varepsilon(\mathbf{Q}) := E_\varepsilon[w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}}]-kI[w]-c_1\varepsilon^2,$$

where $\phi_{\mathbf{Q}}$ has been constructed in Lemma 6.1 and c_1 is given by Proposition 4.2.

First we provide the following estimate.

Lemma 7.1. *For $\varepsilon>0$ sufficiently small the following holds:*

$$M_\varepsilon(\mathbf{Q}) = \alpha(\mathbf{Q})-c_2\varepsilon^3 H(\varepsilon Q_1)+o(\varepsilon^3)$$

uniformly for $\mathbf{Q}\in\overline{\Gamma}_\varepsilon$, where $\alpha:\mathbb{R}^{3k}\to\mathbb{R}$ and $c_2>0$ are given by Proposition 4.2.

Proof. By using d) of Proposition 4.1 and (6.14), for $\varepsilon>0$ sufficiently small we compute

$$\begin{aligned}E_\varepsilon[w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}}] &= \frac12\int_{\Omega_\varepsilon}\big(|\nabla(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})|^2+(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})^2\big)dx-\int_{\Omega_\varepsilon}F(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})dx\\ &\quad+\frac{\delta}{4}\int_{\Omega_\varepsilon}(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})^2T_\varepsilon[(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})^2]dx\\ &= E_\varepsilon(w_{\varepsilon,\mathbf{Q}})-\int_{\Omega_\varepsilon}S_\varepsilon(w_{\varepsilon,\mathbf{Q}})\phi_{\mathbf{Q}}dx+\frac12\|\phi_{\mathbf{Q}}\|^2_{H^1}\\ &\quad-\int_{\Omega_\varepsilon}\big(F(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})-F(w_{\varepsilon,\mathbf{Q}})-f(w_{\varepsilon,\mathbf{Q}})\phi_{\mathbf{Q}}\big)dx+O(\varepsilon^{2+\tau})\end{aligned}$$

uniformly for $\mathbf{Q}\in\overline{\Gamma}_\varepsilon$. By Lemma 4.3 we have $|S_\varepsilon[w_{\varepsilon,\mathbf{Q}}]|\le\varepsilon^\tau w_{\mathbf{Q}}^{\frac{2\sigma-1}{4}}$ for small ε, while $|F(w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}})-F(w_{\varepsilon,\mathbf{Q}})-f(w_{\varepsilon,\mathbf{Q}})\phi_{\mathbf{Q}}|\le C|\phi_{\mathbf{Q}}|^2$; hence, by using again (6.14) we get

$$E_\varepsilon[w_{\varepsilon,\mathbf{Q}}+\phi_{\mathbf{Q}}] = E_\varepsilon(w_{\varepsilon,\mathbf{Q}})+O(\varepsilon^{2\tau})$$

uniformly for $\mathbf{Q}\in\overline{\Gamma}_\varepsilon$. The thesis will follow from Proposition 4.2. □

Next lemma concerns the relation between the critical points of M_ε and those of the energy functional E_ε.

Lemma 7.2. *Let $\mathbf{Q}_\varepsilon \in \Gamma_\varepsilon$ be a critical point of M_ε. Then, provided that $\varepsilon > 0$ is sufficiently small, the corresponding function $u_\varepsilon = w_{\varepsilon,\mathbf{Q}_\varepsilon} + \phi_{\mathbf{Q}_\varepsilon}$ is a solution of (4.1).*

Proof. Fix $\varepsilon_0 > 0$ sufficiently small such that Lemma 6.1 and 6.2 hold for $\varepsilon \in (0, \varepsilon_0)$. According to Lemma 6.1, for every $\varepsilon \in (0, \varepsilon_0)$ and $\mathbf{Q} \in \overline{\Gamma}_\varepsilon$ $\phi_{\mathbf{Q}}$ solves the equation

$$S_\varepsilon(w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}}) = \sum_{i,j} \beta_{ij}(\mathbf{Q}) Z_{\varepsilon,i,j} \text{ in } \Omega_\varepsilon \tag{7.1}$$

for some constants $\beta_{ij}(\mathbf{Q}) \in \mathbb{R}^{3k}$.

Let $\mathbf{Q}_\varepsilon \in \Gamma_\varepsilon$ be a critical point of M_ε:

$$\frac{\partial}{\partial Q_{m,n}}\bigg|_{\mathbf{Q}=\mathbf{Q}_\varepsilon} M_\varepsilon(\mathbf{Q}) = 0, \quad m = 1, \ldots, k, \quad n = 1, 2, 3. \tag{7.2}$$

Using e) of Proposition 4.1 and the C^1 regularity of the map $\mathbf{Q} \in \Gamma_\varepsilon \mapsto \phi_{\mathbf{Q}} \in H_0^1(\Omega_\varepsilon)$, (7.2) may be rewritten as

$$\int_{\Omega_\varepsilon} \left(\nabla u_\varepsilon \nabla \frac{\partial (w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{m,n}} + \big(u_\varepsilon - f(u_\varepsilon) + \delta u_\varepsilon T_\varepsilon[u_\varepsilon^2]\big) \frac{\partial (w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{m,n}} \right) dx \bigg|_{\mathbf{Q}=\mathbf{Q}_\varepsilon} = 0$$

for $m = 1, \ldots, k$ and $n = 1, 2, 3$, which is equivalent, by (7.1), to

$$\sum_{i,j} \beta_{ij}(\mathbf{Q}_\varepsilon) \int_{\Omega_\varepsilon} Z_{\varepsilon,i,j} \frac{\partial (w_{\varepsilon,\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{m,n}} dx \bigg|_{\mathbf{Q}=\mathbf{Q}_\varepsilon} = 0. \tag{7.3}$$

Since $\langle Z_{\varepsilon,i,j}, \phi_{\mathbf{Q}} \rangle_\varepsilon = 0$, differentiating with respect to $Q_{m,n}$ we have that

$$\int_{\Omega_\varepsilon} Z_{\varepsilon,i,j} \frac{\partial \phi_{\mathbf{Q}}}{\partial Q_{m,n}} dx = -\int_{\Omega_\varepsilon} \phi_{\mathbf{Q}} \frac{\partial Z_{\varepsilon,i,j}}{\partial Q_{m,n}} dx = O(\varepsilon^\tau), \tag{7.4}$$

since $|\frac{\partial Z_{\varepsilon,i,j}}{\partial Q_{m,n}}| = |\delta_{im} \frac{\partial Z_{\varepsilon,i,j}}{\partial x_n}| = \delta_{im} |(1 - \Delta) \frac{\partial^2 w_{\varepsilon,i}}{\partial x_j x_n}| \le C e^{-|x - Q_i|}$ by (1.4). Notice that $\frac{\partial w_{\varepsilon,\mathbf{Q}}}{\partial Q_{m,n}} = -\frac{\partial w_{\varepsilon,m}}{\partial x_n}$; hence, combining (5.3), (7.3) and (7.4) we achieve

$$\beta_{m,n}(\mathbf{Q}_\varepsilon) \Big\| \frac{\partial w}{\partial x_1} \Big\|^2_{H^1(\mathbb{R}^3)} + \sum_{i,j} o(1) \beta_{i,j}(\mathbf{Q}_\varepsilon) = 0.$$

So $\beta_{ij}(\mathbf{Q}_\varepsilon) = 0$ for $i = 1, \ldots, k$, $j = 1, 2, 3$. Hence u_ε solves the equation

$$\Delta u_\varepsilon - u_\varepsilon + f(u_\varepsilon) - \delta u_\varepsilon T_\varepsilon[u_\varepsilon^2] = 0. \tag{7.5}$$

It remains to show that $u_\varepsilon > 0$. Indeed, multiplying (7.5) by $u_\varepsilon^- = \max(0, -u_\varepsilon)$, and using (f1) we see that

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^-|^2 \, dx + \int_{\Omega_\varepsilon} |u_\varepsilon^-|^2 \, dx + \delta \int_{\Omega_\varepsilon} (u_\varepsilon^-)^2 T_\varepsilon[u_\varepsilon^2] \, dx = 0$$

which implies, by a) of Proposition 4.1, $u_\varepsilon^- = 0$. By the strong maximum principle $u_\varepsilon > 0$ in Ω_ε. $\square$

8. The Reduced Problem: Proof of Theorem 1.1

In order to complete the proof of Theorem 1.1, we study a maximization problem.

Proposition 8.1. *For $\varepsilon > 0$ sufficiently small, the following maximization problem*

$$\max\{M_\varepsilon(\mathbf{Q}) : \mathbf{Q} \in \overline{\Gamma}_\varepsilon\}$$

has a solution $\mathbf{Q}_\varepsilon = (Q_1^\varepsilon, \ldots, Q_k^\varepsilon) \in \Gamma_\varepsilon$. Furthermore, $H(\varepsilon Q_i^\varepsilon) \to H_0$ and, setting $l_\varepsilon = \min_{i\neq j} |Q_i^\varepsilon - Q_j^\varepsilon|$, the following holds:

$$\lim_{\varepsilon\to 0^+} \frac{l_\varepsilon}{|\log \varepsilon^2|} = 1, \quad \lim_{\varepsilon\to 0^+} \sum_{i\neq j} \frac{l_\varepsilon}{|Q_i^\varepsilon - Q_j^\varepsilon|} = m(k), \tag{8.1}$$

where the number $m(k)$ has been defined in Section 2.

Proof. Let $\mathbf{Q}_\varepsilon \in \overline{\Gamma}_\varepsilon$ be the maximum point of the function M_ε in the set $\overline{\Gamma}_\varepsilon$. First we prove that $H(\varepsilon Q_i^\varepsilon) \to H_0$ for $i = 1, \ldots, k$, which is equivalent, since $\varepsilon|Q_i^\varepsilon - Q_1^\varepsilon| \to 0$, to $H(\varepsilon Q_1^\varepsilon) \to H_0$. Otherwise, there would exist a sequence $\varepsilon_n \to 0^+$ such that $H(\varepsilon_n Q_1^{\varepsilon_n}) > H_0 + a > H_0$. Then let $P_0 \in \Omega$ be a harmonic center, i.e. $H(P_0) = H_0$, and set $\tilde{\mathbf{Q}}_\varepsilon = \mathbf{Q}_\varepsilon - (Q_1^\varepsilon)^k + (\frac{P_0}{\varepsilon})^k$. It is easy to prove that $\tilde{\mathbf{Q}}_\varepsilon \in \overline{\Gamma}_\varepsilon$: indeed $|\tilde{Q}_i^\varepsilon - \tilde{Q}_j^\varepsilon| = |Q_i^\varepsilon - Q_j^\varepsilon|$; furthermore $H(\varepsilon \tilde{Q}_1^\varepsilon) = H(P_0) = H_0$ and then, since $\varepsilon|\tilde{Q}_i - \tilde{Q}_1| \to 0$, for small ε $H(\varepsilon\tilde{Q}_i^\varepsilon) < 2H_0$. By applying Proposition 4.2 and Lemma 7.1 we get

$$\begin{aligned}\alpha(\mathbf{Q}_{\varepsilon_n}) - c_2\varepsilon_n^3(H_0 + a) + o(\varepsilon_n^3) &> M_{\varepsilon_n}(\mathbf{Q}_{\varepsilon_n}) \geq M_{\varepsilon_n}(\tilde{\mathbf{Q}}_{\varepsilon_n}) \\ &= \alpha(\tilde{\mathbf{Q}}_{\varepsilon_n}) - c_2\varepsilon_n^3 H_0 + o(\varepsilon_n^3) = \alpha(\mathbf{Q}_{\varepsilon_n}) - c_2\varepsilon_n^3 H_0 + o(\varepsilon_n^3)\end{aligned}$$

and the contradiction follows.

To show that $\mathbf{Q}_\varepsilon \in \Gamma_\varepsilon$, we first obtain an upper bound for $M_\varepsilon(\mathbf{Q}_\varepsilon)$. Let $\mathbf{P} = (P_1, \ldots, P_k) \in \Sigma_k$ be an optimal configuration given in Lemma 3.1. By the translation invariance of problem (*) we can take $P_1 = \frac{P_0}{\varepsilon\rho_\varepsilon}$, where

$$\rho_\varepsilon = \log\frac{1}{\varepsilon^2} + \log\log\frac{1}{\varepsilon^2}.$$

It is easy to see that $\rho_\varepsilon \mathbf{P}$ belongs to Γ_ε: indeed, for $i \neq j$ $\rho_\varepsilon|P_i - P_j| \geq \rho_\varepsilon > (1-\eta)\log\frac{1}{\varepsilon^2}$ and $\rho_\varepsilon|P_i - P_j| < (\log\frac{1}{\varepsilon^2})^2$ for ε sufficiently small. Furthermore $H(\varepsilon\rho_\varepsilon P_1) = H(P_0) = H_0$ and then, since $\varepsilon\rho_\varepsilon|P_i - P_1| \to 0$, for small ε $H(\varepsilon\rho_\varepsilon P_i) < 2H_0$.

By the definition of ρ_ε and (1.4) for $i \neq j$ we get

$$\frac{1}{\rho_\varepsilon|P_i - P_j|} = \frac{1 + o(1)}{|\log\varepsilon^2|P_i - P_j||}, \qquad w(\rho_\varepsilon|P_i - P_j|) \leq w(\rho_\varepsilon) \leq \frac{2A\varepsilon^2}{|\log\varepsilon^2|^2}. \tag{8.2}$$

Then, by using Proposition 4.2, Lemma 7.1 and (8.2), we have the following estimate

$$\begin{aligned}M_\varepsilon(\mathbf{Q}_\varepsilon) \geq M_\varepsilon(\rho_\varepsilon\mathbf{P}) &= c_3\varepsilon^2\sum_{i\neq j}\frac{1 + o(1)}{\rho_\varepsilon|P_i - P_j|} - c_4\sum_{i\neq j}(1 + o(1))w(\rho_\varepsilon|P_i - P_j|) - c_2\varepsilon^3 H_0 + o(\varepsilon^3) \\ &= c_3\varepsilon^2\sum_{i\neq j}\frac{1 + o(1)}{|\log\varepsilon^2|P_i - P_j||} + o\left(\frac{\varepsilon^2}{|\log\varepsilon^2|}\right) = \frac{c_3\varepsilon^2}{|\log\varepsilon^2|}m(k) + o\left(\frac{\varepsilon^2}{|\log\varepsilon^2|}\right)\end{aligned} \tag{8.3}$$

We are going to prove that $\frac{l_\varepsilon}{|\log\varepsilon^2|}\to 1$ as $\varepsilon\to 0^+$. Assume the existence of a sequence $\varepsilon_n\to 0^+$ such that $\frac{l_{\varepsilon_n}}{|\log\varepsilon_n^2|}>1+a>1$. Setting $\hat{\mathbf{Q}}_\varepsilon=\frac{1}{l_\varepsilon}\mathbf{Q}_\varepsilon\in\Sigma_k$, by using again Proposition 4.2 and Lemma 7.1,

$$\begin{aligned}M_{\varepsilon_n}(\mathbf{Q}_{\varepsilon_n})&\le c_3\varepsilon_n^2\sum_{i\neq j}\frac{1+o(1)}{|Q_i^{\varepsilon_n}-Q_j^{\varepsilon_n}|}+o(\varepsilon_n^3)\le c_3\varepsilon_n^2\sum_{i\neq j}\frac{1+o(1)}{\big((1+a)|\log\varepsilon_n^2||\hat{Q}_i^{\varepsilon_n}-\hat{Q}_j^{\varepsilon_n}|\big)}+o(\varepsilon_n^3)\\&\le\frac{c_3\varepsilon_n^2}{(1+a)|\log\varepsilon_n^2|}m(k)+o\bigg(\frac{\varepsilon_n^2}{|\log\varepsilon_n^2|}\bigg)\end{aligned}$$

in contradiction with (8.3). Without loss of generality, we may assume that $l_\varepsilon=|Q_1^\varepsilon-Q_2^\varepsilon|$. Now suppose the existence of a sequence such that $\frac{l_{\varepsilon_n}}{|\log\varepsilon_n^2|}<1-a<1$. Then, by (1.4),

$$\frac{\varepsilon_n^2}{l_{\varepsilon_n}w(l_{\varepsilon_n})}\le\frac{2}{A}\varepsilon_n^2e^{l_{\varepsilon_n}}\le\frac{2}{A}\varepsilon_n^{2a}\to 0$$

as $n\to+\infty$, by which we deduce

$$\frac{c_3\varepsilon_n^2}{|Q_1^{\varepsilon_n}-Q_2^{\varepsilon_n}|}(1+o(1))-c_4w(Q_1^{\varepsilon_n}-Q_2^{\varepsilon_n})(1+o(1))=\frac{c_3\varepsilon_n^2}{l_{\varepsilon_n}}(1+o(1))-c_4w(l_{\varepsilon_n})(1+o(1))\le 0$$

for large n. Since $l_\varepsilon\ge(1-\eta)\log\frac{1}{\varepsilon^2}$, we get

$$\begin{aligned}M_{\varepsilon_n}(\mathbf{Q}_{\varepsilon_n})&\le c_3\varepsilon_n^2\sum_{i\neq j,(i,j)\neq(1,2)}\frac{1+o(1)}{|Q_i^{\varepsilon_n}-Q_j^{\varepsilon_n}|}+o(\varepsilon_n^3)\\&\le c_3\varepsilon_n^2\sum_{i\neq j,(i,j)\neq(1,2)}\frac{1+o(1)}{(1-\eta)|\log\varepsilon_n^2||\hat{Q}_i^{\varepsilon_n}-\hat{Q}_j^{\varepsilon_n}|}+o(\varepsilon_n^3)\\&\le\frac{c_3\varepsilon_n^2}{(1-\eta)|\log\varepsilon_n^2|}\sum_{i\neq j}(1+o(1))\bigg(\frac{1}{|\hat{Q}_i^{\varepsilon_n}-\hat{Q}_j^{\varepsilon_n}|}-1\bigg)+o(\varepsilon_n^3)\\&\le\frac{c_3\varepsilon_n^2}{(1-\eta)|\log\varepsilon_n^2|}(m(k)-1)+o\bigg(\frac{\varepsilon_n^2}{|\log\varepsilon_n^2|}\bigg)\end{aligned}$$

If we choose $\eta>0$ sufficiently small such that $\frac{1}{1-\eta}(m(k)-1)<m(k)$ we achieve again a contradiction with (8.3). Then the first limit in (8.1) holds. Furthermore we deduce

$$M_\varepsilon(\mathbf{Q}_\varepsilon)\le c_3\varepsilon^2\sum_{i\neq j}\frac{1+o(1)}{|Q_i^\varepsilon-Q_j^\varepsilon|}+o(\varepsilon^3)=\frac{c_3\varepsilon^2}{|\log\varepsilon^2|}\sum_{i\neq j}\frac{1+o(1)}{|\hat{Q}_i^\varepsilon-\hat{Q}_j^\varepsilon|}+o(\varepsilon^3)\le\frac{c_3\varepsilon^2}{|\log\varepsilon^2|}m(k)+o\bigg(\frac{\varepsilon^2}{|\log\varepsilon^2|}\bigg).$$

Comparing this with (8.3) we deduce the second limit in (8.1). Finally notice that $|\hat{Q}_i^\varepsilon-\hat{Q}_j^\varepsilon|\le C$ for every $i\neq j$ and ε. Otherwise we could find a sequence $\varepsilon_n\to 0^+$ and $I,J\subset\{1,\dots,k\}$, $I,\,J\neq\emptyset$, such that $I\cup J=\{1,\dots,k\}$ and $|Q_i^{\varepsilon_n}-Q_j^{\varepsilon_n}|\to+\infty$ for $i\in I$ and $j\in J$. Set $k_1=\#I$, $k_2=\#J$. Then, up to a subsequence,

$$m(k)=\lim_{n\to\infty}\sum_{i\neq j}\frac{1}{|\hat{Q}_i^{\varepsilon_n}-\hat{Q}_j^{\varepsilon_n}|}=\lim_{n\to\infty}\sum_{i\neq j,i,j\in I}\frac{1}{|\hat{Q}_i^{\varepsilon_n}-\hat{Q}_j^{\varepsilon_n}|}+\lim_{n\to\infty}\sum_{i\neq j,i,j\in J}\frac{1}{|\hat{Q}_i^{\varepsilon_n}-\hat{Q}_j^{\varepsilon_n}|}\le m(k_1)+m(k_2)$$

in contradiction with Lemma 3.1. Hence we obtain

$$l_\varepsilon\le|Q_i^\varepsilon-Q_j^\varepsilon|\le Cl_\varepsilon.$$

Taking into account of (8.1) we obtain that $\mathbf{Q}_\varepsilon\in\Gamma_\varepsilon$ for ε sufficiently small. $\square$

Proof of Theorem 1.1.

For every $\varepsilon > 0$ set $u_\varepsilon = w_{\varepsilon,\mathbf{Q}_\varepsilon} + \phi_{\mathbf{Q}_\varepsilon}$ and $\psi_\varepsilon = T_\varepsilon[u_\varepsilon^2]$, where $\mathbf{Q}_\varepsilon$ is given by Proposition 8.1. According to Lemma 7.2, $(u_\varepsilon, \psi_\varepsilon)$ solves the system (4.1). Consider $\varepsilon_n \to 0^+$ a generic sequence. Then, by using again Proposition 8.1, up to a subsequence, $\varepsilon_n Q_i^{\varepsilon_n} \to P_0$ for every i, with $H(P_0) = H_0$. Then, fixed $r > 0$, by (6.14),

$$\int_{|\varepsilon y - P_0| \le r/2} G(\varepsilon x, \varepsilon y) u_\varepsilon^2 dy = (G(\varepsilon x, P_0) + o(1)) \int_{|\varepsilon y - P_0| \le r/2} u_\varepsilon^2 dy = (G(\varepsilon x, P_0) + o(1)) k \int_{\mathbb{R}^3} w^2 dy \tag{8.4}$$

uniformly for $|\varepsilon x - P_0| \ge r$. Furthermore by d) of Proposition 4.1, using also (1.4) and (6.14) and setting $A_\varepsilon = \{y \in \Omega \,||\varepsilon y - P_0| \ge r/2\}$,

$$\varepsilon^3 \int_{A_\varepsilon} G(\varepsilon x, \varepsilon y) u_\varepsilon^2 dy \le C\varepsilon^2 (\|w_{\mathbf{Q}_\varepsilon}^2 + \phi_{\mathbf{Q}_\varepsilon}^2\|_{L^1(A_\varepsilon)} + \|w_{\mathbf{Q}_\varepsilon}^2 + \phi_{\mathbf{Q}_\varepsilon}^2\|_{L^\infty(A_\varepsilon)}) = o(\varepsilon^3) \tag{8.5}$$

uniformly for $x \in \mathbb{R}^3$. Combining (8.4) and (8.5), and using (4.2), we conclude the proof. □

Remark 8.2. *We point out that the result of Theorem 1.1 can be improved in this sense: for every isolated minimum point P_0 for the Robin's function H, there exists a family of solutions $(v_{\varepsilon,P_0}, \phi_{\varepsilon,P_0})$ such that v_{ε,P_0} concentrates at P_0. Indeed, it is sufficient to apply a simple localization process: letting $R > 0$ be such that $H(P) > H_0$ for $0 < |P - P_0| < R$, we replace the configuration set Γ_ε with*

$$\tilde{\Gamma}_\varepsilon = \Gamma_\varepsilon \cap \left\{ \mathbf{Q} = (Q_1, \ldots, Q_k) \in \Omega_\varepsilon^k \,\middle|\, |\varepsilon Q_i - P_0| < R \;\; \forall i \right\}$$

and all the proof developed above continue to work identically. If no information is known on the minima of H, then, according to theorem 1.1, we are able to get just one family of solutions $(v_\varepsilon, \phi_\varepsilon)$ and different sequences $\varepsilon_n \to 0^+$ can give rise to different concentration points among the harmonic centers of Ω.

References

[1] A. Ambrosetti, M. Badiale, S. Cingolani. *Semiclassical states of nonlinear Schrödinger equations,* Arch. Rat. Mech. Anal. **140** (1997), 285–300.

[2] A. Ambrosetti, A. Malchiodi, S. Secchi. *Multiplicity results for some nonlinear Schrödinger equations with potentials,* Arch. Ration. Mech. Anal. **159** , no. 3 (2001), 253–271.

[3] A. Ambrosetti, A. Malchiodi, W. M. Ni. *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, I.* Comm. Math. Phys. **235**, no. 3 (2003), 427–466.

[4] A. Ambrosetti, A. Malchiodi, W. M. Ni. *Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, II.* Indiana Univ. Math. J. **53**, no. 2 (2004), 297–329.

[5] C. Bandle, M. Flucher. *Harmonic radius and concentration of energy, hyperbolic radius and Liouville's equations, $\Delta u = e^u$ and $\Delta u = u^{\frac{n+2}{n-2}}$*, SIAM Review **38**, no. 2 (1996), 191–238.

[6] V. Benci, D. Fortunato. *An eigenvalue problem for the Schrödinger-Maxwell equations,* Top. Meth. Nonlinear Anal. **11**, no. 2 (1998), 283–293.

[7] P. Cardaliguet, T. Rabah. *On the sctrict concavity of the harmonic radius in dimension $N \ge 3$,* J. Math. Pures Appl. **81**, no. 3 (2002), 223–240.

[8] X. Chen, M. Del Pino, M. Kowalczyk. *The Gierer & Meinhardt system: the breaking of homopclinics and multi-bump ground states,* Commun. Contemp. Math. **3**, no. 3 (2001), 419–439.

[9] C. C. Chen, C. S. Lin. *Uniqueness of the ground state solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^N, N \ge 3$,* Comm. Partial Differential Equations **16**, no. 8–9 (1991), 1549–1572.

[10] G. M. Coclite, V. Georgiev. *Solitary waves for Maxwell-Schrödinger equations,* Electronic J. Differential Equations **2004** (2004), no. 94, 1–31.

[11] E. N. Dancer. *A note on asymptotic uniqueness for some nonlinearities which change sign,* Bull. Austral. Math. Soc. **61** (2000), 305–312.

[12] E. N. Dancer, J. Wei. *On the profile of solutions with two sharp layers to a singularly perturbed semilinear Dirichlet problem.* Proc. Roy. Soc. Edinburgh Sect. A **127**, no. 4 (1997), 691–701.
[13] E. N. Dancer, S. Yan. *Multipeak solutions for a singular perturbed Neumann problem,* Pacific J. Math. **189**, no.2 (1999), 241–262.
[14] T. D'Aprile, D. Mugnai. *Existence of solitary waves for the nonlinear Klein-Gordon Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134**, no. 5 (2004), 893–906.
[15] T. D'Aprile, J. Wei. *On bound states concentrating on spheres for the Maxwell-Schrödinger equation*, SIAM J. Math. Anal. **37**, no. 1 (2005), 321–342.
[16] T. D'Aprile, J. Wei. *Standing waves in the Maxwell-Schrodinger equation and an optimal configuration problem*, Calc. Var. Partial Differential Equations **25**, no. 1 (2006), 105–137.
[17] M. Del Pino, P. Felmer. *Local mountain passes for semilinear elliptic problems in unbounded domains,* Calc. Var. Partial Differential Equations **4**, no. 2 (1996), 121–137.
[18] M. Del Pino, P. Felmer. *Multi-peak bound states for nonlinear Schrödinger equations,* Ann. Inst. H. Poincaré Anal. Non Linéaire **15**, no. 2 (1998), 127–149.
[19] M. Del Pino, P. Felmer. *Semi-classical states for nonlinear Schrödinger equations,* J. Funct. Anal. **149**, no. 1 (1997), 245–265.
[20] M. Del Pino, P. Felmer. *Semi-classical states of nonlinear Schrödinger equations: a variational reduction method*, Math. Ann. **324**, no. 1 (2002), 1–32.
[21] M. Del Pino, M. Kowalczyk, J. Wei. *Multi-bump ground states for the Gierer-Meinhardt system in* $\mathbb{R}^2$, Ann. Inst. H. Poincaré Anal. Non Linéaire **20**, no. 1 (2003), 53–85.
[22] L. C. Evans. *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island (1998).
[23] A. Floer, A. Weinstein. *Nonspreading wave pockets for the cubic Schrödinger equation with a bounded potential,* J. Funct. Anal. **69**, no. 3 (1986), 397–408.
[24] B. Gidas, W. M. Ni, L. Nirenberg. Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, *Adv. Math. Suppl Stud.* **7A** (1981), 369–402.
[25] D. Gilbarg, N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, Springer Verlag Berlin Heidelberg 2001.
[26] M. Grossi. *Some results on a class of nonlinear Schrödinger equations,* Math. Z. **235**, no. 4 (2000), 687–705.
[27] C. Gui. *Multipeak solutions for a semilinear Neumann problem*, Duke Math. J. **84**, no. 3 (1996), 739–769.
[28] C. Gui, J. Wei. *On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems*, Canad. J. Math. **52**, no. 3 (2000), 522–538.
[29] C. Gui, J. Wei, M. Winter. *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17**, no. 1 (2000), 47–82.
[30] L. L. Helms. *Introduction to Potential Theory*, John Wiley & sons Inc., New York, 1969.
[31] D. P. Hardin, E. B. Saff, *Discretizing manifolds via minimum energy points,* Notices of Amer. Math. Soc. **51**, no. 10 (2004), 1186–1194.
[32] L. Jeanjean, K. Tanaka. *Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities*, Calc. Var. Partial Differential Equations **21**, no. 3 (2004), 287–318.
[33] X. Kang, J. Wei. *On interacting bumps of semiclassical states of nonlinear Schrödinger equations*, Adv. Differential Equations **5**, no. 7–9 (2000), 899–928.
[34] M. K. Kwong. *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^N$, Arch. Rational Mech. Anal. **105**, no. 3 (1989), 243–266.
[35] Y.Y. Li. *On a singularly perturbed elliptic equation,* Adv. Differential Equations **6**, no. 2 (1997), 955–980.
[36] W. M. Ni, I. Takagi. *On the shape of least-energy solutions to a semilinear Neumann problem,* Comm. Pure App. Math. **44**, no. 7 (1991), 819–851.
[37] Y. J. Oh. *Existence of semi-classical bound states of nonlinear Schrödinger equation with potential on the class* $(V)_a$, Comm. Partial Differential Equations **13**, no. 12 (1988), 1499–1519.
[38] A. Pistoia. *Multi-peak solutions for a class of nonlinear Schrdinger equations*, NoDEA Nonlinear Differential Equations Appl. **9** (2002), no. 1, 69–91.
[39] P. Rabinowitz. *On a class of nonlinear Schrödinger equations,* Z. Angew. Math. Phys. **43**, no. 2 (1992), 270–291.
[40] D. Ruiz. *Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere,* Math. Models Methods Appl. Sci. **15**, no. 1 (2005), 141–164.
[41] X. Wang. *On concentration of positive bound states of nonlinear Schrödinger equations,* Comm. Math. Phys. **153**, no. 2 (1993), 229–244.
[42] J. Wei. *On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem,* J. Differential Equations **129**, no. 2 (1996), 315–333.
[43] J. Wei, M. Winter. *Symmetric and asymmetric multiple clusters in a reaction-diffusion system,* NoDEA Nonlinear Differential Equations Appl., to appear.

[44] J. Wei, M. Winter. *Clustered spots in the FitzHugh-Nagumo system,* J. Differential Equations **213**, no. 1 (2005), 121–145.

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