ON THE NUMBER OF NODAL SOLUTIONS TO A SINGULARLY PERTURBED NEUMANN PROBLEM

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Abstract. We show that for ϵ small, there are arbitrarily many **nodal** solutions for the following nonlinear elliptic Neumann problem

$$\epsilon^2 \Delta u - u + f(u) = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

where Ω is a bounded and smooth domain in $\mathbb{R}^2$ and f grows superlinearly. (A typical $f(u)$ is $f(u) = a_1 u_+^p - a_2 u_-^q, a_1, a_2 > 0, p, q > 1$.) More precisely, for any positive integer K, there exists $\epsilon_K > 0$ such that for $0 < \epsilon < \epsilon_K$, the above problem has a nodal solution with K positive local maximum points and K negative local minimum points. This solution has at least $K + 1$ nodal domains. The locations of the maximum and minimum points are related to the mean curvature on $\partial\Omega$. The solutions are constructed as critical points of some finite dimensional reduced energy functional. No assumption on the symmetry, nor the geometry, nor the topology of the domain is needed.

1. Introduction

Of concern is the following nonlinear elliptic equation

$$\epsilon^2 \Delta u - u + f(u) = 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $\epsilon > 0$ is a small parameter, and $f(u) = f_1(u_+) - f_2(u_-)$ where $u_+ = \max(u, 0), u_- = \max(-u, 0)$ and both f_1 and f_2 satisfy the following conditions:

(f1) $f_1, f_2 \in C^{1+\sigma}(R) \cap C^2_{loc}(0, +\infty)$ with $0 < \sigma \leq 1, f_1(0) = f_2(0) = f_1'(0) = f_2'(0) = 0$ and $f_1(t) = f_2(t) = 0$ for $t \leq 0$.

(f2) For $i = 1, 2$, the problem in the whole space

$$\begin{cases} \Delta w^i - w^i + f_i(w^i) = 0, w^i > 0 \text{ in } \mathbb{R}^2, \\ w^i(0) = \max_{y \in \mathbb{R}^2} w^i(y), \lim_{|y| \to +\infty} w^i(y) = 0, \end{cases} \tag{1.2}$$

has a radially symmetric solution w^i, which is nondegenerate, i.e.

$$\text{Kernel}(\Delta - 1 + f_i'(w^i)) = \text{span}\left\{\frac{\partial w^i}{\partial y_1}, \frac{\partial w^i}{\partial y_2}\right\}. \tag{1.3}$$

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Note that $f_1(u) = f_2(u) = u^p$ with $1 < p < +\infty$ is a special example. Note also that we can allow different nonlinearities for the positive and negative part of f. For example, we may have $f(u) = a_1 u_+^p - a_2 u_-^q$ for arbitrary $p, q > 1$ and $a_1, a_2 > 0$.

Problem (1.1) arises in the study of some mathematical models in chemotaxis ([15]) and pattern formation ([12]) and has been studied by numerous authors. In [18], Ni and Takagi showed that, under some conditions on $f(u)$, as $\epsilon \to 0$, the least energy solution for (1.1) has a unique maximum point, say P_ϵ, on $\partial\Omega$. Moreover, $H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P)$, where H is the mean curvature function on $\partial\Omega$. Since then, many papers further investigated the higher energy solutions of (1.1). These solutions are called spike layer solutions. A general principle is that the interior spike layer solutions are generated by distance functions. We refer the reader to the articles [1], [6], [7], [8], [9], [11], [14], [19], [22], and the references therein. On the other hand, the boundary peaked solutions are related to the boundary mean curvature function. This aspect is discussed in the papers [2], [5], [10], [13], [23], [24], [25], and the references therein.

All the above results are concerned with positive solutions. Concerning the existence and asymptotic behavior of nodal solutions, the first result was due to Noussair and Wei [20]. In [20], it showed that for ϵ sufficiently small, (1.1) has a least energy nodal solution, which has two peaks-a positive maximum and a negative minimum. Furthermore, these two peaks must approach the global maximum points of the mean curvature. In [21], the corresponding result for the Dirichlet problem is established. The effect of domain geometry on solutions with two nodal domains for the Dirichlet problem is studied in [3] and the existence of three nodal solutions for the Dirichlet problem is established in [4]. The nodal symmetry of two-peak solutions in a ball is established in [26]. As far as the authors know, there are very few results on the multiplicity of nodal solutions. The main purpose of this paper is to show the existence of **arbitrarily many** nodal solutions for (1.1) in a two-dimensional domain. Moreover, we show the existence of nodal solutions having **arbitrarily many nodal domains**.

Now we can state the main theorem of the paper:

Theorem 1.1. *Suppose that Ω is a two-dimensional bounded smooth domain and f satisfies (f1) and (f2). Let K be any fixed positive integer. Then there exists $\epsilon_K > 0$ such that for $\epsilon < \epsilon_K$, problem (1.1) admits a nodal solution u_ϵ with precisely K local maximum points $P_1, P_3, ..., P_{2K-1} \in \partial\Omega$ and K local minimum points $P_2, P_4, ..., P_{2K} \in \partial\Omega$, where $u_\varepsilon(P_i) > 0$ for i odd and $u_\varepsilon(P_i) < 0$ for i even. Moreover, u_ε has at least $K+1$ nodal domains. Furthermore, as $\varepsilon \to 0$,*

$$\Big\| u_\epsilon(x) - \sum_{j=1}^{K} \Big(w^1\Big(\frac{x - P_{2j-1}^\epsilon}{\epsilon}\Big) - w^2\Big(\frac{x - P_{2j}^\epsilon}{\epsilon}\Big) \Big) \Big\|_{L^\infty(\Omega)} \to 0, \tag{1.4}$$

and

$$H(P_j^\epsilon) \to \max_{Q\in\partial\Omega} H(Q) \qquad \text{for} \quad j = 1, ..., 2K. \tag{1.5}$$

The novelty of Theorem 1.1 lies in the fact that no assumption on the symmetry, nor the geometry, nor the topology of the domain is needed. By the results of [21], the least energy nodal solutions can only have one local maximum point and one local minimum point. Thus the solutions in Theorem 1.1 must have higher energy. It seems difficult to use direct variational method to obtain such solutions. To capture higher energy solutions, we use the so-called **"localized energy method"**-a combination of Liapunov-Schmidt reduction method and variational techniques. Namely, we first use Liapunov-Schmidt reduction method to reduce the problem to a finite dimensional one with some **reduced energy**. Then the solutions in Theorem 1.1 are local minimizers of the reduced energy functional. Such an idea has been successfully used in [8] in the study of interior spike solutions of problem (1.1) and in [10] in the study of existence of clustered spikes at a local minimum point of the mean curvature. We shall follow the approaches used in [10]. To shorten the presentation, we shall state without proof most of the reduction procedure. The reader may consult [10] for more details. The assumption on the dimension is only used at the last step. We believe that this assumption should be dropped, at least for a result which does not include the estimate on the number of nodal domains. In fact, it is possible to generalize the results of Theorem 1.1 to high-dimensional domains with symmetry.

The organization of this paper is as follows: In Section 2, we construct approximate nodal solutions and study its properties. Then in Section 3 we state without proof the reduction process. In Section 4 we complete the proof of the existence of a nodal solution u_ε satisfying (1.4) and (1.5). Finally, in Section 5 we prove that u_ε has at least $K+1$ nodal domains.

Throughout this paper, the letters C, c will denote various constants independent of ϵ small.

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2. Approximate Nodal Solutions

In this section, we construct approximate $K-$nodal solutions.

As in [10], we first need to project the ground state solutions w^1, w^2 to $H^1(\Omega)$ with homogeneous Neumann boundary condition.

It is known that w^1 and w^2 are radially symmetric and have the following asymptotics:

$$\lim_{|y|\to\infty} w^i(y)|y|^{\frac{1}{2}}e^{|y|} = A_i \tag{2.1}$$

Here $A_i, i=1,2$ are two positive generic constants. The energies of w^i are defined as

$$I^i[w^i] = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla w^i|^2 + (w^i)^2) - \int_{\mathbb{R}^2} F^i(w^i), \qquad i=1,2, \tag{2.2}$$

where $F^i(u) = \int_0^u f_i(s)ds$.

For any smooth bounded domain $U \subset \mathbb{R}^2$ we set $P_U w$ to be the unique solution of

$$\Delta u - u + f_i(w(y)) = 0 \text{ in } U, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U. \tag{2.3}$$

Without loss of generality, we may assume that $0 \in \partial\Omega$. We set

$$\Omega_\epsilon = \{z | \epsilon z \in \Omega\}.$$

We consider the energy functional

$$J_\epsilon[u] = \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 + \frac{1}{2}\int_{\Omega_\epsilon} u^2 - \int_{\Omega_\epsilon} F(u) \tag{2.4}$$

where $F(u) = \int_0^u f(s)ds$. It is known that a critical point of J_ϵ is a solution of the equation $\Delta u - u + f(u) = 0$ in Ω_ϵ, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega_\epsilon$, and thus corresponds to a solution of (1.1) by rescaling. Next we define our configuration set

$$\Lambda := \left\{ \mathbf{P} = (P_1, \dots, P_{2K}) \;\middle|\; \begin{array}{l} P_j \in \partial\Omega, \quad j = 1, \dots, 2K, \\ w^1(\frac{P_i - P_j}{\epsilon}) + w^2(\frac{P_i - P_j}{\epsilon}) < \epsilon \quad \text{for} \quad i \neq j \end{array} \right\} \tag{2.5}$$

Thus Λ is a relatively open subset of $\partial\Omega^{2K}$, and we denote by $\overline{\Lambda}$ the relative closure of Λ in $\partial\Omega^{2K}$.

Fix $\mathbf{P} = (P_1, ..., P_{2K}) \in \overline{\Lambda}$. We set

$$w^1_{\epsilon, P_i}(z) = P_{\Omega_\epsilon} w^1\left(z - \frac{P_i}{\epsilon}\right), \qquad w^2_{\epsilon, P_i}(z) = P_{\Omega_\epsilon} w^2\left(z - \frac{P_i}{\epsilon}\right). \tag{2.6}$$

Our approximate nodal solution is

$$w_{\epsilon, \mathbf{P}}(z) = \sum_{j=1}^{K} \left[w^1_{\epsilon, P_{2j-1}}(z) - w^2_{\epsilon, P_{2j}}(z) \right], \qquad z \in \Omega_\epsilon. \tag{2.7}$$

Then we have the following energy computations:

Lemma 2.1. *For any* $\mathbf{P} \in \overline{\Lambda}$ *and* ϵ *sufficiently small*

$$J_\epsilon(w_{\epsilon,\mathbf{P}}) = \frac{K}{2}(I^1[w^1] + I^2[w^2]) - \epsilon\gamma_0 \sum_{i=1}^{2K} H(P_i) - \sum_{i\neq j, i,j=1}^{K} \gamma_1\, w^1\left(\frac{P_{2i-1} - P_{2j-1}}{\epsilon}\right) - \sum_{i\neq j, i,j=1}^{K} \gamma_2\, w^2\left(\frac{P_{2i} - P_{2j}}{\epsilon}\right) + \sum_{i,j=1}^{K}\left[\gamma_2\, w^1\left(\frac{P_{2j} - P_{2i-1}}{\epsilon}\right) + \gamma_1\, w^2\left(\frac{P_{2j} - P_{2i-1}}{\epsilon}\right)\right] + o(\epsilon), \tag{2.8}$$

where $\gamma_0 > 0$ *is a generic constant, and*

$$\gamma_1 = \int_{\mathbb{R}^2} f(w^1(y))e^{y_1}\, dy \; > \; 0, \qquad \gamma_2 = \int_{\mathbb{R}^2} f(w^2(y))e^{y_1}\, dy \; > \; 0.$$

Let us explain the meanings of the five terms in the right hand side of (2.8). The first term represents the total energy. The second term represents the boundary effect. The third term gives the interaction of positive spikes while the fourth term represents the interaction of negative spikes. The last term represents the interaction between positive and negative spikes. The proof of Lemma 2.1 is similar to that of Lemma 2.8 of [10]. Thus we omit the most of the details. The only difference is that we use the following lemma in place of [10, Lemma 2.4].

Lemma 2.2. *For any* $\mathbf{P} \in \overline{\Lambda}$, ϵ *sufficiently small we have*

$$\int_{\Omega_\epsilon} f_1\Big(w^1(y - \frac{P_{2i-1}}{\varepsilon})\Big)w^1(y - \frac{P_{2j-1}}{\varepsilon}) = \gamma_1 w^1\left(\frac{P_{2i-1} - P_{2j-1}}{\varepsilon}\right) + o(\epsilon), \tag{2.9}$$

$$\int_{\Omega_\epsilon} f_2\Big(w^2(y - \frac{P_{2i}}{\varepsilon})\Big)w^2(y - \frac{P_{2j}}{\varepsilon}) = \gamma_2 w^2\left(\frac{P_{2i} - P_{2j}}{\varepsilon}\right) + o(\epsilon), \tag{2.10}$$

$$\int_{\Omega_\epsilon} f_1\Big(w^1(y - \frac{P_{2i-1}}{\varepsilon})\Big)w^2(y - \frac{P_{2j}}{\varepsilon}) = \gamma_1 w^2\left(\frac{P_{2j} - P_{2i-1}}{\varepsilon}\right) + o(\epsilon), \tag{2.11}$$

$$\int_{\Omega_\epsilon} f_2\Big(w^2(y - \frac{P_{2j}}{\varepsilon})\Big)w^1(y - \frac{P_{2i-1}}{\varepsilon})) = \gamma_2 w^1\left(\frac{P_{2j} - P_{2i-1}}{\varepsilon}\right) + o(\epsilon), \tag{2.12}$$

Proof: We only show (2.11), the proof of the other relations is similar. As $\varepsilon \to 0$, we have $\frac{P_{2i-1} - P_{2j}}{\varepsilon} \to \infty$, since $\mathbf{P} \in \overline{\Lambda}$. Let $\mathbb{R}^2_+ = \{(y_1, y_2) \, : \, y_2 > 0\}$. By straightening the boundary at P_{2i-1} we find

$$\int_{\Omega_\epsilon} f_1\Big(w^1(y-\frac{P_{2i-1}}{\varepsilon})\Big)w^2(y-\frac{P_{2j}}{\varepsilon}) = \int_{\mathbb{R}^2_+} f_1(w^1(y))w^2(y-\frac{P_{2j}-P_{2i-1}}{\varepsilon})(1+o(1))$$

$$= w^2(\frac{P_{2j}-P_{2i-1}}{\varepsilon})(1+o(1))\int_{\mathbb{R}^2_+} f_1(w^1(y))$$

$$\times \left(w^2(\frac{P_{2j}-P_{2i-1}}{\varepsilon})\right)^{-1} w^2(y-\frac{P_{2j}-P_{2i-1}}{\varepsilon})$$

$$= w^2(\frac{P_{2j}-P_{2i-1}}{\varepsilon})(1+o(1))\int_{\mathbb{R}^2_+} f_1(w^1(y))e^{\langle b,y\rangle}\,dy \tag{2.13}$$

for some

$$b = \lim_{\varepsilon\to 0}\frac{P_{2j}-P_{2i-1}}{|P_{2j}-P_{2i-1}|}\in\mathbb{R}^2, \qquad |b|=1.$$

Here we have used (2.1). Now if $|P_{2j}-P_{2i-1}|$ stays bounded away from zero, we directly deduce from (2.1) and (2.13) that

$$\left|\int_{\Omega_\epsilon} f_1\Big(w^1(y-\frac{P_{2i-1}}{\varepsilon})\Big)w^2(y-\frac{P_{2j}}{\varepsilon})\right| \le C\left|w^1\left(\frac{P_{2j}-P_{2i-1}}{\varepsilon}\right)\right| = o(\varepsilon).$$

Hence we may assume $P_{2j}-P_{2i-1}\to 0$, which implies $b=(\pm 1,0)$. Consequently,

$$\int_{\Omega_\epsilon} f_1(w^1(y-\frac{P_{2i-1}}{\varepsilon}))w^2(y-\frac{P_{2j}}{\varepsilon})) = w^2(\frac{P_{2j}-P_{2i-1}}{\varepsilon})\int_{\mathbb{R}^2_+} f_1(w^1(y))e^{\pm y_1}\,dy + o(\varepsilon)$$

$$= w^2(\frac{P_{2j}-P_{2i-1}}{\varepsilon})\frac{1}{2}\int_{\mathbb{R}^2} f_1(w^1(y))e^{\pm y_1}\,dy + o(\varepsilon)$$

$$= w^2(\frac{P_{2j}-P_{2i-1}}{\varepsilon})\frac{1}{2}\int_{\mathbb{R}^2} f_1(w^1(y))e^{y_1}\,dy + o(\varepsilon)$$

In the last step we used that w^1 is radially symmetric. The proof is finished. □

3. Reduction Process

In this section, we reduce problem (1.1) to finite dimensions by the Liapunov-Schmidt method. Since this is similar to [10], we shall state all the results without proofs. The reader may consult [10] for details.

We first introduce some notations.

Let $H^2_\nu(\Omega_\epsilon)$ be the Hilbert space defined by

$$H^2_\nu(\Omega_\epsilon) = \left\{u\in H^2(\Omega_\epsilon)\,\middle|\,\frac{\partial u}{\partial\nu}=0 \text{ on } \partial\Omega_\epsilon\right\}.$$

Define

$$S_\epsilon[u] = \Delta u - u + f(u) \qquad \text{for} \quad u\in H^2_\nu(\Omega_\epsilon).$$

Then solving equation (1.1) is equivalent to solving

$$S_\epsilon[u]=0 \qquad \text{for} \quad u\in H^2_\nu(\Omega_\epsilon).$$

Fix $\mathbf{P} \in \overline{\Lambda}$. We define $Z_i \in L^2(\Omega_\varepsilon)$, $i = 1, ..., 2K$ by

$$Z_i = \begin{cases} \dfrac{\partial w^1_{\epsilon,P_i}}{\partial \tau_{P_i}} & \text{for } i \text{ odd}, \\ \dfrac{\partial w^2_{\epsilon,P_i}}{\partial \tau_{P_i}} & \text{for } i \text{ even}, \end{cases}$$

where τ_{P_i} represents the tangential derivative at the point P_i.

We summarize the following results

Lemma 3.1. *If $\varepsilon > 0$ is sufficiently small, then for every $\mathbf{P} \in \overline{\Lambda}$ there exists a unique $\varphi = \varphi_{\epsilon,\mathbf{P}} \in H^2_\nu(\Omega_\epsilon)$ satisfying*

$$S_\epsilon[w_{\epsilon,\mathbf{P}} + \varphi] \in \operatorname{span}\{Z_1, ..., Z_{2K}\}, \qquad \int_{\Omega_\epsilon} \varphi Z_i = 0, \quad i = 1, ..., 2K, \tag{3.1}$$

and $\|\varphi_{\epsilon,\mathbf{P}}\|_{H^2(\Omega_\epsilon)} \leq C\epsilon^{\frac{1+\sigma}{2}}$. Moreover, the map $\mathbf{P} \to \varphi_{\epsilon,\mathbf{P}}$ is C^1.

Proof: See the proof of Lemma 3.4 and Lemma 3.6 of [10]. □

The next lemma is our main tool. The proof is similar to that of Lemma 3.5 of [10].

Lemma 3.2. *Let $\varphi_{\epsilon,\mathbf{P}}$ be defined by Lemma 3.1. Then we have*

$$J_\epsilon(w_{\epsilon,\mathbf{P}} + \varphi_{\epsilon,\mathbf{P}}) = J_\epsilon(w_{\epsilon,\mathbf{P}}) + o(\epsilon).$$

Moreover, if we define the map $M_\varepsilon : \overline{\Lambda} \to \mathbb{R}$ by

$$M_\epsilon(\mathbf{P}) = J_\epsilon(w_{\epsilon,\mathbf{P}} + \varphi_{\epsilon,\mathbf{P}}), \tag{3.2}$$

then a critical point of M_ϵ in Λ gives rise to a critical point of J_ϵ and thus a solution to (1.1).

4. Existence of a K-nodal solution

By Lemma 3.2, we just need to prove that the reduced energy functional $M_\epsilon : \overline{\Lambda} \to \mathbb{R}$ has a critical point in the open set $\Lambda \subset \partial\Omega^{2K}$. Here we will exploit the two dimensional character of our problem. Let $\Gamma \subset \partial\Omega$ be a connected component of $\partial\Omega$ where the curvature H attains its maximum. This component Γ can be parametrized by a homeomorphism $h : S^1 \to \Gamma$, where $S^1 \subset \mathbb{C}$ is the unit circle. Without loss of generality, we may assume that h preserves the arc length. In the following we say that $P_1, P_2, P_3 ... \in \Gamma$ are in cyclic order whenever the points $h^{-1}(P_1), h^{-1}(P_2), h^{-1}(P_3), ... \in S^1$ are in cyclic order on S^1. We now consider the following subset of Λ:

(4.1)

$$\Lambda^* := \left\{ \mathbf{P} = (P_1, \ldots, P_{2K}) \in \Gamma^{2K} \;\middle|\; \begin{array}{l} P_1, P_2, ..., P_{2K} \text{ are in cyclic order}, \\ w^1(\frac{P_i - P_j}{\epsilon}) + w^2(\frac{P_i - P_j}{\epsilon}) < \epsilon \quad \text{for} \quad i \neq j \end{array} \right\}$$

Thus Λ^* is a relatively open subset of $\partial\Omega^{2K}$, and we denote by $\overline{\Lambda^*}$ the relative closure of Λ^* in $\partial\Omega^{2K}$. We shall prove

Lemma 4.1. *There is $\epsilon_K > 0$ such that for $0 < \varepsilon < \varepsilon_K$ the minimization problem*

$$\min\{M_\epsilon(\mathbf{P}) : \mathbf{P} \in \overline{\Lambda^*}\} \tag{4.2}$$

has a solution $\mathbf{P}^\epsilon \in \Lambda^*$. Moreover, as $\epsilon \to 0$,

$$H(P_i^\epsilon) \to \max_{Q\in\partial\Omega} H(Q) \qquad \text{for } i = 1, ..., 2K. \tag{4.3}$$

An important observation central to the proof of Lemma 4.1 is the following simple lemma, whose proof is easy and thus omitted.

Lemma 4.2. *There exist two constants $c_0 > 0$ and $\delta_0 > 0$ such that if $Q_1, Q_2, Q_3 \in \Gamma$ are in cyclic order with $|Q_1 - Q_2| + |Q_2 - Q_3| < \delta_0$, then*

$$|Q_1 - Q_3| > (1 + c_0)\min(|Q_1 - Q_2|, |Q_2 - Q_3|). \tag{4.4}$$

Lemma 4.2 implies that for $\mathbf{P} = (P_1, ..., P_{2K}) \in \overline{\Lambda^*}$ and i, j with $|i-j| \geq 2$ we have

$$|P_i - P_j| \geq \min\Big\{(1 + c_0)\min_{k\neq l}|P_k - P_l|, \delta_0\Big\} \tag{4.5}$$

Combining this with (2.1), we obtain the estimate

$$w^1\big((P_i - P_j)/\epsilon\big) \leq C\epsilon^{1+c_0}, \qquad w^2\big((P_i - P_j)/\epsilon\big) \leq C\epsilon^{1+c_0} \qquad \text{if } |i-j| \geq 2. \tag{4.6}$$

From Lemma 2.1, Lemma 3.2 and (4.6), we deduce that

$$\begin{aligned} M_\epsilon(\mathbf{P}) = {} & \frac{K}{2}(I^1[w^1] + I^2[w^2]) - \epsilon\gamma_0\sum_{i=1}^{2K} H(P_i) \\ & + \gamma_2\left(\sum_{i=1}^{K} w^1\left(\frac{P_{2i} - P_{2i-1}}{\epsilon}\right) + \sum_{i=1}^{K-1} w^1\left(\frac{P_{2i} - P_{2i+1}}{\epsilon}\right)\right) \\ & + \gamma_1\left(\sum_{i=1}^{K} w^2\left(\frac{|P_{2i} - P_{2i-1}|}{\epsilon}\right) + \sum_{i=1}^{K-1} w^2\left(\frac{P_{2i} - P_{2i+1}}{\epsilon}\right)\right) + o(\epsilon), \end{aligned} \tag{4.7}$$

hence in particular

$$M_\epsilon(\mathbf{P}) \geq \frac{K}{2}(I^1[w^1] + I^2[w^2]) - \epsilon\gamma_0\sum_{i=1}^{2K} H(P_i) + o(\varepsilon). \tag{4.8}$$

We are now ready to prove Lemma 4.1.

Proof of Lemma 4.1: Since $J_\epsilon(w_{\epsilon,\mathbf{P}} + \varphi_{\epsilon,\mathbf{P}})$ is continuous in $\mathbf{P}$, the minimizing problem has a solution in $\overline{\Lambda^*}$. Let $M_\epsilon(\mathbf{P}^\epsilon)$ be the minimum value, where $\mathbf{P}^\epsilon \in \overline{\Lambda^*}$. We claim that $\mathbf{P}^\epsilon \in \Lambda^*$. Suppose not. We assume that $\mathbf{P}^\epsilon \in \partial\Lambda^* = \overline{\Lambda^*} \setminus \Lambda^*$ to obtain a contradiction. To this end, we first obtain an upper bound for $M_\epsilon(\mathbf{P}^\epsilon)$. In fact, let $Q_0 \in \Gamma$ be such that $H(Q_0) = \max_{Q\in\partial\Omega} H(Q)$. We let $P_j \in \Gamma$, $j = 1, ..., 2K$ be defined by

$$h^{-1}(P_j) = e^{ik\sqrt{\epsilon}}h^{-1}(Q_0) \in S^1.$$

For ε small we then have $\frac{P_i-P_j}{\epsilon} \geq \frac{1}{2\sqrt{\varepsilon}}$ for $i \neq j$, and therefore $w^1(\frac{P_i-P_j}{\epsilon})+w^2(\frac{P_i-P_j}{\epsilon}) = o(\epsilon)$ by (2.1). Consequently, $\mathbf{P} = (P_1, ..., P_{2K}) \in \Gamma^*$. We compute

$$M_\epsilon(\mathbf{P}) = \frac{K}{2}(I^1[w^1] + I^2[w^2]) - 2K\gamma_0 H(Q_0)\epsilon + o(\epsilon) \tag{4.9}$$

which implies that

$$M_\epsilon(\mathbf{P}^\epsilon) \leq \frac{K}{2}(I^1[w^1] + I^2[w^2]) - 2K\gamma_0 H(Q_0)\epsilon + o(\epsilon). \tag{4.10}$$

Note that, if $\mathbf{P}^\epsilon \in \partial\Lambda$, we have

$$w^1\left(\frac{P_k^\epsilon - P_l^\epsilon}{\epsilon}\right) + w^2\left(\frac{P_k^\epsilon - P_l^\epsilon}{\epsilon}\right) = \epsilon$$

for some $k \neq l$. By (4.6) we must have $|k-l| = 1$. Without loss of generality, we may assume that $k = 2i$ and $l = 2i-1$ for some $i \in \{1, ..., K\}$. Then from (4.7) we deduce that

$$\begin{aligned} M_\epsilon(\mathbf{P}^\epsilon) &\geq \frac{K}{2}(I^1[w^1] + I^2[w^2]) - 2K\gamma_0 \max_{Q\in\partial\Omega} H(Q)\epsilon \\ &+ \gamma_2 w^1\left(\frac{P_{2i}^\epsilon - P_{2i-1}^\epsilon}{\epsilon}\right) + \gamma_1 w^2\left(\frac{P_{2i}^\epsilon - P_{2i-1}^\epsilon}{\epsilon}\right) + o(\epsilon) \\ &\geq \frac{K}{2}(I^1[w^1] + I^2[w^2]) - 2K\gamma_0 H(Q_0)\epsilon + \left(\min_{j=1,2}\gamma_j\right)\epsilon + o(\epsilon). \end{aligned}$$

For $\epsilon > 0$ sufficiently small, this contradicts (4.10). Consequently there is $\epsilon_K > 0$ such that $\mathbf{P}^\epsilon \in \Lambda^*$ for $0 < \epsilon < \epsilon_K$.
To prove (4.3), we now suppose by contradiction that for a sequence $\epsilon_n \to 0$ and some i, $1 \leq i \leq 2K$ we have

$$H(P_i^{\epsilon_n}) \leq \max_{Q\in\partial\Omega} H(Q) - c = H(Q_0) - c, \tag{4.11}$$

where $c > 0$ is a constant. Then (4.8) implies

$$M_{\epsilon_n}(\mathbf{P}^{\epsilon_n}) \geq \frac{K}{2}(I^1[w^1] + I^2[w^2]) - 2K\gamma_0 H(Q_0)\epsilon_n + c\epsilon_n + o(\epsilon_n)$$

which contradicts (4.10) for n large. The proof is finished. □

Remark 4.1. *By a slightly more careful argument one can construct solutions of this type concentrating at any strict local maximum of the curvature H on $\partial\Omega$.*

5. A lower estimate for the number of nodal domains

Let u_ε be a solution as constructed in Section 4, with maximum points $P_1, P_3, ..., P_{2K-1}$ and minimum points $P_2, P_4, ..., P_{2K}$. To estimate the number of nodal domains of u_ε from below, we consider the graph $\mathcal{G}$ formed by the $2K$ vertices $P_1, ..., P_{2K}$ under the only defining rule that P_i is adjacent to P_j if and only if P_i and P_j lie in the same nodal domain of u_ε. Then $\mathcal{G}$ has at most as many connected components as u_ε has nodal domains. We remark that in general $\mathcal{G}$ is not a planar graph. We claim:

(5.1) The graph $\mathcal{G}$ has at least $K+1$ connected components.

Indeed, note first that the properties of u_ε (i.e. the special character of our problem) lead to the following observations.

(I) P_i is adjacent to P_j iff $i = j \mod 2$.
(II) If P_i is adjacent to P_j for some numbers $i < j$, then no vertex in $\{P_k : i < k < j\}$ is adjacent to a vertex in $\{P_k : k < i \text{ or } k > j\}$.
(III) If P_i is adjacent to P_j and P_j is adjacent to P_k, then P_i is adjacent to P_k.

We now prove the more general statement that any graph $\mathcal{G}_n$ formed by n vertices $P_1, ..., P_n$, $n \in \mathbb{N}$ and obeying the rules (I)–(III) has at least $\left[\frac{n}{2}\right]+1$ connected components. We proceed by induction. For $n = 1$ the statement is trivial. Now let $n > 1$, and consider a graph $\mathcal{G}_n$ with vertices $P_1, ..., P_n$ satisfying (I)–(III). Suppose first that the vortex P_n is isolated in $\mathcal{G}_n$. Then the subgraph formed by removing P_n from $\mathcal{G}_n$ has at least $\left[\frac{n-1}{2}\right]+1$ connected components by induction. Hence $\mathcal{G}_n$ has at least $\left[\frac{n-1}{2}\right]+2 \geq \left[\frac{n}{2}\right]+1$ connected components, as claimed. Suppose next that P_n is adjacent to some P_j, $j < n$, and let j be minimal with this property. We distinguish the following cases.

Case I: $j = 1$. Then n must be odd. The graph $\mathcal{G}'$ arising by reducing P_1 and P_n to one point has $n-1$ vertices $P_1, ..., P_{n-1}$ and still obeys the rules (I)– (III). By induction, $\mathcal{G}'$ has at least $\left[\frac{n-1}{2}\right]+1 = \frac{n-1}{2} = \left[\frac{n}{2}\right]+1$ connected components, and so does $\mathcal{G}_n$.

Case II: $j > 1$. As a consequence of (II), none of the points $P_1, ..., P_{j-1}$ is adjacent to any of the points $P_j, ..., P_n$. Hence $\mathcal{G}_n$ splits into two disjoint and disconnected subgraphs $\mathcal{G}_1$ resp. $\mathcal{G}_2$ formed by the vertices $P_1, ..., P_{j-1}$ and $P_j, ..., P_n$ respectively. By induction, $\mathcal{G}_1$ has at least $\left[\frac{j-1}{2}\right]+1$ connected components, and $\mathcal{G}_2$ has at least $\left[\frac{n-j+1}{2}\right]+1$ connected components. Hence $\mathcal{G}_n$ has at least $\left[\frac{j-1}{2}\right]+\left[\frac{n-j+1}{2}\right]+2 \geq \left[\frac{n}{2}\right]+1$ connected components, as claimed.

The induction is complete, and we in particular conclude (5.1). Consequently, u_ε has at least $K+1$ connected components.

Remark 5.1. *A somewhat similar argument is used in* [17] *in a different context.*

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