A double bubble in a ternary system with long range interaction

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April 29, 2012

Abstract

The free energy of a ternary system with a self-organization property includes an interface energy and a longer ranging interaction energy. In a planar domain, if the two energies are properly balanced and two of the three constituents make up an equal but small fraction, the free energy admits a local minimizer that is shaped like a perturbed double bubble. One fixes a reference frame and considers a class of restrictedly perturbed double bubbles, where the problem may be formulated as a fixed point in a Hilbert space. This is achieved by two techniques: a special way to parametrize the boundaries of perturbed double bubbles and a new set of variables that are subject to only linear constraints. Within each restricted class of perturbed double bubbles arises a local minimizer of the free energy, which satisfies three of the four equations for critical points of the free energy. The unsatisfied equation is the 120 degree angle condition on triple junction points. Perform another minimization among the local minimizers from all the restricted classes to obtain a minimum of minimizers, which solves all the equations for critical points.

1 Introduction

Exquisitely structured patterns arise in many multi-constituent physical and biological systems as orderly outcomes of self-organization principles. Examples include morphological phases in block copolymers, animal coats, and skin pigmentation. Common in these pattern-forming systems is that a deviation from homogeneity has a strong positive feedback on its further increase. On its own, it would lead to an unlimited increase and spreading. Pattern formation requires in addition a longer ranging confinement of the locally self-enhancing process.

The simplest multi-constituent system is a binary system. Let such a system be in the domain $D$, an open and bounded set in $\mathbb{R}^n$. If the two constituent components are totally separated, $D$ is divided into two subsets: $\Omega$ occupied by one constituent and $D \setminus \Omega$ occupied by the other constituent. These sets may or may not be connected. If the system is governed by its free energy, it is natural to postulate that the size of the interfaces separating $\Omega$ from $D \setminus \Omega$ contributes to this energy. Mathematically this quantity is termed the perimeter of $\Omega$ in $D$, denoted by $P_D(\Omega)$ whose precise definition will be given in the next section. When $\Omega$ is bounded by a smooth boundary $\partial \Omega$, then $P_D(\Omega)$ is the length of $\partial \Omega \cap D$ if $D \subset \mathbb{R}^2$ or the area of $\partial \Omega \cap D$ if $D \subset \mathbb{R}^3$. Note that $\partial \Omega$ stands for the boundary of $\Omega$ in $\mathbb{R}^n$, not the boundary in $D$. The latter is $\partial \Omega \cap D$.

However the perimeter term $P_D(\Omega)$ alone is not sufficient in a pattern forming, self-organizing system. For pattern formation to take place, another longer ranging term is needed. On example of this comes from the Ohta-Kawasaki density functional theory for diblock copolymers [20]. There the longer range term takes the form $\int_D |(-\Delta)^{-1/2}(\chi_\Omega - \omega)|^2 dx$. The function $\chi_\Omega$ is the characteristic function of $\Omega$: $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \in D \setminus \Omega$. The constant $\omega \in (0, 1)$ is prescribed to fix the size of $\Omega$. It is required that $|\Omega| = \omega |D|$ where $|\Omega|$ is the Lebesgue measure of $\Omega$ and $|D|$ the Lebesgue measure of $D$. The operator $(-\Delta)^{-1/2}$ is the positive square root of the inverse of $-\Delta$ with the Neumann boundary condition.

The free energy $J_B$ of the binary system combines these two terms:

$$J_B(\Omega) = \frac{1}{n-1} P_D(\Omega) + \frac{\gamma}{2} \int_D |(-\Delta)^{-1/2}(\chi_\Omega - \omega)|^2 dx. \quad (1.1)$$
The interfaces separating $\Omega_1$ exist: $\partial_1 \Omega$. In (1.2) $|\Omega| = \omega |D|$. Having the coefficients $\frac{1}{n-1}$ and $\frac{7}{2}$ in (1.1) helps generate a simpler looking equation (1.2). By minimizing $J_B$, or more generally by solving the equation (1.2), one discovers patterns that match the ones observed in nature.

The equation (1.2) holds on the part of the boundary of $\Omega$ that is inside $D$, namely $\partial \Omega \cap D$. If $\partial \Omega$ intersects $\partial D$, then the two hyper-surfaces must meet perpendicularly.

This problem is solved completely in one-dimension [23]. There are countably infinitely many solutions to (1.2) and every solution is a local minimizer of the energy functional $J_B$. Among these local minimizers, two or four of them are global minimizers. In higher dimensions several types of solutions have been found [24, 27, 29, 28, 30, 31, 21, 13, 22, 14]. But many questions remain. Even the global minimizer has not been completely identified. Nevertheless progress has been made in [2, 34, 18].

In this work we are concerned with ternary systems. A ternary system has three constituents that occupy subsets $\Omega_1$, $\Omega_2$ and $\Omega_3 = D \backslash (\Omega_1 \cup \Omega_2)$ of $D$, where $\Omega_1$ and $\Omega_2$ are disjoint. In a binary system, part of the free energy of the ternary system is the size of the interfaces separating the three domains $\Omega_1$, $\Omega_2$ and $\Omega_3$. Three types of interfaces exist: $\partial \Omega_1 \cap \partial \Omega_2$, $\partial \Omega_2 \cap \partial \Omega_1$, $\partial \Omega_3 \cap \partial \Omega_2$, $\partial \Omega_1 \cap \partial \Omega_3$, $\partial \Omega_2 \cap \partial \Omega_3$, $\partial \Omega_1 \cap \partial \Omega_3$. Each of these hyper-surfaces must meet perpendicularly.

For the long range interaction we use $\int_D ((-\Delta)^{-1/2}(\chi_{ij} - \omega))(\chi_{ij} - \omega) \, dx$ to model the interaction between $\Omega_i$ and $\Omega_j$. The overall free energy of the ternary system takes the form

$$J_T(\Omega) = \frac{1}{2(n-1)} \sum_{i=1}^3 P_D(\Omega_i) + \sum_{i,j=1}^2 \int_D \frac{\gamma_{ij}}{2} \left((-\Delta)^{-1/2}(\chi_{ij} - \omega_i)(\chi_{ij} - \omega_j)\right) \, dx. \quad (1.3)$$

This model derives from a density functional theory for triblock copolymers proposed by Nakazawa and Ohta [19]. A more mathematical treatment of their theory can be found in [25]. The original theory is more general and allows the constituents to mix, a point that will be discussed in the last section. Here we only consider the strong segregation limit where the three constituents are completely separated. The passage from the original theory to $J_T$ relies on the $\Gamma$-convergence theory of De Giorgi; see [26] for a detailed explanation.

Although experimentally a far larger number of architectures can be synthetically accessed in ternary systems like triblock copolymers than in binary systems [5, Figure 5 and the magazine cover], mathematical study of $J_T$ is still in an early stage. The cyclic lamellar phase is shown to exist in [26]: a diblock copolymer - homopolymer blend problem (a special ternary system) is investigated in [6, 36, 37]. All these papers deal with $J_T$ in one-dimension.

In higher dimensions there are ternary patterns that are obviously analogous to some binary patterns, such as the lamellar and core-shell phases [5, Figure 4: a, b]. However the unique and the most interesting phenomenon of triple junction in ternary systems is not shared by binary systems. Triple junction means that the three constituents may meet at a co-dimension 2 surface. In two dimensions triple junction occurs at points, and in three dimension it occurs at curves. Here we investigate one particular such pattern: a double bubble, or more precisely a perturbed double bubble in two dimensions that serves as a local minimizer of $J_T$. 

Figure 1: An equal area double bubble on the left and an unequal area double bubble on the right. The equal area one is used in this work.
The standard double bubble arises as the optimal configuration of the two component isoperimeric problem. Let $m_1 > 0$ and $m_2 > 0$. Find two disjoint sets $E_1$ and $E_2$ in $\mathbb{R}^n$ such that $|E_1| = m_1$, $|E_2| = m_2$, and the area of $\partial E_1 \cup \partial E_2$, i.e. $\frac{1}{2}(P(E_1) + P(E_2) + P(E_3))$ where $P(E_i)$ is the perimeter of $E_i$ in $\mathbb{R}^n$, is minimum. It is known that for $n = 2$ and 3, the standard double is the unique solution to this isoperimetric problem by the works of Almgren [3], Taylor [35], Foisy et al [10], and Hutchings et al [12].

In $\mathbb{R}^2$ the standard double bubble consists of two regions $E_1$ and $E_2$ separated by an arc and bounded by two other arcs; see Figure 1. In the case that the two areas $m_1$ and $m_2$ are equal, the middle arc becomes a line segment, i.e. a degenerate arc of infinite radius. Triple junction occurs at two points at which the three arcs meet at 120 degrees. The radii of the three arcs $r_1$, $r_2$ and $r_3$ satisfy $\frac{1}{r_1} = \frac{1}{r_2} = \frac{1}{r_3}$, where $r_0$ is the radius of the center arc, $r_1$ is the radius of the larger bubble and $r_2$ the radius of the smaller bubble. In $\mathbb{R}^3$ the double bubble is bounded by two spherical caps and separated by another spherical cap; when $m_1 = m_2$, the last spherical cap becomes a disc. The three spheres meet at 120 degree at the triple junction circle, and the radii of the three spheres satisfy the same equation as in $\mathbb{R}^2$. In this paper we tell a story related to the equal area, two dimensional double bubble, i.e. the case of $m_1 = m_2$ and $n = 2$.

Because of the long range interaction term in (1.3), the standard double bubble is not a critical point of $\mathcal{J}_T$. Nevertheless we will prove the following result.

**Main result.** If the domain $D$ is two dimensional and $\omega_1 = \omega_2$, then in a proper parameter range for $\omega_1 = \omega_2$ and $\gamma$, the functional $\mathcal{J}_T$ has a stable critical point that is shaped approximately like a double bubble of equal area in $D$.

In a proper sense this critical point is considered a local minimizer of $\mathcal{J}_T$, hence the claim that the solution is stable. The precise statement of this result including the range for $\omega_1 = \omega_2$ and $\gamma$ is in Theorem 2.1. There is also an estimate, (9.28), that reveals how much this critical point deviates from the standard double bubble of equal area.

Without the nonlocal term, being the minimizer of the isoperimetric problem any standard double bubble inside $D$ is a critical point of the local part of $\mathcal{J}_T$. However this critical point is highly degenerate, since any translation or rotation of the double bubble is also a critical point. The nonlocal term in (1.3) removes this degeneracy. The perturbed double bubble critical point we construct must appear in a particular place of $D$ with a particular direction.

The novelty in our treatment of ternary systems is the invention of a restricted perturbation class, an idea tailored to suit the triple junction phenomenon in ternary systems. It is a unique method that has no analogy in binary systems. The use of the restricted class neatly breaks our approach into two steps. The first step solves an infinite dimensional problem by the contraction mapping argument, and the second step solves a finite dimensional problem by minimization. Another new idea is the use of internal variables in the first step. They allowed the first infinite dimensional problem to be cast as an equation for a fixed point in a Hilbert space.

The unequal area case $\omega_1 \neq \omega_2$ seems to be harder than the equal area case $\omega_1 = \omega_2$ for several reasons. We hope to deal with it in a future work.

The proof of our result consists of several steps. First we fix a point $\xi$ and a direction, specified by an angle $\theta$, in $D$ to set up a reference frame with the point being the center and the direction being the horizontal direction of the frame. Place a standard double bubble $B$ in $D$ such that $\xi$ is the middle point of the two centers of the two bubbles and one center points to the other center along the direction specified by $\theta$. Consider a special class of perturbed standard bubble. A perturbation described by this class only allows the triple junction points of $B$ to move vertically with respect to the reference frame, and the two points can only move in opposite directions by the same distance.

Next we set up a particular parametrization with three functions $u_1$, $u_2$, $u_0$, and two numbers $A$ and $a$ to describe a perturbed double bubble in the restricted class. Two polar coordinate systems are used here. Each system is centered at the center of one of the two bubbles of $B$. The function $u_1$ (and $u_2$ respectively) is the radius of $\partial \Omega_1 \setminus \partial \Omega_2$ ($\partial \Omega_2 \setminus \partial \Omega_1$ respectively) of a perturbed double bubble. The number $A$ specifies the range of the corresponding angle. The curve $\partial \Omega_1 \cap \partial \Omega_2$ is described by the graph of $u_0$ and $a$ specifies the width of this graph. Geometric quantities, such as normal vectors and curvatures can all be expressed in terms of $u_1$, $A$ and $a$.

Although $u_1$, $A$ and $a$ amply describe the picture of a perturbed double bubble, this is not a convenient setting for analytic work. The reason is that $u_1$, $A$ and $a$ satisfy nonlinear constraints. Instead in the third step we introduce three new functions $\phi_1$, $\phi_2$, $\phi_0$ and one new number $\alpha$, termed internal variables. The original $(u_1, A, \alpha)$ can be transformed into and from $(\phi_1, \alpha)$, but $\phi_1$ and $\alpha$ satisfy linear constraints in the form of linear boundary conditions. Consequently $(\phi_1, \alpha)$ is placed in a Hilbert space, and $\mathcal{J}_T$ becomes a functional on this Hilbert space.

In step 4, we find a local minimizer $(\phi^*_1, \alpha^*)$ of $\mathcal{J}_T$ in this Hilbert space. This is done by a contraction mapping argument and the key step here is to prove a positivity result on the second derivative of $\mathcal{J}_T$, a quadratic form, at the standard double bubble. This is achieved in Lemma 7.4.
However since $J_T$ is only minimized in a restricted class, specified by the $\xi$ and $\theta$, of perturbed double bubbles, the local minimizer $(\phi_i^*,\alpha^*)$ found in Lemma 8.2 only satisfies equations (2.3)-(2.5), but not (2.6), the 120 degree angle condition on triple junction points. In step 5 we let $\xi$ and $\theta$ vary. Note that $(\phi_i^*,\alpha^*)$ depends on $\xi$ and $\theta$, i.e. $\phi_i^* = \phi_i^*(\xi,\theta)$ and $\alpha^* = \alpha^*(\xi,\theta)$. It turns out that one can minimize $J_T(\phi_i^*(\xi,\theta),\alpha^*(\xi,\theta))$ with respect to $(\xi,\theta)$ and find a minimum, say at $(\bar{0},0)$. This particular perturbed double bubble $(\phi_i^*(\cdot,\bar{0},0),\alpha^*(\bar{0},0))$, a minimum of minimizers, is what we need. For $(\xi,\theta)$ close to $(\bar{0},0)$ the local minimizer $(\phi_i^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta))$ is considered a deformation of $(\phi_i^*(\cdot,\bar{0},0),\alpha^*(\bar{0},0))$. This deformation is outside the restricted class where $(\phi_i^*(\cdot,\bar{0},0),\alpha^*(\bar{0},0))$ is.

The fact that $(\bar{0},0)$ is a minimum of $J_T(\phi_i^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta))$ with respect to $(\xi,\theta)$ implies that

$$
\frac{\partial J_T(\phi_i^*(\cdot,\xi,\theta),\alpha^*(\xi,\theta))}{\partial \xi^l}(\bar{0},0) = 0, \quad l = 1, 2,
$$

where $\xi = (\xi^1,\xi^2)$. From (1.4) we prove that the 120 degree angle condition (2.6) holds for $(\phi_i^*(\cdot,\bar{0},0),\alpha^*(\bar{0},0))$.

In [29] the authors proved the existence of a single bubble solution for the binary system (1.1). It is a critical point of the functional $J_B$ and is shaped like a disc inside $D$. The double bubble problem in a ternary system is much harder, not just because of the triple junction issue. The standard double bubble is in some sense a generalization of the standard disc because the disc minimizes the perimeter among all sets of the same area, the one component isoperimetric problem. However the disc is more compatible with the nonlocal term in $J_B$ than the double bubble is with the nonlocal term in $J_T$. In the binary case, a standard disc is a solution of a related profile equation

$$
\mathcal{H}(\partial E) + \gamma N(E) = \lambda \quad (1.5)
$$

which holds on the boundary of $E$. In (1.5) $\mathcal{N}$ is the Newtonian potential operator defined by $N(E)(x) = \frac{1}{n} \log \frac{1}{|x-y|} dy$ for each $x \in \mathbb{R}^n$. This profile equation may be regarded as an asymptotic limit of the equation (1.2). More results on this profile equation can be found in [32, 33]. If a standard disc is inserted into the boundary, is a piecewise $C^1$ curve, then $P_D(E)$ is just the length of $\partial E \cap D$. Hence if $\Omega_1$ and $\Omega_2$ have piecewise $C^1$ boundaries, the first term in (2.1) is merely the length of the set $(\partial \Omega_1 \cup \partial \Omega_2) \cap D$.

The interaction matrix $\gamma = [\gamma_{ij}]$ is symmetric and positive definite. It balances the interaction strength between the three constituents. It also weighs the perimeter part of $J$ against the long range part of $J$. A uniform positivity condition, (2.9), will be imposed on all the $\gamma$’s in the parameter range of interest. The functions $\chi_{\Omega_1}$ and $\chi_{\Omega_2}$ are the characteristic functions of the sets $\Omega_1$ and $\Omega_2$ respectively. The operator $(-\Delta)^{-1/2}$ will be explained shortly.

2 Ternary system

Henceforth we consider (1.3) on a bounded and smooth open subset $D$ of $\mathbb{R}^2$ and simplify the notation $J_T$ to $J$. Let $\omega_1$ and $\omega_2$ be two positive numbers such that $\omega_1 + \omega_2 < 1$. For two measurable subsets $\Omega_1$ and $\Omega_2$ of $D$ satisfying $|\Omega_1| = \omega_1|D|$, $|\Omega_2| = \omega_2|D|$, and $|\Omega_1 \cap \Omega_2| = 0$. Let $\Omega_3 = D \setminus (\Omega_1 \cup \Omega_2)$ and set $\Omega = (\Omega_1, \Omega_2)$. The energy of the system is

$$
J(\Omega) = \frac{1}{2} \sum_{i=1}^{3} P_D(\Omega_i) + \sum_{i,j=1}^{2} \int_D \frac{\gamma_{ij}}{2} \left((-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i)\right) \left((-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j)\right) dx. \quad (2.1)
$$

In (2.1) $P_D(\Omega_i)$ is the perimeter of $\Omega_i$ in $D$. The perimeter may be defined for any measurable subset $E$ of $D$ by

$$
P_D(E) = \text{sup} \left\{ \int_E \text{div} \ g(x) \ dx : g \in C^1_{0}(D, \mathbb{R}^2), \ |g(x)| \leq 1 \ \forall x \in D \right\} \quad (2.2)
$$

where $\text{div} \ g$ is the divergence of the vector field $g$ on $D$ with compact support and $|g(x)|$ stands for the Euclidean norm of the vector $g(x) \in \mathbb{R}^2$. A subset $E$ of $D$ has finite perimeter if and only $\chi_E$ is a function of bounded variation on $D$. Here $\chi_E$ is the characteristic function of $E$, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. See [9] or [38] for more discussion on the notion of perimeter. For this paper it suffices to know that when $E$ is open and its boundary is a piecewise $C^1$ curve, then $P_D(E)$ is just the length of $\partial E \cap D$. Hence if $\Omega_1$ and $\Omega_2$ have piecewise $C^1$ boundaries, the first term in (2.1) is merely the length of the set $(\partial \Omega_1 \cup \partial \Omega_2) \cap D$.

The interaction matrix $\gamma = [\gamma_{ij}]$ is symmetric and positive definite. It balances the interaction strength between the three constituents. It also weighs the perimeter part of $J$ against the long range part of $J$. A uniform positivity condition, (2.9), will be imposed on all the $\gamma$’s in the parameter range of interest. The functions $\chi_{\Omega_1}$ and $\chi_{\Omega_2}$ are the characteristic functions of the sets $\Omega_1$ and $\Omega_2$ respectively. The operator $(-\Delta)^{-1/2}$ will be explained shortly.
A critical point $\Omega = (\Omega_1, \Omega_2)$ of $J$ is a solution of the following equations:

\[
\begin{align*}
\kappa_1 + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} &= \lambda_1 \text{ on } \partial \Omega_1 \setminus \partial \Omega_2 \\
\kappa_2 + \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} &= \lambda_2 \text{ on } \partial \Omega_2 \setminus \partial \Omega_1 \\
\kappa_0 + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{21} - \gamma_{22}) I_{\Omega_2} &= \lambda_1 - \lambda_2 \text{ on } \partial \Omega_1 \cap \partial \Omega_2 \\
\nu_1 + \nu_2 + \nu_0 &= 0 \text{ at } \partial \Omega_1 \cap \partial \Omega_2 \cap \partial \Omega_3
\end{align*}
\]  

(2.3) (2.4) (2.5) (2.6)

Here we assume that $\Omega_1$ and $\Omega_2$ do not share boundaries with $D$. Otherwise we need to add another condition that when the boundary of $\Omega_1$ (or $\Omega_2$) meets the boundary of $D$, it does so perpendicularly.

In (2.3)-(2.5) $\kappa_1$, $\kappa_2$, and $\kappa_0$ are the curvatures of the curves $\partial \Omega_1 \setminus \partial \Omega_2$, $\partial \Omega_2 \setminus \partial \Omega_1$, and $\partial \Omega_1 \cap \partial \Omega_2$, respectively. These are signed curvatures defined with respect to a choice of normal vectors. On $\partial \Omega_1 \setminus \partial \Omega_2$ the normal vector points inward into $\Omega_1$. For $\partial \Omega_2 \setminus \partial \Omega_1$, the normal vector points inward into $\Omega_2$. For $\partial \Omega_1 \cap \partial \Omega_2$, the normal vector points from $\Omega_2$ towards $\Omega_1$. If a curve bends in the direction of the normal vector, then the curvature is positive.

Also in (2.3) and (2.4) $I_{\Omega_1}$ and $I_{\Omega_2}$ are two functions on $D$ determined from $\Omega_1$ and $\Omega_2$ respectively. The function $I_{\Omega_1}$ is the solution of

\[
-\Delta I_{\Omega_1} = \chi_{\Omega_1} - \omega_1 \text{ in } D, \quad \partial_n I_{\Omega_1} = 0 \text{ on } \partial D, \quad \int_D I_{\Omega_1}(x) \, dx = 0,
\]  

(2.7)

where $\partial_n I_{\Omega_1}$ stands for the outward normal derivative of $I_{\Omega_1}$ on $\partial D$. Note that the constraint $|\Omega_1| = \omega_1|D|$ implies that the integral of the right side of the PDE in (2.7) is zero, so the PDE together with the boundary condition is solvable. The solution is unique up to an additive constant. The last condition $\int_D I_{\Omega_1}(x) \, dx = 0$ fixes this constant and selects a particular solution. One can also write $I_{\Omega_1} = (-\Delta)^{-1}(\chi_{\Omega_1} - \omega_1)$, as the outcome of the operator $(-\Delta)^{-1}$ on $\chi_{\Omega_1} - \omega_1$. The operator $(-\Delta)^{-1/2}$ in (2.1) is the positive square root of $(-\Delta)^{-1}$.

The constants $\lambda_1$ and $\lambda_2$ are Lagrange multipliers corresponding to the constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$. They are unknown and are to be found with $\Omega_1$ and $\Omega_2$.

In the last equation (2.6), $\nu_1$, $\nu_2$, and $\nu_0$ are the inward pointing, unit tangent vectors of the curves $\partial \Omega_1 \setminus \partial \Omega_2$, $\partial \Omega_2 \setminus \partial \Omega_1$, and $\partial \Omega_1 \cap \partial \Omega_2$ at triple junction points. The requirement that the three unit vectors sum to zero is equivalent to the condition the three curves meet at 120 degree angles.

Since we only consider the equal area case, the area constraints $|\Omega_1| = \omega_1|D|$ and $|\Omega_2| = \omega_2|D|$ take the form

\[
|\Omega_1| = |\Omega_2| = m \rho^2 \quad \text{where } m = \frac{4\pi}{3} \pm \frac{\sqrt{3}}{2} \text{ and } \rho > 0.
\]  

(2.8)

The number $m \rho^2$ is chosen because this is the area of a bubble in the equal area double bubble if the bubble radius is $\rho$; see Figure 2. Our problem has $\rho$ and $\gamma$ as the parameters. They reside in a particular range. Within this range the $\gamma$’s satisfy a uniform positivity condition. Let $\lambda(\gamma)$, and $\Lambda(\gamma)$ be the two eigenvalues of $\gamma$ such that $0 < \lambda(\gamma) \leq \Lambda(\gamma)$. For a fixed $b \in (0, 1]$, a bound of the form

\[
\frac{b}{2} \lambda(\gamma) \leq \lambda(\gamma)
\]  

(2.9)

holds for all $\gamma$. The norm of $\gamma$ is denoted by $|\gamma|$ and given by $|\gamma| = \sum_{i,j=1}^2 |\gamma_{ij}|$. The main result of this paper is the following existence theorem.

**Theorem 2.1** Let $b \in (0, 1]$. There exist $\delta > 0$ and $\sigma > 0$ depending on $D$ and $b$ only, such that if $\rho < \delta$, $|\gamma| \rho^3 < \sigma$, and (2.9) holds, then a perturbed double bubble exists as a stable solution of the problem (2.3)-(2.6). Each of the two perturbed bubbles is bounded by a continuous curve that is $C^\infty$ except at the two triple junction points.

The standard double bubble is described in section 3 with an estimate of its energy. The first variation of $J$ is calculated in section 4. Section 5 introduces the classes of restrictedly perturbed double bubbles, and a special way to parametrize a perturbed double bubble’s boundaries. The internal variables and the internal representation are defined in section 6, and the positivity of the second variation of the standard double bubble under restricted deformations is proved in section 7. In section 8 one finds a local minimizer of $J$ within each restricted class by a contraction mapping argument. Also in this section the question how much the solution in the theorem as a perturbed double bubble differs from a standard double bubble is answered in Lemma 8.2. In section 9 $J$ is minimized among the local minimizers of the restricted classes. A minimum exists and it is the critical point of $J$ claimed in Theorem 2.1. A few remarks are included in the last section.

The perimeter part of $J$ and the long range part of $J$ are respectively denoted by $J_L$ and $J_N$, where the subscripts stand for “local” and “nonlocal”:

\[
J_L(\Omega) = \frac{1}{2} \sum_{i=1}^3 \mathcal{P}_D(\Omega_i) \quad \text{and} \quad J_N(\Omega) = \sum_{i,j=1}^2 \int_D \frac{\gamma_{ij}}{2} \left( (-\Delta)^{-1/2}(\chi_{\Omega_i} - \omega_i) \right) \left( (-\Delta)^{-1/2}(\chi_{\Omega_j} - \omega_j) \right) \, dx.
\]  

(2.10)
Figure 2: The equal area double bubble $(E_1, E_2)$ where the radius of the two bubbles is 1. In each bubble the area of the sector is $\frac{2\pi}{3}$ and the area of the triangle is $\frac{\sqrt{3}}{4}$.

The two component isoperimetric problem is just the minimization of $J_L$.

For simpler notations, $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$ and the complex multiplication is often employed. For instance we write $e^{i\theta} \tilde{x}$ to denote the vector resulted from rotating $\tilde{x} \in \mathbb{R}^2$ counterclockwise by an angle $\theta$. The $O$ notation will be used frequently. For a quantity, say $\gamma$, that depends on $\rho$ and/or $\gamma$, if one sees something like $Q = O(\rho^2)$, then there exists $C > 0$ independent of $\rho$ and $\gamma$ such that $|Q| \leq C\rho^2$ for all $\rho$ and $\gamma$. If in addition to $\rho$ and $\gamma$, $Q$ also depends on another variable, say $t$, then $Q = O(\rho^2)$ always means $|Q| \leq C\rho^2$ uniformly with respect to all $t$.

## 3 Double bubble

We use $E = (E_1, E_2)$ to denote a particular equal area double bubble in $\mathbb{R}^2$. As shown in Figure 2 the set $E_1$ is open and is bounded by the vertical line segment $\{ti : -\frac{\sqrt{3}}{2} \leq t \leq \frac{\sqrt{3}}{2}\}$ and the arc $\{\zeta \in \mathbb{C} : |\zeta + \frac{1}{2}| = 1, \text{Re}(\zeta) \leq 0\}$, and $E_2$ is open and bounded by the same vertical line segment and the arc $\{\zeta \in \mathbb{C} : |\zeta - \frac{1}{2}| = 1, \text{Re}(\zeta) \geq 0\}$. The three arcs meet at 120 degree angles. Each $E_i$ is the union of a sector whose area is $\frac{2\pi}{3}$ and a triangle whose area is $\frac{\sqrt{3}}{4}$, so the area of $E_i$ is $\frac{m\pi}{2}$ where $m$ is given in (2.8).

The double bubble $E$ will be scaled by a factor $\rho > 0$, rotated by an angle $\theta \in S^1$, and translated by a vector $\xi \in \mathbb{R}^2$. An angle $\theta$ in $[0, 2\pi)$ is regarded as a point on the unit circle $S^1$, so $\theta \in S^1$. Define

$$B_i = B_i(\rho, \xi, \theta) = \{\rho e^{i\theta} \tilde{x} + \xi : \tilde{x} \in E_i\}, \quad B = (B_1, B_2).$$

(3.1)

The Green’s function of $-\Delta$ on $D$ with the Neumann boundary condition is denoted by $G = G(x, y)$. It satisfies

$$-\Delta G(\cdot, y) = \delta(\cdot - y) - \frac{1}{|D|} \text{ in } D, \quad \partial_n G(\cdot, y) = 0 \text{ on } \partial D, \quad \int_D G(x, y) \,dx = 0,$$

(3.2)

for every $y \in D$. Here $\partial_n G$ stands for the outward normal derivative at $\partial D$ of $G$ with respect to its first argument $x$. One can write

$$G(x, y) = \frac{1}{2\pi} \log \frac{1}{|x - y|} + R(x, y)$$

(3.3)

where $R$ is the regular part of $G$, a smooth function on $D \times D$. It is known that

$$R(z, z) \to \infty \quad \text{as } z \to \partial D.$$

(3.4)

Let $\tilde{\delta} > 0$ and set $D_{\tilde{\delta}} = \{x \in D : \text{dist}(x, \partial D) > \tilde{\delta}\}$. Because of (3.4), we can find $\tilde{\delta}$ small enough so that

$$\min_{z \in D} R(z, z) < \min_{z \in D \setminus D_{\tilde{\delta}}} R(z, z).$$

(3.5)
Fix a $\delta$ satisfying (3.5) throughout this paper. Next take $\delta$ such that

$$0 < 3\delta < \delta.$$  

(3.6)

For the moment we only assume that $\delta$ satisfies (3.6). Later more conditions on $\delta$ will be imposed.

The parameter $\rho$ stays in the range

$$\rho \in (0, \delta).$$

(3.7)

Let $\bar{\delta} = 3 - 3 \delta > 0$ and $D_{\bar{\delta}} = \{ x \in D : \text{dis}(x, \partial D) > \bar{\delta} \}. $ If a double bubble $B(\rho, \xi, \theta) = (B_1, B_2)$ satisfies (3.7) and $\xi \in \overline{D_2}$, then $B_1 \cup B_2 \subset \overline{D_0}$. Actually since The distance from the furthest points in $B_1 \cup B_2$ to $\xi$ is $3 \rho < \frac{3 \bar{\delta}}{2}$, there is at least a distance of $\frac{3 \bar{\delta}}{2}$ from $B_1 \cup B_2$ to the boundary of $D_{\bar{\delta}}$. The last property is needed later when we modify $B = (B_1, B_2)$ to form a perturbed double bubble $\Omega = (\Omega_1, \Omega_2)$ and wish to keep $\Omega_1 \cup \Omega_2$ in $D_{\bar{\delta}}$.

Also let $\sigma > 0$ and assume that $\gamma$ satisfies

$$|\gamma| \rho^3 \in (0, \sigma).$$

(3.8)

The existence Theorem 2.1 will be proved for $\rho$ and $\gamma$ in the parameter range (3.7) and (3.8), and the bounds $\delta$ and $\sigma$ must be sufficiently small.

**Lemma 3.1.** The energy of the double bubble $B(\rho, \xi, \theta)$ is estimated as

$$\left| J(B(\rho, \xi, \theta)) - \left\{ \rho(2m) + \sum_{i,j=1}^2 \frac{\gamma_{ij}}{2} \left( \frac{\rho^4}{2 \pi} \log \left( \frac{m}{\rho} \right)^2 \right) + \rho^4 \epsilon_{ij} + \rho^4 \left( \frac{m}{\rho} \right)^2 R(\xi, \theta) \right\} \right| \leq \frac{3|\gamma| \rho^3 m^2}{2} \max_{x,y \in \partial D_{\bar{\delta}}} |\nabla R(x, y)|$$

where the constants $\epsilon_{ij}$ are given by $\epsilon_{ij} = \int_{E_i} \int_{E_j} \frac{1}{4 \pi} \log \frac{1}{|x-y|} \, dx\, dy$, and $\nabla R$ denotes the gradient of $R(x, y)$ with respect to its first variable $x$.

**Proof.** The lengths of $\partial B_1 \setminus \partial B_2, \partial B_2 \setminus \partial B_1$, and $\partial B_1 \cap \partial B_2$ are respectively $\frac{4\pi}{\rho}, \frac{4\pi}{\rho} \rho$ and $\sqrt{3} \rho$. Hence

$$J_L(B) = \frac{1}{2} \left( P_D(B_1) + P_D(B_2) + P_D(B_3) \right) = \rho \left( \frac{8\pi}{3} + \sqrt{3} \right) = \rho(2m).$$

(3.9)

To estimate $J_N(B)$ note that

$$\int_D \left( (-\Delta)^{-1/2} (\chi_{B_i(\rho, \xi, \theta)} - \frac{\rho^2 m_i}{|D|}) \right) \left( (-\Delta)^{-1/2} (\chi_{B_j(\rho, \xi, \theta)} - \frac{\rho^2 m_j}{|D|}) \right) \, dx$$

$$= \int_{B_i(\rho, \xi, \theta)} \int_{B_j(\rho, \xi, \theta)} G(x, y) \, dx \, dy = \int_{B_i(\rho, \xi, \theta)} \int_{B_j(\rho, \xi, \theta)} \left( \frac{1}{2 \pi} \log \frac{1}{|x-y|} + R(x, y) \right) \, dx \, dy$$

$$= \rho^4 \frac{1}{2 \pi} \log \frac{1}{\rho^2} \left( \frac{m}{\rho} \right)^2 + \rho \int_{E_i} \int_{E_j} \frac{1}{2 \pi} \log \frac{1}{|x-y|} \, dx \, dy + \rho^4 \int_{E_i} \int_{E_j} R(\rho e^{i\theta} \bar{x} + \xi, \rho e^{i\theta} \bar{y} + \xi) \, dx \, dy.$$  

(3.10)

For the last term note that by the symmetry $R(x, y) = R(y, x)$, there exists $\tau \in (0, 1)$ such that

$$|R(\rho e^{i\theta} \bar{x} + \xi, \rho e^{i\theta} \bar{y} + \xi) - R(\xi, \xi)|$$

$$= |\nabla R(\tau \rho e^{i\theta} \bar{x} + \xi, \tau \rho e^{i\theta} \bar{y} + \xi) \cdot (\rho e^{i\theta} \bar{x}) + \nabla R(\tau \rho e^{i\theta} \bar{x} + \xi, \tau \rho e^{i\theta} \bar{y} + \xi) \cdot (\rho e^{i\theta} \bar{y})|$$

$$\leq \left( \max_{x,y \in \partial D_{\bar{\delta}}} \|\nabla R(x, y)\| \right) \left( |\rho \bar{x}| + |\rho \bar{y}| \right) \leq 3\rho \max_{x,y \in \partial D_{\bar{\delta}}} \|\nabla R(x, y)\|$$  

(3.11)

where $\nabla$ denotes the gradient of $R(x, y)$ with respect to its second variable $y$. The lemma follows from (3.9), (3.10) and (3.11).

\[ \Box \]

4 Variations

We derive the criticality condition for the functional $J$ in this section.

Let $r(t) = (x(t), y(t)), t \in [-1, 1]$, be a planner curve. The length element of the curve is $ds = |r'| dt$. Let the unit tangent vector be

$$T(t) = \frac{r'(t)}{|r'(t)|}. $$

(4.1)
There are two choices of a unit normal vector both of which we use:
\[ \mathbf{N}(t) = \mathbf{T}(t)i, \quad \text{or} \quad \mathbf{N}(t) = \mathbf{T}(t)(-i). \] (4.2)

The signed curvature \( \kappa \) corresponding to \( \mathbf{N} \) is
\[ \kappa(t) = \begin{cases} \frac{\det[\mathbf{r}', \mathbf{r}'']}{|\mathbf{r}'|^3} & \text{if } \mathbf{N} = \mathbf{T}i \\ -\frac{\det[\mathbf{r}', \mathbf{r}'']}{|\mathbf{r}'|^3} & \text{if } \mathbf{N} = \mathbf{T}(-i) \end{cases}. \] (4.3)

Under this convention,
\[ \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \] (4.4)
holds, regardless which \( \mathbf{N} \) one picks. The vector \( \kappa \mathbf{N} \) is termed the curvature vector, which is independent of the parametrization of the curve.

The following two lemmas can be proved by direct computation.

**Lemma 4.1** Let \( \mathbf{r}'(t) \) be a deformation of \( \mathbf{r}(t) \) so that, \( \mathbf{r}'(t) \) is \( C^2 \) with respect to \( t \) and \( C^1 \) with respect to \( \epsilon \), and \( \mathbf{r}^0 = \mathbf{r} \). Let \( \mathbf{X} \) be its infinitesimal element: \( \mathbf{X}(t) = \frac{\partial \mathbf{r}'(t)}{\partial \epsilon} |_{\epsilon = 0} \). Then
\[ \frac{d}{d\epsilon} \int_{-1}^{1} |\mathbf{r}'| \, dt \bigg|_{\epsilon = 0} = \mathbf{T} \cdot \mathbf{X} |_{-1}^{1} - \int_{-1}^{1} \kappa \mathbf{N} \cdot \mathbf{X} \, ds \]
where \( \int_{-1}^{1} |\mathbf{r}'| \, dt \) is the length of \( \mathbf{r}' \).

**Lemma 4.2** Suppose that a bounded domain \( U \) is enclosed by a piecewise \( C^1 \) curve, and \( U^\epsilon \) is a deformation of \( U \) with piecewise \( C^1 \) boundary. Also the deformation of \( \partial U \) to \( \partial U^\epsilon \) is \( C^1 \) with respect to \( \epsilon \) so the infinitesimal element \( \mathbf{X} \) exists and is continuous on \( \partial U \). Then
\[ \frac{d}{d\epsilon} \int_{\partial U^\epsilon} f(x) \, dx = -\int_{\partial U} f(x) \mathbf{N} \cdot \mathbf{X} \, ds \]
where \( \mathbf{N} \) is the inward unit normal vector on \( \partial U \).

We denote a perturbed double bubble by \( \Omega \), which consists of two disjoint open sets \( \Omega_1 \) and \( \Omega_2 \). The two sets share part of their boundaries, i.e. \( \partial \Omega_1 \cap \partial \Omega_2 \neq \emptyset \). Assume that \( \partial \Omega_1 \cap \partial \Omega_2 \) is parametrized by \( \mathbf{r}_0(t) \). The rest of the boundary of \( \Omega_1 \), i.e. \( \partial \Omega_1 \setminus \partial \Omega_2 \), is parametrized by \( \mathbf{r}_1(t) \), and the rest of the boundary of \( \Omega_2 \), i.e. \( \partial \Omega_2 \setminus \partial \Omega_1 \), is parametrized by \( \mathbf{r}_2(t) \). The argument \( t \) is in \([-1, 1]\) in all the three cases. For now we assume that the \( \mathbf{r}_i \)'s are \( C^2 \) vector valued functions. Since the three curves meet at two triple junction points, the conditions
\[ \mathbf{r}_1(1) = \mathbf{r}_2(1) = \mathbf{r}_0(1) \quad \text{and} \quad \mathbf{r}_1(-1) = \mathbf{r}_2(-1) = \mathbf{r}_0(-1) \] (4.5)
must hold. Sometimes we write \( \mathbf{r} \) collectively for \( \mathbf{r}_1, \mathbf{r}_2, \) and \( \mathbf{r}_0 \), i.e. treat \( \mathbf{r} \) as the union of \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_0 \). Then \( \mathbf{r} \) becomes a piecewise \( C^2 \) vector field on \( \partial \Omega_1 \cup \partial \Omega_2 \).

Let \( \mathbf{N}_1, \mathbf{N}_2, \) and \( \mathbf{N}_0 \) be unit normal vectors to \( \mathbf{r}_1, \mathbf{r}_2, \) and \( \mathbf{r}_0 \) respectively. We adopt the following direction convention: \( \mathbf{N}_1 \) points inward with respect to \( \Omega_1 \), \( \mathbf{N}_2 \) points inward with respect to \( \Omega_2 \), and \( \mathbf{N}_0 \) points from \( \Omega_2 \) towards \( \Omega_1 \), i.e. inward with respect to \( \Omega_1 \) and outward with respect to \( \Omega_2 \). We also write \( \mathbf{N} \) collectively for \( \mathbf{N}_1, \mathbf{N}_2, \) and \( \mathbf{N}_0 \). However one must be careful that when viewed as a single vector field on \( \partial \Omega_1 \cup \partial \Omega_2 \), \( \mathbf{N} \) is usually not single valued at triple junction points.

A deformation \( \Omega^\epsilon \) of \( \Omega \) is a family of perturbed double bubbles parametrized by \( \epsilon \) in a neighborhood of 0. The three curves \( \partial \Omega_1^\epsilon \setminus \partial \Omega_2^\epsilon, \partial \Omega_2^\epsilon \setminus \partial \Omega_1^\epsilon, \) and \( \partial \Omega_1^\epsilon \cup \partial \Omega_2^\epsilon \) that enclose \( \Omega^\epsilon \) are parametrized respectively by \( \mathbf{r}_1^\epsilon, \mathbf{r}_2^\epsilon, \) and \( \mathbf{r}_0^\epsilon \). Every \( \mathbf{r}_i^\epsilon(t) \) is \( C^2 \) with respect to \( t \) and \( C^1 \) with respect to \( \epsilon \); at \( \epsilon = 0 \), we require that \( \mathbf{r}_i^0 = \mathbf{r}_i \); \( \mathbf{r}_j^\epsilon \) also satisfy the compatibility condition (4.5). Define
\[ \mathbf{X}_i(t) = \frac{\partial \mathbf{r}_i^\epsilon(t)}{\partial \epsilon} |_{\epsilon = 0} \] (4.6)
which is the infinitesimal element of the deformation \( \mathbf{r}_i^\epsilon \). Again we write \( \mathbf{X} \) for \( \mathbf{X}_1, \mathbf{X}_2, \) and \( \mathbf{X}_0 \). Because \( \mathbf{r}_j^\epsilon \) satisfy (4.5), unlike \( \mathbf{N} \) the vector field \( \mathbf{X} \) on \( \partial \Omega_1 \cup \partial \Omega_2 \) is still single valued at the triple junction points.
We proceed to find \(\frac{d\mathcal{J}(\Omega^\varepsilon)}{de}|_{e=0}\). Recall \(I_\Omega\) from (2.7) which can be written as

\[
I_\Omega(x) = \int_{\Omega_i} G(x, y) \, dy, \quad i = 1, 2, \tag{4.7}
\]

in terms of the Green’s function. Then the product rule of differentiation implies that

\[
\frac{d}{de}|_{e=0} \int_{\Omega_1^e} \int_{\Omega_2^e} G(x, y) \, dx \, dy = \frac{d}{de}|_{e=0} \int_{\Omega_1^e} I_{\Omega_1}(x) \, dx + \frac{d}{de}|_{e=0} \int_{\Omega_2^e} I_{\Omega_2}(x) \, dx. \tag{4.8}
\]

However Lemma 4.2 shows

\[
\frac{d}{de}|_{e=0} \int_{\Omega_1^e} I_{\Omega_1}(x) \, dx = - \int_{\partial\Omega_1 \setminus \partial\Omega_2} I_{\Omega_1} N_1 \cdot X_1 \, ds - \int_{\partial\Omega_1 \cap \partial\Omega_2} I_{\Omega_1} N_0 \cdot X_0 \, ds \tag{4.9}
\]

\[
\frac{d}{de}|_{e=0} \int_{\Omega_2^e} I_{\Omega_2}(x) \, dx = - \int_{\partial\Omega_2 \setminus \partial\Omega_1} I_{\Omega_2} N_2 \cdot X_2 \, ds + \int_{\partial\Omega_1 \cap \partial\Omega_2} I_{\Omega_2} N_0 \cdot X_0 \, ds. \tag{4.10}
\]

Therefore

\[
\frac{d}{de}|_{e=0} \int_{\Omega_1^e} \int_{\Omega_2^e} G(x, y) \, dx \, dy = -2 \int_{\partial\Omega_1 \setminus \partial\Omega_2} I_{\Omega_1} N_1 \cdot X_1 \, ds - 2 \int_{\partial\Omega_1 \cap \partial\Omega_2} I_{\Omega_1} N_0 \cdot X_0 \, ds \tag{4.11}
\]

Combining (4.12) with Lemma 4.1 we obtain the following.

**Lemma 4.3** Let \(\Omega^\varepsilon\) be a deformation of a perturbed double bubble \(\Omega\). The three curves \(\partial\Omega_1^e \setminus \partial\Omega_2^e\), \(\partial\Omega_2^e \setminus \partial\Omega_1^e\) and \(\partial\Omega_1^e \cap \partial\Omega_2^e\), are parametrized by \(r_1^e(t), r_2^e(t)\) and \(r_0^e(t)\) respectively, which are \(C^2\) with respect to \(t\) and \(C^1\) with respect to \(e\), and satisfy (4.5). Then

\[
\frac{d\mathcal{J}(\Omega^\varepsilon)}{de}|_{e=0} = (T_1 + T_2 + T_0) \cdot X \bigg|_{p^+} - \int_{\partial\Omega_1 \setminus \partial\Omega_2} (\kappa + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2}) N_1 \cdot X \, ds
\]

\[
- \int_{\partial\Omega_2 \setminus \partial\Omega_1} (\kappa + \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2}) N_2 \cdot X \, ds
\]

\[
- \int_{\partial\Omega_1 \cap \partial\Omega_2} (\kappa + \gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{21} - \gamma_{22}) I_{\Omega_2} N_0 \cdot X \, ds.
\]

where the \(N_i\)’s are the unit normal vectors of \(\Omega\) conforming to the direction convention, and \(X\) is the infinitesimal element of the deformation given in (4.6).

If \(\Omega\) is a critical point of the functional \(\mathcal{J}\), then the quantity on the right side of Lemma 4.3 equals 0 for the infinitesimal element \(X\) of any permissible deformation \(r^\varepsilon\). A deformation is permissible if the area of each \(\Omega_i^e\) remains
Figure 3: A standard double bubble is enclosed by the dashed curves in a \((\xi, \theta)\)-frame; the solid curves bound a restrictedly perturbed double bubble; also plotted are the centers \(O_1\) and \(O_2\), triple junction points \(P^+\) and \(P^-\), and the height \(a\) of \(P^+\) and the angle \(A\); the triangles \(\Delta O_1 P^+ P^-\) and \(\Delta O_2 P^+ P^-\) are mentioned in section 6.

unchanged under the deformation. Lemma 4.2 shows that

\[
\frac{d|\Omega_1|}{de} \bigg|_{e=0} = -\int_{\partial \Omega_1 \setminus \partial \Omega_2} \mathbf{N}_1 \cdot \mathbf{X}_1 \, ds - \int_{\partial \Omega_1 \cap \partial \Omega_2} \mathbf{N}_0 \cdot \mathbf{X}_0 \, ds
\]

\[
\frac{d|\Omega_2|}{de} \bigg|_{e=0} = -\int_{\partial \Omega_2 \setminus \partial \Omega_1} \mathbf{N}_2 \cdot \mathbf{X}_2 \, ds + \int_{\partial \Omega_1 \cap \partial \Omega_2} \mathbf{N}_0 \cdot \mathbf{X}_0 \, ds
\]

Hence for a permissible deformation, the right sides of (4.13) and (4.14) must vanish.

5 Restricted perturbations

Given a double bubble \(B(\rho, \xi, \theta)\) we also adopt a reference frame in relation to \(B(\rho, \xi, \theta)\). The point \(\xi\) becomes the origin of the new frame whose horizontal axis points from the center of \(B(\rho, \xi, \theta)\) to the center of \(B(\rho, \xi, \theta)\). The vertical axis is obtained by rotating the horizontal axis 90 degrees counterclockwise. This new coordinate system is termed the \((\xi, \theta)\)-frame. The centers of the left and right bubbles of \(B(\rho, \xi, \theta)\) are \(O_1 = (-\rho^2, 0)\) and \(O_2 = (\rho^2, 0)\) respectively, in the \((\xi, \theta)\)-frame, and the triple junction points are \((0, \sqrt{3}/2 \rho)\) and \((0, -\sqrt{3}/2 \rho)\). Henceforward \(B(\rho, \xi, \theta)\) stands for the double bubble in the \((\xi, \theta)\)-frame. Under this frame we describe perturbed double bubbles. However we only consider a restricted class of perturbed double bubbles at this point. To form a restrictedly perturbed double bubble, the upper and lower triple junction points of \(B(\rho, \xi, \theta)\) are allowed to move to new positions \(P^+\) and \(P^-\) only vertically and only by the same distance in opposite directions; namely there is \(a > 0\) such that in the \((\xi, \theta)\)-frame \(P^+ = (0, a)\) and \(P^- = (0, -a)\). Here \(a\) is close to \(\sqrt{3}/2 \rho\). Between now and the end of section 8 we will only consider restrictedly perturbed double bubbles in a fixed \((\xi, \theta)\)-frame. Deformations of a restrictedly perturbed double bubble will also lie in the same restricted class. In section 9 deformations outside the restricted class will be explored.

In general curves are described by parametrization up to diffeomorphism. Say the three curves of a perturbed double bubble \(\Omega\) in the restricted class are parametrized by \(r_1(t)\), \(r_2(t)\) and \(r_0(t)\). Then any reparametrizations of \(r_i\) will give the same curves. To eliminate this freedom, we set up a particular way of parametrization. Introduce three functions \(u_1\), \(u_2\), and \(u_0\), all defined on \([-1, 1]\), so that in the \((\xi, \theta)\)-frame the three curves on the left, right
and center, of the perturbed double bubble $\Omega$ are parametrized respectively by
\[
\mathbf{r}_1(t) = u_1(t)e^{i(\pi - At)} + O_1, \quad \mathbf{r}_2(t) = u_2(t)e^{iAt} + O_2, \quad \mathbf{r}_0(t) = (u_0(t), at).
\] (5.1)

A polar coordinate system centered at $O_1$ with the angle starting from the negative horizontal direction of the $(\xi, \theta)$-frame describes the curve $\mathbf{r}_1$. In this polar coordinate system the radius and angle of $P^+$ are $u_1(1)$ and $A$; the radius and angle of $P^-$ are $u_1(-1)$ and $-A$. For $\mathbf{r}_2$ another polar coordinate system centered at $O_2$ with the angle starting from the positive horizontal direction of the $(\xi, \theta)$-frame is adopted, $P^+$ is described by the radius $u_2(1)$ and the angle $A$, and $P^-$ by $u_2(-1)$ and $-A$ accordingly. The center curve $\mathbf{r}_0$ is given as a graph, up to a factor $a$, in the $(\xi, \theta)$-frame, with the vertical direction representing the independent variable and the horizontal direction representing the dependent variable.

From now on $\Omega$ will denote a perturbed double bubble in the $(\xi, \theta)$-frame. Obviously that $\Omega$ can be described by (5.1) means more restrictions. As a graph, the center curve $\partial\Omega_1 \cap \partial\Omega_2$ must intersect every horizontal line whose height is between $-a$ and $a$ (with respect to the $(\xi, \theta)$-frame) exactly once. The left curve $\partial\Omega_1 \setminus \partial\Omega_2$ must be a graph in the polar coordinate system centered at $O_1$; namely each ray between $-A$ and $A$ must intersect $\partial\Omega_1 \setminus \partial\Omega_2$ exactly once. A similar condition holds for the right curve $\partial\Omega_2 \setminus \partial\Omega_1$.

The two numbers $A$ and $a$ in (5.1) are related:
\[
A = \pi - \arctan \left( \frac{2a}{\rho} \right), \quad \text{where } A \in \left( \frac{\pi}{2}, \pi \right) \text{ and } a \in (0, \infty).
\] (5.2)

At the two triple junction points,
\[
u_1(1) = u_1(-1) = u_2(1) = u_2(-1) = \sqrt{a^2 + \left( \frac{\rho}{2} \right)^2}, \quad u_0(1) = u_0(-1) = 0.
\] (5.3)

To maintain the proper shape of a perturbed double bubble with two triple junction points, $u_1$, $A$ and $a$ need to be close to the corresponding variables of the standard double bubble. The value of $(u_1, u_2, u_0, A, a)$ for the standard double bubble $B(\rho, \xi, \theta)$ is $(\rho, \rho, 0, 2\pi, \sqrt{3}\rho)$. Let $\tilde{c} > 0$ such
\[
\|u_i - \rho\|_{C^1} \leq \tilde{c}\rho, \quad i = 1, 2, \quad \|u_0\|_{C^1} \leq \tilde{c}\rho, \quad |A - \frac{2\pi}{3}| \leq \tilde{c}, \quad |a - \frac{\sqrt{3}\rho}{2}| \leq \tilde{c}\rho.
\] (5.4)

By choosing $\tilde{c}$ suitably small, we are guaranteed two sets sharing part of their boundaries. The use of the $C^4$ norm in (5.4) makes the curves meeting at the two triple junction points properly. Moreover, according to the remark following (3.7), with a small $\tilde{c}$, $\Omega_1 \cup \Omega_2$ always stays inside $\overline{D_{\rho'}}$, a compact subset of the domain $D$.

In summary the class of restricted perturbations only include perturbed double bubbles described by (5.1) subject to (5.2), (5.3) and (5.4), plus the constraints (2.9). A member in this class is termed a restrictedly perturbed double bubble if no possibility of ambiguity exists. Despite the many conditions, a member in the restricted class may deviate quite a bit from the standard double bubble as seen in Figure 3.

For $r_1$, the length element is $ds = |r'_1| dt = (u'_1)^2 + A^2 u_1^2 dt$, and the unit tangent vector and the unit normal vectors are
\[
\mathbf{T}_1(t) = \frac{u'_1 e^{i(\pi - At)} + Au_1 e^{i(\pi - At)(-1)}}{\sqrt{(u'_1)^2 + A^2 u_1^2}}, \quad \mathbf{N}_1(t) = \mathbf{T}_1(t)(-i) = \frac{-Au_1 e^{i(\pi - At)} + u'_1 e^{i(\pi - At)(-1)}}{\sqrt{(u'_1)^2 + A^2 u_1^2}}.
\] (5.5)

The signed curvature $\kappa_1$ is given by
\[
\kappa_1(t) = -Au_1 u'_1 + 2Au_1^2 + 3u_1^2 \left( (u'_1)^2 + A^2 u_1^2 \right)^{3/2}.
\] (5.6)

For a deformation $r'_1(t) = u'_1(t)e^{i(\pi - A(t))} + O_1$ of $r_1(t)$, we set
\[
u'_1(0) = u_1(t), \quad \frac{\partial u'_1(t)}{\partial \epsilon} \bigg|_{\epsilon = 0} = v_1(t), \quad A(0) = A, \quad A'(0) = A',
\] (5.7)

so that
\[
X_1(t) = \frac{\partial r'_1(t)}{\partial \epsilon} \bigg|_{\epsilon = 0} = v_1 e^{i(\pi - At)} + u_1 A't e^{i(\pi - At)(-1)}.
\] (5.8)
For $r_2$ given in (5.1), $ds = \sqrt{(u'_2)^2 + A^2 u_2^2} dt$, and the tangent vector, normal vector and curvature are

$$T_2(t) = \frac{u_2 e^{i\alpha} + Au_2 e^{i\alpha t}}{\sqrt{(u'_2)^2 + A^2 u_2^2}}, \quad N_2(t) = T_2(t)i = -\frac{Au_2 e^{i\alpha} + u_2 e^{i\alpha t}}{\sqrt{(u'_2)^2 + A^2 u_2^2}}, \quad \kappa_2(t) = -\frac{Au_2 u'_2 + 2A(u'_2)^2 + A^3 u_2^2}{((u'_2)^2 + A^2 u_2^2)^{3/2}}. \quad (5.9)$$

For a deformation $r_2^\varepsilon(t) = u_2^\varepsilon(t) e^{i\alpha(t)} + O_2$,

$$X_2(t) = \frac{\partial e^\varepsilon(t)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = v_2 e^{i\alpha} + u_2 A't e^{i\alpha t}, \quad \text{where} \quad v_2 = \frac{\partial u_2^\varepsilon}{\partial \varepsilon} \bigg|_{\varepsilon=0}. \quad (5.10)$$

For the curve $r_0$,

$$ds = \sqrt{(u'_0)^2 + a^2} dt, \quad T_0(t) = \frac{(u'_0, a)}{\sqrt{(u'_0)^2 + a^2}}, \quad N_0(t) = T_0(t)i = \frac{(-a, u'_0)}{\sqrt{(u'_0)^2 + a^2}}, \quad \kappa_0(t) = \frac{-au'_0}{((u'_0)^2 + a^2)^{3/2}}. \quad (5.11)$$

For a deformation $r_0^\varepsilon(t) = (u_0^\varepsilon(t), a(\varepsilon) t)$,

$$X_0(t) = \frac{\partial e^\varepsilon(t)}{\partial \varepsilon} \bigg|_{\varepsilon=0} = (v_0, a(t)), \quad \text{where} \quad v_0 = \frac{\partial u_0^\varepsilon}{\partial \varepsilon} \bigg|_{\varepsilon=0}. \quad (5.12)$$

# 6 Hilbert Spaces

While capturing the geometric picture of a restrictedly perturbed double bubble, the variables $u_i$, $A$, and $a$ are not very convenient for analytic techniques, such as the contraction mapping theorem, because (5.2), (5.3) and (2.8) are nonlinear constraints. We introduce a new set of variables to describe perturbed double bubbles. These so-called internal variables will be elements in a Hilbert space, so that our problem becomes a nonlinear equation between Hilbert spaces, which can also be formulated in a fixed point form.

Let $\Omega = (\Omega_1, \Omega_2)$ be a restrictedly perturbed double bubble. Figure 3 shows that the area of $\Omega_1$ (and also of $\Omega_2$) is the sum of the areas of three regions: a triangle, a sector, and a strip. The triangle is formed from the two triple junction points $P^+$, $P^-$ and the center $O_1$ of $\Omega_1$; the sector is the part of $\Omega_1$ described by points whose angle in the polar coordinates is between $-A$ and $A$; the strip is the (signed) region bounded by the line segment connecting $P^+$ and $P^-$ and the curve $r_0$. Hence the area of $\Omega_1$ and the area of $\Omega_2$ are written as

$$|\Omega_1| = \frac{a\rho}{2} + \int_{-1}^{1} A \frac{u_1^2}{2} dt + \int_{-1}^{1} au_0 dt = \int_{-1}^{1} \left(\frac{a\rho}{4} + \frac{A}{2} u_1^2\right) dt + \int_{-1}^{1} au_0 dt \quad (6.1)$$

$$|\Omega_2| = \frac{a\rho}{2} + \int_{-1}^{1} A \frac{u_2^2}{2} ds - \int_{-1}^{1} au_0 ds = \int_{-1}^{1} \left(\frac{a\rho}{4} + \frac{A}{2} u_2^2\right) dt - \int_{-1}^{1} au_0 dt \quad (6.2)$$

Inspired by (6.1) and (6.2), introduce $\phi_1$, $\phi_2$, and $\phi_0$ such that

$$\phi_1(\pm 1) = \phi_2(\pm 1) = 0 \quad \text{and} \quad \phi_0(\pm 1) = 0. \quad (6.5)$$

With these new variables, we find that

$$|\Omega_1| = \frac{m\rho^2}{2} + \int_{-1}^{1} \phi_1 dt + \int_{-1}^{1} \phi_0 dt, \quad |\Omega_2| = \frac{m\rho^2}{2} + \int_{-1}^{1} \phi_2 dt - \int_{-1}^{1} \phi_0 dt. \quad (6.6)$$

The area constraints (2.8) become

$$\int_{-1}^{1} \phi_1 dt + \int_{-1}^{1} \phi_0 dt = 0, \quad \int_{-1}^{1} \phi_2 dt - \int_{-1}^{1} \phi_0 dt = 0. \quad (6.7)$$
again linear conditions.

Henceforth we use the \( \phi_i \)'s and \( \alpha \) as our primary variables, called internal variables. Collectively write \( \phi \) for \((\phi_1, \phi_2, \phi_0)\), and use \((\phi, \alpha)\) to describe a restrictedly perturbed double bubble. In terms of internal variables the standard double bubble \( \mathcal{B}(\rho, \xi, \theta) \) is represented by \((0, 0)\). This way of representing double bubbles and perturbed double bubbles is termed the internal representation. The previous functions \( u_i \) and the numbers \( A \) and \( a \) can be derived from \( \phi_i \) and \( \alpha \). Of course this transformation between the two sets of variables can only be done for \((\phi, \alpha)\) close to \((0, 0)\), i.e. \((\phi(t), \alpha)\) must be uniformly (with respect to \( t \)) within a certain distance of order \( \rho^2 \) from \((0, 0)\).

Because of the linear conditions (6.5) and (6.7) on \((\phi, \alpha)\), it is very easy to define deformations within the restricted class in this setting. Let a perturbed double bubble be parametrized by \( r_i \) defined by the original variables \( u_i \), \( A \) and \( a \). Transform them to the new variables \( \phi_i \) and \( \alpha \). For \( \phi_i \in C^1[-1, 1] \) and \( \alpha \in \mathbb{R} \) satisfying (6.5) and (6.7), let \( \psi_i \in C^1[-1, 1], \beta \in \mathbb{R} \), satisfying (6.5) and (6.7) as well, and \( \epsilon \in \mathbb{R} \). Then for \( \epsilon \) sufficiently close to 0, \((\phi', \alpha') = (\phi, \alpha) + \epsilon(\psi, \beta)\) also describes a perturbed double bubble. Transforming them back to the original variables \( u_i', A' \), and \( a' \), we obtain a deformation \( r_i' \) of \( r_i \). Let \( X_i = \frac{\partial r_i'}{\partial \epsilon} \big|_{\epsilon=0} \) be the infinitesimal element of the deformation \( r_i' \). Calculations show that in terms of the new variables

\[
-N_i \cdot X_i \, ds = (\psi_i + e_i(\phi_i, \alpha)\beta) \, dt, \quad i = 1, 2, 0.
\]  

(6.8)

where \( e_1, e_2 \) and \( e_0 \) act on \((\phi_1, \alpha), (\phi_2, \alpha), \) and \((\phi_0, \alpha)\) respectively as follows.

\[
e_i(\phi_i, \alpha) = -\frac{\rho}{4} \frac{da}{d\alpha} - \frac{1}{A} \left( \frac{m_i \rho^2}{4} - \frac{a \rho}{4} + \phi_i + t\phi'_i \right) \frac{dA}{d\alpha}, \quad i = 1, 2; \quad e_0(\phi_0, \alpha) = -\frac{1}{a} \left( \phi_0 + t\phi'_0 \right) \frac{da}{d\alpha}.
\]  

(6.9)

Implicit differentiation from (5.2) and (6.4) gives that

\[
\frac{da}{d\alpha} = \frac{1}{aA}, \quad \frac{dA}{d\alpha} = -\left( \frac{2\rho}{\rho^2 + 4a^2} \right) \frac{1}{aA}.
\]  

(6.10)

Using the new variables we derive a necessary and sufficient condition for \(-N \cdot X \, ds\) to be associated with a deformation \( r_i' \).

**Lemma 6.1** Let \( r_i \) parametrize a perturbed double bubble via \( u_i, A \) and \( a \).

1. Suppose that \( r_i'(t) \) be a deformation of \( r_i \) within the restricted class and is \( C^1 \) with respect to both \( t \) and \( \epsilon \). If

\[
-N_i \cdot X_i \, ds = f_i(t) \, dt
\]

then the following two properties hold:

(a) \( \int_{-1}^{1} f_1 \, dt + \int_{-1}^{1} f_0 \, dt = \int_{-1}^{1} f_2 \, dt - \int_{-1}^{1} f_0 \, dt = 0; \)

(b) there exists \( k \in \mathbb{R} \) such that

\[
\begin{pmatrix}
  f_1(1) \\
  f_2(1) \\
  f_0(1) \\
  f_1(-1) \\
  f_2(-1) \\
  f_0(-1)
\end{pmatrix}
= k
\begin{pmatrix}
  aA - u_1'(1) \cos A \\
  aA - u_2'(1) \cos A \\
  -u_0'(1) \\
  aA + u_1(-1) \cos A \\
  aA + u_2(-1) \cos A \\
  u_0(-1)
\end{pmatrix}
.
\]

2. Conversely if \( r_i(t) \) is a \( C^2 \) parametrization of a perturbed double bubble and \( f_i \in C^1[-1, 1], \quad i = 1, 2, 0, \) satisfy the two conditions in part 1, then there is a deformation \( r_i' \) of \( r_i \), \( C^1 \) with respect to both \( t \) and \( \epsilon \), within the restricted class such that

\[
-N_i \cdot X_i \, ds = f_i(t) \, dt
\]

for \( i = 1, 2, 0, \) where \( X_i = \frac{\partial r_i'}{\partial \epsilon} \big|_{\epsilon=0} \).

**Proof.** To show part 1, note that (a) follows from the area constraint (2.8) and the formulas (4.13) and (4.14). For (b), note that deformations in the restricted class satisfy that \( X_1(1) = X_2(1) = X_0(1) = (0, k) \) and \( X_1(-1) = X_2(-1) = X_0(-1) = (0, -k) \) for some \( k \in \mathbb{R} \). Since \( X_0(1) = (v_0(1), a') \) by (5.12), \( v_0(1) = 0 \) and \( k = a' \). Also

\[
-N_0 \cdot X_0 \, ds = (a\psi_0 - tu_0' a) \, dt
\]

by (5.11) and (5.12). Hence \( f_0(1) = -u_0'(1)k \). Similarly \( X_1(1) = v_1(1)e^{i(\pi - A)} + u_1(1)A' e^{i(\pi - A)} (-i) = (0, k) \) by (5.8) implies that \( v_1(1) = \frac{-ak}{\psi_1(-1)} \). Then

\[
-N_1 \cdot X_1 \, ds = (Au_1v_1 - u_1' u_1 A' t) \, dt
\]

shows that \( f(1) = (aA - u_1'(1) \cos A)k \). Other equations in (b) are derived in the same way.

To prove part 2, transform \( u_i \), \( A \) and \( a \) to the new variables \( \phi_i \) and \( \alpha \). Set

\[
\beta = aA k \quad \text{and} \quad \psi_i = f_i - e_i(\phi_i, \alpha)\beta, \quad i = 1, 2, 0,
\]

(6.11)
where \( k \) is given in condition (b). By (6.3), (6.4), (6.9), and (b), we can show that \( \psi_0(1) = \psi_0(-1) = 0 \) and \( \psi_1(1) = \psi_1(-1) = \psi_2(1) = \psi_2(-1) = \beta \), i.e. \( (\psi, \beta) \) satisfies the boundary condition (6.5). By (6.4) and (6.5), we see that

\[
\int_{-1}^{1} e_i(\phi, \alpha) \, dt = 0, \quad i = 1, 2, 0. \tag{6.12}
\]

Then condition (a) implies that the \( \psi_i \)'s satisfy the constraint (6.7). Consider \( (\phi', \alpha') = (\phi, \alpha) + \epsilon(\psi, \beta) \) as an internal representation and transform it to \( u_i', A^i, \) and \( a_i^\alpha \) of \( u_i, A, \) and \( a, \alpha, \) and consequently \( r_i^\alpha \) of \( r, \) to be a restricted deformation. Then (6.8) and (6.11) show that \( -\mathbf{N}_i \cdot \mathbf{X}_i \, ds = f_i(t) \, dt. \]

In the case of a perfect double bubble, i.e. \( (\phi, \alpha) = (0, 0) \), it follows from (6.9) that

\[
e_i(0, 0) = 0, \quad i = 1, 2, 0. \tag{6.13}
\]

Hence in this case \( f_i \) in Lemma 6.1 is just \( \psi_i \).

The functional \( \mathcal{J} \) is now considered a functional of \( (\phi, \alpha) \). To specify the domain of \( \mathcal{J} \) let

\[
\mathcal{Y} = \{ (\phi, \alpha) \in H^1(-1, 1) \times H^1(-1, 1) \times H^1(-1, 1) \times \mathbb{R} : \phi_1(\pm 1) - \alpha = \phi_2(\pm 1) - \alpha = \phi_0(\pm 1) = 0, \int_{-1}^{1} \phi_1 \, dt + \int_{-1}^{1} \phi_0 \, dt = \int_{-1}^{1} \phi_2 \, dt = \int_{-1}^{1} \phi_0 \, dt = 0 \}. \tag{6.14}
\]

The space \( \mathcal{Y} \) is equipped with the norm \( \| \cdot \|_{\mathcal{Y}} \) given by

\[
\| (\phi, \alpha) \|_{\mathcal{Y}}^2 = \| \phi_1 \|_{H^1}^2 + \| \phi_2 \|_{H^1}^2 + \| \phi_0 \|_{H^1}^2 + \alpha^2. \tag{6.15}
\]

In (6.15) \( \| \cdot \|_{H^1} \) denotes the usual norm of the Sobolev space \( H^1(-1, 1) \); namely \( \| f \|_{H^1}^2 = \int_{-1}^{1} \| f' \|^2 + f^2 \, dt \). Since the transformation between \( (\phi, \alpha) \) and \( (u, a, A) \) is valid only if \( (\phi, \alpha) \) is in a neighborhood of \((0, 0)\) of order \( \rho^2 \), there exists \( \bar{c} > 0 \) such that the domain of \( \mathcal{J} \) is the closed ball of radius \( \bar{c} \rho^2 \) centered at \((0, 0)\) in \( \mathcal{Y} \):

\[
\mathcal{D}(\mathcal{J}) = \{ (\phi, \alpha) \in \mathcal{Y} : \| (\phi, \alpha) \|_{\mathcal{Y}} \leq \bar{c} \rho^2 \}. \tag{6.16}
\]

With \( \bar{c} \) being sufficiently small, the variables \( u_i, A, \) and \( a \) corresponding to \( (\phi, \alpha) \in \mathcal{D}(\mathcal{J}) \) meet the requirement (5.4).

In addition to \( \mathcal{Y} \), two more spaces are needed:

\[
\mathcal{X} = \{ (\phi, \alpha) \in H^2(-1, 1) \times H^2(-1, 1) \times H^2(-1, 1) \times \mathbb{R} : \phi_1(\pm 1) - \alpha = \phi_2(\pm 1) - \alpha = \phi_0(\pm 1) = 0, \int_{-1}^{1} \phi_1 \, dt + \int_{-1}^{1} \phi_0 \, dt = \int_{-1}^{1} \phi_2 \, dt = \int_{-1}^{1} \phi_0 \, dt = 0 \}. \tag{6.17}
\]

\[
\mathcal{Z} = \{ (\psi, \beta) \in L^2(-1, 1) \times L^2(-1, 1) \times L^2(-1, 1) \times \mathbb{R} : \int_{-1}^{1} \psi_1 \, dt + \int_{-1}^{1} \psi_0 \, dt = \int_{-1}^{1} \psi_2 \, dt + \int_{-1}^{1} \psi_0 \, dt = 0 \}. \tag{6.18}
\]

Note that

\[
\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z} \subset L^2(-1, 1) \times L^2(-1, 1) \times L^2(-1, 1) \times \mathbb{R}. \tag{6.19}
\]

Define the inner product on \( L^2(-1, 1) \times L^2(-1, 1) \times L^2(-1, 1) \times \mathbb{R} \) by

\[
\langle (\psi, \beta), (\tilde{\psi}, \tilde{\beta}) \rangle = \sum_{i=1, 2, 0} \int_{-1}^{1} \psi_i \tilde{\psi}_i \, dt + \beta \tilde{\beta}. \tag{6.20}
\]

Then \( L^2(-1, 1) \times L^2(-1, 1) \times L^2(-1, 1) \times \mathbb{R} \) becomes a Hilbert space and \( \mathcal{Z} \) a closed subspace. The norm in \( \mathcal{Z} \) inherited from \( L^2(-1, 1) \times L^2(-1, 1) \times L^2(-1, 1) \times \mathbb{R} \) is denoted by \( \| \cdot \|_{\mathcal{Z}} \) given by

\[
\| (\psi, \beta) \|_{\mathcal{Z}}^2 = \int_{-1}^{1} \psi_1^2 \, dt + \int_{-1}^{1} \psi_0^2 \, dt + \int_{-1}^{1} \psi_2^2 \, dt + \beta^2. \tag{6.21}
\]

Denote the orthogonal projection of \( L^2(-1, 1) \times L^2(-1, 1) \times L^2(-1, 1) \times \mathbb{R} \) onto \( \mathcal{Z} \) by \( \Pi \). Namely

\[
\Pi(\psi, \beta) = (\psi_1, \psi_2, \psi_0, \beta) - \left( \frac{1}{3} \int_{-1}^{1} \psi_1 \, dt + \frac{1}{6} \int_{-1}^{1} \psi_2 \, dt + \frac{1}{6} \int_{-1}^{1} \psi_0 \, dt \right) (1, 0, 1, 0) - \left( \frac{1}{6} \int_{-1}^{1} \psi_1 \, dt + \frac{1}{3} \int_{-1}^{1} \psi_2 \, dt - \frac{1}{6} \int_{-1}^{1} \psi_0 \, dt \right) (0, 1, -1, 0). \tag{6.22}
\]
In $\mathcal{X}$ we use the norm

$$\|(\phi, \alpha)\|_{X}^2 = \|\phi_1\|_{H^2}^2 + \|\phi_2\|_{H^2}^2 + \|\phi_0\|_{H^2}^2 + \alpha^2,$$  

(6.22)

where $\| \cdot \|_{H^2}$ is the norm of the Sobolev space $H^2(-1; 1)$: $\|f\|_{H^2}^2 = \int_{-1}^{1} ((f')^2 + (f'')^2) \, dt$. Note that both $\mathcal{X}$ and $\mathcal{Y}$ are also Hilbert spaces under their respective norms.

We now introduce the gradient of $J$, which is an operator $S$ from a neighborhood of $(0, 0)$ in $\mathcal{X}$ to $\mathbb{Z}$ such that

$$\frac{d}{de} \Big|_{e=0} J((\phi, \alpha) + e(\psi, \beta)) = \langle S(\phi, \alpha), (\psi, \beta) \rangle$$  

(6.23)

for all $(\psi, \beta) \in \mathcal{X}$. Any local minimizer $(\phi, \alpha)$ of $J$ in a restricted class is a solution of the equation $S(\phi, \alpha) = (0, 0)$. In section 8 we will find such a local minimizer by solving the equation. The domain of $S$ is taken to be

$$\mathcal{D}(S) = \{ (\phi, \alpha) \in \mathcal{X} : \| (\phi, \alpha) \|_{X} \leq c \rho^2 \}$$  

(6.24)

where $c$ in (6.24) is the same as the $\tilde{c}$ in (6.16). Consequently $\mathcal{D}(S) \subset \mathcal{D}(J)$. This nonlinear operator is written as the sum of two operators,

$$S = S_L + S_N,$$  

(6.25)

where $S_L$ and $S_N$ correspond to the two parts $J_L$ and $J_N$ of $J$ given in (2.10).

To find $S_L$ we express the length of each curve in terms of $\phi_i$ and $\alpha$:

$$J_L(\phi, \alpha) = \int_{-1}^{1} L_1(\phi_1', \phi_1, \alpha) \, dt + \int_{-1}^{1} L_2(\phi_2', \phi_2, \alpha) \, dt + \int_{-1}^{1} L_0(\phi_0', \alpha) \, dt$$  

(6.26)

where the three integrals are the length of $\partial \Omega_1 \setminus \partial \Omega_2$, $\partial \Omega_2 \setminus \partial \Omega_1$, and $\partial \Omega_1 \cap \partial \Omega_2$ respectively. The $L_i$’s are given by

$$L_i(\phi_i', \phi_i, \alpha) = \sqrt{\frac{(\phi_i')^2}{2A(\phi_i + \frac{mp^2}{4} - \frac{ap}{4})} + 2A(\phi_i + \frac{mp^2}{4} - \frac{ap}{4})}, \quad i = 1, 2, \quad \text{and} \quad L_0(\phi_0', \alpha) = \sqrt{\frac{(\phi_0')^2}{a^2} + a^2}.$$  

(6.27)

In these formulas $A$ and $a$ are functions of $\alpha$. We regard curvature as an operator on $\phi_i$ and $\alpha$. Let

$$\kappa_i(\phi_i, \alpha) = -\frac{\partial}{\partial t} \left( \frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \phi_i'} \right) + \frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \phi_i}, \quad i = 1, 2, \quad \text{and} \quad \kappa_0(\phi_0, \alpha) = -\frac{\partial}{\partial t} \left( \frac{\partial L_0(\phi_0', \alpha)}{\partial \phi_0} \right).$$  

(6.28)

Define another operator from $\mathcal{D}(S)$ to $\mathbb{R}$ by

$$\kappa_s(\phi, \alpha) = \sum_{i=1}^{2} \frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \phi_i'} \Big|_{-1}^{1} + \sum_{i=1}^{2} \int_{-1}^{1} \frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \alpha} \, dt + \int_{-1}^{1} \frac{\partial L_0(\phi_0', \alpha)}{\partial \alpha} \, dt.$$  

(6.29)

Now we set $S_L$ to be

$$S_L(\phi, \alpha) = \Pi \left( \begin{array}{c} \kappa_1(\phi_1, \alpha) \\ \kappa_2(\phi_2, \alpha) \\ \kappa_0(\phi_0, \alpha) \\ \kappa_s(\phi, \alpha) \end{array} \right).$$  

(6.30)

This operator is the gradient of $J_L$ in the sense that for every $(\phi, \alpha) \in \mathcal{D}(S)$ and $(\psi, \beta) \in \mathcal{X}$

$$\frac{d}{de} \Big|_{e=0} J_L((\phi, \alpha) + e(\psi, \beta)) = \langle S_L(\phi, \alpha), (\psi, \beta) \rangle.$$  

(6.31)

For $S_N$ we have

$$S_N(\phi, \alpha) = \Pi \left( \begin{array}{c} \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2} \\ \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2} \\ (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{21} - \gamma_{22}) I_{\Omega_2} \\ h(\phi, \alpha) \end{array} \right).$$  

(6.32)

A few remarks regarding the $I_{\Omega_i}$’s in (6.32) are in order. Recall that each $I_{\Omega_i}$, $i = 1, 2$, is a function on $D$ given in (2.7), and the set $\Omega_i$ is specified by the internal variables $\phi_i$, $\phi_0$ and $\alpha$ for $i = 1, 2$. The $I_{\Omega_i}$’s $(i = 1, 2)$ in the first three components on the right side of (6.32) are the outcomes of the operators

$$I_{ij} : (\phi_i, \phi_0, \alpha) \to I_{\Omega_j}(e^{it} r_j(t) + \xi), \quad i = 1, 2, \quad j = 1, 2, 0.$$  

(6.33)
where \( j = 1, 2, 0 \) corresponds to the first, second, and third component in (6.32) respectively.

The last component \( h \) in (6.32) is a scalar valued operator given by

\[
h(\phi, \alpha) = \int_{-1}^{1} (\gamma_{11}I_{\Omega_1} + \gamma_{12}I_{\Omega_2}) e_1 dt + \int_{-1}^{1} (\gamma_{21}I_{\Omega_1} + \gamma_{22}I_{\Omega_2}) e_2 dt + \int_{-1}^{1} [(\gamma_{11} - \gamma_{12})I_{\Omega_1} + (\gamma_{21} - \gamma_{22})I_{\Omega_2}] e_0 dt. \tag{6.34}
\]

Consequently by (4.12) and (6.8),

\[
\frac{d}{de} \bigg|_{e=0} J_N((\phi, \alpha) + e(\psi, \beta)) = (S_N(\phi, \alpha), (\psi, \beta)). \tag{6.35}
\]

By (6.30) and (6.32), one obtains the expression of the operator \( S \):

\[
S(\phi, \alpha) = \Pi \begin{pmatrix} \kappa_1(\phi_1, \alpha) + \gamma_{11}I_{\Omega_1} + \gamma_{12}I_{\Omega_2} \\ \kappa_2(\phi_2, \alpha) + \gamma_{21}I_{\Omega_1} + \gamma_{22}I_{\Omega_2} \\ \kappa_0(\phi_0, \alpha) + (\gamma_{11} - \gamma_{12})I_{\Omega_1} + (\gamma_{21} - \gamma_{22})I_{\Omega_2} \\ \kappa_s(\phi, \alpha) + h(\phi, \alpha) \end{pmatrix}. \tag{6.36}
\]

**Lemma 6.2** It holds uniformly with respect to \( t \) that

\[
S(0, 0) = \begin{pmatrix} O(|\gamma| \rho^2) \\ O(|\gamma| \rho^2) \\ O(|\gamma| \rho^2) \\ 0 \end{pmatrix}. 
\]

Consequently there exists \( \tilde{C} > 0 \) such that \( \|S(0, 0)\|_x \leq \tilde{C}|\gamma| \rho^2 \).

**Proof.** Calculations show that

\[
\kappa_i(0, 0) = \frac{1}{\rho} \ (i = 1, 2), \quad \kappa_0(0, 0) = 0, \quad \kappa_s(0, 0) = 0 \tag{6.37}
\]

which follow from

\[
\frac{\partial L_i(0, 0, 0)}{\partial \phi_i} = 0, \quad \frac{\partial L_i(0, 0, 0)}{\partial \alpha} = \frac{1}{\rho}, \quad \frac{\partial L_i(0, 0, 0)}{\partial \phi_0} = 0, \quad \frac{\partial L_0(0, 0)}{\partial \alpha} = \frac{\sqrt{3}}{\pi \rho} \tag{6.38}
\]

with the help of (6.10). Consequently

\[
S(0, 0) = \Pi \begin{pmatrix} \frac{1}{\rho}, \frac{1}{\rho}, 0, 0 \end{pmatrix} = (0, 0, 0, 0). \tag{6.39}
\]

Because of (6.13),

\[
h(0, 0) = 0. \tag{6.40}
\]

When \( \Omega_i \) becomes \( B_i(\rho, \xi, \theta) \), for every \( x \in \overline{B_1} \cup \overline{B_2} \)

\[
I_{B_i(\rho, \xi, \theta)}(e^{i\theta}x + \xi) = \int_{B_i(\rho, \xi, \theta)} \frac{1}{2\pi} \log \frac{1}{|x - y|} dy + \int_{B_i(\rho, \xi, \theta)} R(e^{i\theta}x + \xi, e^{i\theta}y + \xi) dy = \frac{m\rho^2}{4\pi} \log \frac{1}{\rho} + O(\rho^2) \tag{6.41}
\]

holds uniformly with respect to such \( x \). Note that here \( B_1 \) and \( B_2 \) are considered sets under the \((\xi, \theta)\)-frame so \( x \) and \( y \) are in this frame, but the arguments of \( R \) are still in the original coordinate system of \( \mathbb{R}^2 \); hence the composition in \( R(e^{i\theta}x + \xi, e^{i\theta}y + \xi) \). Therefore, uniformly with respect to \( t \),

\[
S_N(0, 0) = \Pi \begin{pmatrix} (\gamma_{11} + \gamma_{12})\frac{m\rho^2}{4\pi} \log \frac{1}{\rho} + O(|\gamma| \rho^2) \\ (\gamma_{21} + \gamma_{22})\frac{m\rho^2}{4\pi} \log \frac{1}{\rho} + O(|\gamma| \rho^2) \\ (\gamma_{11} - \gamma_{12})\frac{m\rho^2}{4\pi} \log \frac{1}{\rho} + O(|\gamma| \rho^2) \\ 0 \end{pmatrix} = \begin{pmatrix} O(|\gamma| \rho^2) \\ O(|\gamma| \rho^2) \\ O(|\gamma| \rho^2) \\ 0 \end{pmatrix}. \tag{6.42}
\]

The lemma follows from (6.39) and (6.42). \( \square \)
7 Positivity

In this section we study the linear operator $S'(0,0) : \mathcal{X} \to \mathcal{Z}$, and show that $S'(0,0)$ is positive definite and invertible. A few simple estimates regarding functions in $H^1_0(-1,1)$ are needed. Note that $H^1_0(-1,1) = \{ f \in H^1(-1,1) : f(\pm 1) = 0 \}$.

**Lemma 7.1** 1. For all $f \in H^1_0(-1,1)$, $\int_{-1}^{1} (f')^2 \, dt \geq \frac{(\pi^2)}{4} \int_{-1}^{1} f^2 \, dt$.

2. For all $f \in H^1_0(-1,1)$, $\int_{-1}^{1} (f')^2 \, dt \geq \frac{1}{2} \int_{-1}^{1} f^2 \, dt$.

3. Let $\mu > 0$. Then for all $f \in H^1_0(-1,1)$, $\int_{-1}^{1} (f')^2 \, dt + \mu \int_{-1}^{1} f^2 \, dt \geq S_\mu \int_{-1}^{1} f^2 \, dt$, where $S_\mu$ is the smallest positive solution of $\tan \sqrt{S} = 1 - \frac{S}{2\mu}$ if $\mu < \frac{\pi^2}{2}$, and $S_\mu = \pi^2$ if $\mu \geq \frac{\pi^2}{2}$.

**Proof.** For part 1 we minimize $\int_{-1}^{1} |f'|^2 \, dt$ among $f \in H^1_0(-1,1)$ subject to the constraint $\int_{-1}^{1} f^2 \, dt = 1$. The minimizer exists and is a solution of the eigenvalue problem $-f'' = Sf$ where $S$ is the principal eigenvalue. The solution is $S = (\frac{\pi^2}{2})^2$ and, up to normalization, $f(t) = \cos \frac{\pi t}{2}$. This $S$ is also the best constant in the desired inequality, achieved by $\cos \frac{\pi t}{2}$.

To prove part 2, we minimize $\int_{-1}^{1} (f')^2 \, dt$ among all $f \in H^1_0(-1,1)$ subject to the constraint $\int_{-1}^{1} f^2 \, dt = 1$. The minimizer exits and is a solution of $-f'' = Sf$ for some $S \in \mathbb{R}$. Then the solution is $f(t) = \frac{\sqrt{2}}{2}(1 - t^2)$ and consequently $1 = \int_{-1}^{1} \frac{\sqrt{2}}{2}(1 - t^2) \geq \frac{3\sqrt{2}}{4}$. Hence $S = \frac{\sqrt{2}}{2}$, $f(t) = \frac{\sqrt{2}}{2}(1 - t^2)$, and $\int_{-1}^{1} |f'|^2 \, dt = \frac{3\sqrt{2}}{4}$.

For part 3 we minimize $\int_{-1}^{1} (f')^2 \, dt + \mu \int_{-1}^{1} f^2 \, dt$ among all $f \in H^1_0(-1,1)$ under the constraint $\int_{-1}^{1} f^2 \, dt = 1$. The minimizer exists and satisfies the integro-differential equation $-f'' + \mu f = Sf$. In this eigenvalue problem $S$ is the principal eigenvalue. For $f$ to be a non-trivial solution, $S$ is necessarily positive. This can be seen by multiplying the equation by $f$ and integrating on $(-1,1)$. Let $h = \mu \int_{-1}^{1} f \, dt$. Then $f(t) = c_1 \cos \sqrt{S}t + c_2 \sin \sqrt{S}t + \frac{h}{S}$. Therefore $h = \mu \int_{-1}^{1} f \, dt = \mu(\frac{2 \sin \sqrt{S}}{\sqrt{S}} c_1 + \frac{h}{S})$, which is coupled to the boundary conditions $c_1 \cos \sqrt{S} \pm c_2 \sin \sqrt{S} + \frac{h}{S} = 0$. They form a system of three linear homogeneous equations for $c_1$, $c_2$ and $h$. Its determinant must be 0 for a non-trivial solution $f$ to exist, i.e.

$$\det \left[ \begin{array}{ccc} \frac{2 \sin \sqrt{S}}{\sqrt{S}} & \cos \sqrt{S} & 0 \\ \cos \sqrt{S} & \sin \sqrt{S} & \frac{2}{\sqrt{S}} - \frac{1}{\mu} \\ -\sin \sqrt{S} & -\cos \sqrt{S} & \frac{2}{\sqrt{S}} \end{array} \right] = 0.$$  

There are two possibilities: $\sin \sqrt{S} = 0$ and $\sin \sqrt{S} \neq 0$. In the first case $S = (n\pi)^2$, $n = 1, 2, 3, \ldots$ In the second case $S$ must be a positive solution of the equation $\tan \sqrt{S} = 1 - \frac{S}{2\mu}$. Since only the principal eigenvalue is needed, we just compare the smallest possible $S$ from the first case, which is $\pi^2$, with the smallest positive solution of the equation from the second case. As $\mu$ increases from 0 to $\infty$, the smallest positive solution of the second case increases from $(\frac{\pi^2}{2})^2$ to $x^*_2$ where $x_2 \approx 4.4934$ is the smallest positive solution of $\tan x = x$. Note that $x_2 > \pi$. So which of $\pi^2$ and the smallest solution of the second case is smaller depends on $\mu$. Because $S = \pi^2$ is the smallest positive solution of $\tan \sqrt{S} = 1 - \frac{\sqrt{S}}{2\mu}$ when $\mu = \frac{\sqrt{S}}{2}$, $S_\mu$ is the smallest positive solution of $\tan \sqrt{S} = 1 - \frac{\sqrt{S}}{2\mu}$ if $\mu < \frac{\pi^2}{2}$, and $S_\mu = \pi^2$ if $\mu \geq \frac{\pi^2}{2}$. 

The derivative of the first part of $S$ is studied in the next lemma. The lemma is also valuable for the equal area, two component isoperimetric problem. See the last section for a discussion on this point.

**Lemma 7.2** There exists a universal constant $d > 0$ such that

$$\langle S_L^*(0,0)(\psi, \beta), (\psi, \beta) \rangle \geq \frac{2d}{\rho^2} \| (\psi, \beta) \|_{S}^2$$

for all $(\psi, \beta) \in \mathcal{X}$.

**Proof.** Define some constants:

$$t^{11} = \frac{\partial^2 L_0(0,0,0)}{\partial \phi_i^2}, \quad t^{00} = \frac{\partial^2 L_0(0,0,0)}{\partial \phi_i^2}, \quad t^{1s} = \frac{\partial^2 L_i(0,0,0)}{\partial \phi_i^2}, \quad t^{0s} = \frac{\partial^2 L_i(0,0,0)}{\partial \phi_i^2}, \quad i = 1, 2,$$

$$t^{00} = \frac{\partial^2 L_0(0,0,0)}{\partial \phi_i \partial \phi_i}, \quad t^{1s} = \frac{\partial^2 L_i(0,0,0)}{\partial \phi_i \partial \phi_i}, \quad t^{00} = \frac{\partial^2 L_i(0,0,0)}{\partial \phi_i \partial \phi_i}, \quad i = 1, 2,$$
At this point we impose our first condition on

\[ l_0^{11} = \frac{\partial^2 L_0(0,0)}{\partial (\varphi_0')^2}, \quad l_0^s = \frac{\partial^2 L_0(0,0)}{\partial \alpha^2}, \quad l_0^s = \frac{\partial^2 L_0(0,0)}{\partial \varphi_0' \partial \alpha}. \]  

(7.1)

After some lengthy calculations, we find that

\[ l_0^{11} = \frac{27}{8\pi^2 \rho^3}, \quad l_0^{00} = -\frac{3}{2\pi \rho^3}, \quad l_s^0 = \frac{14\pi \sqrt{3} - 9}{8\pi^3 \rho^3}, \quad l_0^{11} = l_0^{s} = l_0^{00} = 0, \quad l_0^s = \frac{8\pi \sqrt{3} - 9}{4\pi^3 \rho^3}, \quad l_0^s = 0. \]  

(7.2)

The linearized operators of \( \kappa_i \) at \((0,0)\) are

\[ \kappa_i'(0,0) : (\psi, \beta) \rightarrow -l_0^{11} \psi'' + l_0^{00} \psi_i, \quad i = 1, 2, \quad \kappa_0'(0,0) : (\psi_0, \beta) \rightarrow -l_0^{11} \psi_0''. \]  

(7.3)

For \( \kappa_s'(0,0) \) calculations show that the linearized operator is

\[ \kappa_s'(0,0) : (\psi, \beta) \rightarrow \sum_{i=1}^{2} l_0^{11} \psi_i'_{i-1} + (4l_s^{ss} + 2l_0^{ss}) \beta. \]  

(7.4)

Hence

\[ S'_L(0,0) : \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_0 \\ \beta \end{pmatrix} \rightarrow \Pi \begin{pmatrix} -l_0^{11} \psi''_1 + l_0^{00} \psi_1 \\ -l_0^{11} \psi''_2 + l_0^{00} \psi_2 \\ -l_0^{11} \psi''_0 \\ l_0^{11} (\psi_1' + \psi_2')_{i-1} + (4l_s^{ss} + 2l_0^{ss}) \beta \end{pmatrix}. \]  

(7.5)

Define a quadratic form

\[ B(\psi, \beta) = \langle S'_L(0,0)(\psi, \beta), (\psi, \beta) \rangle - 2d \rho^{-3} \| (\psi, \beta) \|_2^2 \]

\[ = (l_0^{11} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_1'')^2 + (\psi_2'')^2 \, dt + (l_0^{00} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_1^2 + \psi_2^2) \, dt \]

\[ + (l_0^{11} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_0'')^2 \, dt - 2d \rho^{-3} \int_{-1}^{1} \psi_0^2 \, dt + (4l_s^{ss} + 2l_0^{ss} - 2d \rho^{-3}) \beta^2. \]  

(7.6)

where \( d \) is a small positive number, independent of \( \rho \), to be specified later. By Lemma 7.1 part 1, we have

\[ B(\psi, \beta) \geq (l_0^{11} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_1'')^2 + (\psi_2'')^2 \, dt + (l_0^{00} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_1^2 + \psi_2^2) \, dt \]

\[ + (l_0^{11} - 2d \rho^{-3} - 2d \left( \frac{2}{\pi} \right)^2 \rho^{-3}) \int_{-1}^{1} (\psi_0'')^2 \, dt + (4l_s^{ss} + 2l_0^{ss} - 2d \rho^{-3}) \beta^2. \]  

(7.7)

At this point we impose our first condition on \( d \): it must be small enough so that

\[ l_0^{11} - 2d \rho^{-3} > 0 \quad \text{and} \quad l_0^{11} - 2d \rho^{-3} - 2d \left( \frac{2}{\pi} \right)^2 \rho^{-3} > 0. \]  

(7.8)

Now we hold \( \psi_1, \psi_2 \) and \( \beta \) fixed and minimize the right side of (7.7) with respect to \( \psi_0 \) subject to the constraints \( \int_{-1}^{1} \psi_1 \, dt + \int_{-1}^{1} \psi_0 \, dt = 0 \) and \( \int_{-1}^{1} \psi_2 \, dt - \int_{-1}^{1} \psi_0 \, dt = 0 \). In other words we minimize the right side of (7.7) among \( \psi_0 \in H_0^1(-1,1) \) with the fixed value of \( \int_{-1}^{1} \psi_0 \, dt \). Then part 2 of Lemma 7.1 implies that

\[ B(\psi, \beta) \geq (l_0^{11} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_1'')^2 + (\psi_2'')^2 \, dt + (l_0^{00} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_1^2 + \psi_2^2) \, dt \]

\[ + \frac{3}{2} (l_0^{11} - 2d \rho^{-3} - 2d \left( \frac{2}{\pi} \right)^2 \rho^{-3}) (\int_{-1}^{1} \psi_0 \, dt)^2 + (4l_s^{ss} + 2l_0^{ss} - 2d \rho^{-3}) \beta^2. \]  

(7.9)

Define two more quadratic forms:

\[ B_1(\psi_1, \beta) = (l_0^{11} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_1')^2 \, dt + (l_0^{00} - 2d \rho^{-3}) \int_{-1}^{1} \psi_1^2 \, dt \]

\[ + \frac{3}{4} (l_0^{11} - 2d \rho^{-3} - 2d \left( \frac{2}{\pi} \right)^2 \rho^{-3}) (\int_{-1}^{1} \psi_1 \, dt)^2 + (4l_s^{ss} + l_0^{ss} - d \rho^{-3}) \beta^2. \]  

(7.10)

\[ B_2(\psi_2, \beta) = (l_0^{11} - 2d \rho^{-3}) \int_{-1}^{1} (\psi_2')^2 \, dt + (l_0^{00} - 2d \rho^{-3}) \int_{-1}^{1} \psi_2^2 \, dt \]

\[ + \frac{3}{4} (l_0^{11} - 2d \rho^{-3} - 2d \left( \frac{2}{\pi} \right)^2 \rho^{-3}) (\int_{-1}^{1} \psi_2 \, dt)^2 + (4l_s^{ss} + l_0^{ss} - d \rho^{-3}) \beta^2. \]  

(7.11)
By the constraints on $\psi_1$, $\psi_2$ and $\psi_0$, we deduce, following (7.9), that
\[ B(\psi, \beta) \geq B_1(\psi_1, \beta) + B_2(\psi_2, \beta). \] (7.12)

Introduce $g_1 \in H^1_0(-1,1)$ so that $\psi_1(t) = g_1(t) + \beta$. Then
\[ B_1(\psi_1, \beta) = (l^{11} - 2d\rho^{-3}) \int_{-1}^{1} (g_1')^2 \, dt + (l^{00} - 2d\rho^{-3}) \int_{-1}^{1} g_1^2 \, dt + \frac{3}{4} \left( \int_{0}^{1} (t^{11})^2 - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right)^2 \int_{-1}^{1} g_1 \, dt^2 \]
\[ + \left[ 2(l^{00} - 2d\rho^{-3}) + 3 \left( l^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right) \right] \beta \int_{-1}^{1} g_1 \, dt \]
\[ + \left[ 2(l^{00} - 2d\rho^{-3}) + 3 \left( l^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right) + 2l^{ss} + l_0^{ss} - d\rho^{-3} \right] \beta^2. \] (7.13)

To apply Lemma 7.1 part 3, we will choose a proper $\mu > 0$ so that
\[ S_\mu = \frac{-l^{00} + 2d\rho^{-3}}{l^{11} - 2d\rho^{-3}}. \] (7.14)

Note that if $d$ were 0 then $S_\mu$ would be $\frac{l^{00}}{l^{11}} = \frac{4\pi^2}{9}$. Since $\frac{4\pi^2}{9} \in (\frac{1}{2}, \pi^2)$, we can make $d$ small so that
\[ S_\mu \in \left( \frac{1}{2}, \pi^2 \right). \] (7.15)

According to Lemma 7.1 part 3, $S_\mu$ has to be the smallest positive solution of $\frac{\tan(\sqrt{S})}{\sqrt{S}} = 1 - \frac{S}{2d}$. Therefore by taking
\[ \mu = \frac{S_\mu}{2 \left( 1 - \frac{S_\mu}{2d} \right)} \] (7.16)
we achieve (7.14). With this choice of $\mu$ the first three terms in (7.13) give
\[ (l^{11} - 2d\rho^{-3}) \int_{-1}^{1} (g_1')^2 \, dt + (l^{00} - 2d\rho^{-3}) \int_{-1}^{1} g_1^2 \, dt + \frac{3}{4} \left( l^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right)^2 \int_{-1}^{1} g_1 \, dt^2 \]
\[ \geq \frac{3}{4} \left( l^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right) \int_{-1}^{1} g_1 \, dt^2 \] (7.17)
and consequently (7.13) becomes
\[ B_1(\psi_1, \beta) \geq \frac{3}{4} \left( l^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right) \int_{-1}^{1} g_1 \, dt^2 \]
\[ + \left[ 2(l^{00} - 2d\rho^{-3}) + 3 \left( l^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right) \right] \beta \int_{-1}^{1} g_1 \, dt \]
\[ + \left[ 2(l^{00} - 2d\rho^{-3}) + 3 \left( l^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} \right) + 2l^{ss} + l_0^{ss} - d\rho^{-3} \right] \beta^2. \] (7.18)

To check the sign of the coefficient of $(\int_{-1}^{1} g_1 \, dt)^2$ on the right side of (7.18), replace $d$ by 0 in $S_\mu$ of (7.14) so that $S_\mu$ becomes $\frac{l^{00}}{l^{11}} = \frac{4\pi^2}{9}$, $\mu$ of (7.16) becomes
\[ \frac{4\pi^2}{9 \left( 1 - \frac{4\pi^2}{9} \right)}, \] (7.19)
and the coefficient of $(\int_{-1}^{1} g_1 \, dt)^2$ becomes
\[ \frac{3}{4} l^{11} - \frac{4\pi^2}{9 \left( 1 - \frac{4\pi^2}{9} \right)} l^{11} = (1.0240...) \rho^{-3}. \] (7.20)
We add another condition on $d$: it must be sufficiently small so that the coefficient of $(\int_{-1}^{1} g_1\,dt)^2$ stays positive, i.e.

$$\frac{3}{4}\left( l^{11}_0 - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3}\right) - \mu(l^{11}_1 - 2d\rho^{-3}) > 0. \quad (7.21)$$

Completing the square of the right side of (7.18) we obtain

$$B_1(\psi_1, \beta) \geq \left[- \frac{(2l_0^{10} - 3l_{11}^{0})^2}{4\left(\frac{3}{4} l_0^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3}\right)} - \mu(l^{11}_1 - 2d\rho^{-3})\right] \beta^2$$

To check the sign of the quantity in the brackets, we again set $d$ to be 0 and replace $\mu$ by (7.19). Then this quantity becomes

$$- \frac{(2l_0^{10} + 3l_{11}^{0})^2}{4\left(\frac{3}{4} l_0^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3}\right)} + 2l_0^{10} + 2l_{11}^{0} + l_{11}^{ss} = (0.6499...)\rho^{-3}. \quad (7.23)$$

Therefore we choose $d$ so small that

$$- \frac{(2l_0^{10} - 3l_{11}^{0})^2}{4\left(\frac{3}{4} l_0^{11} - 2d\rho^{-3} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3}\right)} - \mu(l^{11}_1 - 2d\rho^{-3}) + 2l_0^{10} + 3l_{11}^{0} - 2d\left(\frac{2}{\pi}\right)^2 \rho^{-3} + 2l_{11}^{ss} + l_{11}^{ss} - d\rho^{-3} > 0 \quad (7.24)$$
in addition to (7.8), (7.15) and (7.21).

In summary we have proved that when $d$ is sufficiently small $B_1(\psi_1, \beta) \geq 0$. Similarly one can show that $B_2(\psi_2, \beta) \geq 0$. Then by (7.12) we obtain that $B(\psi, \beta) \geq 0$ from which the lemma follows. \qed

**Lemma 7.3** There exists $\bar{C} > 0$ depending on $D$ only such that

$$\|\mathcal{S}|_{N}(0, 0)(\psi, \beta)||_z \leq \bar{C}||\gamma||_{(\psi, \beta)}||_z$$

for all $(\psi, \beta) \in \mathcal{X}$.

*Proof.* Recall that for $r_1$, $r_2$ and $r_0$ parametrizing the boundaries of the perturbed double bubble $\Omega$ as in (5.1) with $(\phi, \alpha) \in \mathcal{X}$ being its internal variables, the terms $I_{01}$ and $I_{02}$ in the first, second, or third component of (6.32) (corresponding to $j = 1, 2, 0$) are the outcomes of the operators $I_{ij}$ given in (6.33).

To compute the Fréchet derivatives of $I_{ij}$, deform $(\phi, \alpha)$ to $(\phi, \alpha) + \epsilon(\psi, \beta)$ and denote the corresponding deformation of $r_1$, $r_2$ and $r_0$ by $r'_1$, $r'_2$ and $r'_0$ respectively. Then for $i = 1, 2$ and $j = 1, 2, 0$,

$$I'_{ij}(\phi, \alpha) : (\psi, \beta) \to \frac{\partial}{\partial \epsilon}|_{\epsilon = 0} \int_{\Omega_i^j} G(e^{\imath \theta} r_j(t) + \xi, e^{\imath \theta} y + \xi)\, dy + \frac{\partial}{\partial \epsilon}|_{\epsilon = 0} \int_{\Omega_i} G(e^{\imath \theta} r_j(t) + \xi, e^{\imath \theta} y + \xi)\, dy. \quad (7.25)$$

Apply Lemma 4.2 to the first term on the left side of (7.25) with $\Omega = B$ whose boundaries are parametrized by $r$ to obtain

$$\frac{\partial}{\partial \epsilon}|_{\epsilon = 0} \int_{\Omega_i} G(e^{\imath \theta} r_j(t) + \xi, e^{\imath \theta} y + \xi)\, dy$$

$$= \left\{ - \int_{\partial B_1 \setminus \partial B_2} G(e^{\imath \theta} r_j(t) + \xi, e^{\imath \theta} r_1(\tau) + \xi) N_1 \cdot X \, ds(t) - \int_{\partial B_1 \cap \partial B_2} G(e^{\imath \theta} r_j(t) + \xi, e^{\imath \theta} r_0(\tau) + \xi) N_0 \cdot X \, ds(t) \right\} \quad (7.26)$$
where the first line holds if \( i = 1 \) and the second holds if \( i = 2 \). We calculated earlier that \(-\mathbf{N}_i \cdot \mathbf{x} ds(\tau) = (\psi_1 + e_1(\phi_1, \alpha)\beta) d\tau\). Since \( e_1(0, 0) = 0 \) at \((\phi, \alpha)\) by (6.13),

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \int_{\Omega_1} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} y + \xi) dy
\]

\[
= \begin{cases} \int_{\partial B_1 \setminus \partial B_2} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} \mathbf{r}_1(\tau) + \xi) \psi_1(\tau) d\tau + \int_{\partial B_1 \cap \partial B_2} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} \mathbf{r}_0(\tau) + \xi) \psi_0(\tau) d\tau \\
\int_{\partial B_2 \setminus \partial B_1} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} \mathbf{r}_2(\tau) + \xi) \psi_2(\tau) d\tau - \int_{\partial B_1 \cap \partial B_2} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} \mathbf{r}_0(\tau) + \xi) \psi_0(\tau) d\tau \end{cases}
\]

(7.27)

To estimate the right side of (7.27) we write \( G \) as the sum of the fundamental solution and the regular part, and treat the two parts separately. First

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\pi} \log \frac{1}{|\mathbf{r}_j(t) - \mathbf{r}_i(\tau)|} \psi_1(\tau) d\tau = \frac{1}{2\pi} \log \frac{1}{\rho} \int_{-1}^{1} \psi_1(\tau) d\tau + \frac{1}{2\pi} \log \frac{1}{\rho} \int_{-1}^{1} \psi_0(\tau) d\tau + O(1) \frac{1}{\|\psi_1\|_{L^2}}
\]

(7.28)

holds uniformly with respect to \( t \). Next for the regular part it suffices to note

\[
\int_{-1}^{1} R(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} \mathbf{r}_i(\tau) + \xi) \psi_1(\tau) d\tau = O(1) \frac{1}{\|\psi_1\|_{L^2}}.
\]

(7.29)

By (7.28) and (7.29) we deduce that

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \int_{\Omega_1} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} y + \xi) dy
\]

\[
= \begin{cases} \frac{1}{2\pi} \log \frac{1}{\rho} \int_{-1}^{1} \psi_1(\tau) d\tau + \frac{1}{2\pi} \log \frac{1}{\rho} \int_{-1}^{1} \psi_0(\tau) d\tau + O(1) \frac{1}{\|\psi_1\|_{L^2} + \|\psi_0\|_{L^2}} \\
\frac{1}{2\pi} \log \frac{1}{\rho} \int_{-1}^{1} \psi_2(\tau) d\tau - \frac{1}{2\pi} \log \frac{1}{\rho} \int_{-1}^{1} \psi_0(\tau) d\tau + O(1) \frac{1}{\|\psi_2\|_{L^2} + \|\psi_0\|_{L^2}} \end{cases}
\]

(7.30)

holds uniformly with respect to \( t \).

The second part on the right side of (7.25), for \((\phi, \alpha) = (0, 0)\), is written as

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \int_{B_i} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} y + \xi) dy = \int_{B_i} \nabla G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} y + \xi) \cdot e^{i\theta} \mathbf{X}_j(t) dy
\]

(7.32)

where \( \nabla G \) stands for the gradient of \( G \) with respect to its first argument, and \( \mathbf{X}_j(t) = \frac{\partial \mathbf{r}_j(t)}{\partial \epsilon} \bigg|_{\epsilon=0} \). Clearly

\[
\int_{B_i} |\nabla G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} y + \xi)| dy = O(\rho)
\]

(7.33)

holds uniformly with respect to \( t \). Calculations show that

\[
\mathbf{X}_j(t) = \begin{cases} \frac{3\psi_1}{2\pi\rho} e^{i(\tau - \frac{2\pi}{3} t)} + \left( -\frac{\sqrt{3}t\beta}{2\pi\rho} \right) e^{i\left(\tau - \frac{2\pi}{3} t\right)} (-i), & j = 1 \\
\frac{3\psi_2}{2\pi\rho} e^{i\frac{2\pi}{3} t} + \left( -\frac{\sqrt{3}t\beta}{2\pi\rho} \right) e^{i\frac{2\pi}{3} t} i, & j = 2 \\
\frac{2\psi_0}{\sqrt{3}\rho} - \frac{2\beta}{\beta\rho} \frac{\sqrt{3}t\beta}{\pi\rho}, & j = 0 \end{cases}
\]

(7.34)
Hence
\[ \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} \int_{B_i} G(e^{i\theta} \mathbf{r}_j(t) + \xi, e^{i\theta} \mathbf{y} + \xi) \, dy = O(1)(\|\psi_j\|_{L^2} + |\beta|) \]
holds uniformly with respect to $t$.

By (7.31) and (7.35) we find that
\[ \| (I_{ij}^*(0,0))(\psi, \beta) \|_{L^2} = O(1)\|\psi, \beta\|_{Z} \]
for the $I_{ij}$ terms in the first three components in (6.32) of $\mathcal{S}_N(\phi, \alpha)$.

Finally consider $h(\phi, \alpha)$, the last component of $\mathcal{S}_N(\phi, \alpha)$. Since $e_i(0,0) = 0$ for $i = 1, 2, 0$,
\[ h'(0,0)(\psi, \beta) = \int_{-1}^{1} (\gamma_{11} I_{B_1} + \gamma_{12} I_{B_2}) e_i'(0,0)(\psi_1, \beta) \, dt + \int_{-1}^{1} (\gamma_{21} I_{B_1} + \gamma_{22} I_{B_2}) e_i'(0,0)(\psi_2, \beta) \, dt 
+ \int_{-1}^{1} ((\gamma_{11} - \gamma_{12}) I_{B_1} + (\gamma_{21} - \gamma_{22}) I_{B_2}) e_i'(0,0)(\psi_0, \beta) \, dt. \]

Note that (6.12) implies that
\[ \int_{-1}^{1} e_i'(0,0)(\psi, \beta) = 0. \]

By (6.9) we can write
\[ e_i'(0,0)(\psi, \beta) = p_i(\psi_1 + t\psi_1) + q_i \beta, \quad \frac{\partial(p_i t\psi_1 + q_i t\beta)}{\partial t} \]
where $p_i$ and $q_i$ are constants that depend on $\rho$ only. Then (7.38) implies that
\[ p_i t\psi_1 + q_i t\beta \bigg|_{-1}^{1} = 0. \]

Calculations from (6.9) show that
\[ p_i = O(\rho^{-2}) \quad \text{and} \quad q_i = O(\rho^{-2}). \]

Consider a general term in (7.37):
\[ \int_{-1}^{1} I_{B_1} e_i'(0,0)(\psi_j, \beta) \, dt = \int_{-1}^{1} I_{B_1} (e^{i\theta} \mathbf{r}_j(t) + \xi) \frac{\partial(p_j t\psi_j + q_j t\beta)}{\partial t} \, dt 
- \int_{-1}^{1} \frac{\partial I_{B_1}(\mathbf{r}_j(t))}{\partial t} (p_j t\psi_j + q_j t\beta) \, dt. \]

The two terms on the last line are estimated below.

Since
\[ I_{B_1}(e^{i\theta} \mathbf{r}_j(t) + \xi) = \frac{m \rho^2}{4\pi} \log \frac{1}{\rho} + O(\rho^2) \]
holds uniformly with respect to $t$, by (7.40) and (7.41)
\[ I_{B_1}(\mathbf{r}_j(t))(p_j t\psi_j + q_j t\beta) \bigg|_{-1}^{1} = O(1)|\beta|. \]

For the second term note that
\[ \frac{\partial I_{B_1}(e^{i\theta} \mathbf{r}_j(t) + \xi)}{\partial t} = \int_{B_i} \nabla G(e^{i\theta} \mathbf{r}_j(t) + \xi, y) \cdot e^{i\theta} \mathbf{r}'_j(t) \, dy = O(\rho^2) \]
holds uniformly with respect to $t$. Then
\[ \int_{-1}^{1} \frac{\partial I_{B_1}(e^{i\theta} \mathbf{r}_j(t) + \xi)}{\partial t} (p_j t\psi_j + q_j t\beta) \, dt = O(1)(\|\psi_j\|_{L^2} + |\beta|). \]

By (7.42), (7.44) and (7.46) we deduce that
\[ \int_{-1}^{1} I_{B_1} e_i'(0,0)(\psi_j, \beta) \, dt = O(1)(\|\psi_j\|_{L^2} + |\beta|). \]
There exist $\psi, \beta$ for all $(\psi, \beta)$ in $\mathcal{X}$. Hence one estimates (7.37) and deduces that

$$\|\mathcal{S}'(0,0)(\psi, \beta)\|_z \leq \tilde{C}|\gamma| \|\psi, \beta\|_z$$

(7.49)

for all $(\psi, \beta) \in \mathcal{X}$. □

While $\mathcal{S}'(0,0)$ is an unbounded self-adjoint operator on $\mathcal{Z}$ with a dense domain $\mathcal{X} \subset \mathcal{Z}$, Lemma 7.3 shows that $\mathcal{S}'(0,0)$ may be extended to a bounded self-adjoint operator on $\mathcal{Z}$.

**Lemma 7.4** There exist $d > 0$ and $\sigma > 0$ such that when $|\gamma| \rho^3 < \sigma$,

$$\langle \mathcal{S}'(0,0)(\psi, \beta), (\psi, \beta) \rangle \geq \frac{d}{\rho^3} \|\psi, \beta\|_z^2$$

for all $(\psi, \beta) \in \mathcal{X}$.

**Proof.** Let $d$ be the positive number given in Lemma 7.2 and $\sigma = \frac{d}{C}$ where $\tilde{C}$ comes from Lemma 7.3. Then Lemma 7.3 shows that for $|\gamma| \rho^3 < \sigma$,

$$\|\mathcal{S}'(0,0)(\psi, \beta)\|_z \leq \tilde{C}|\gamma| \|\psi, \beta\|_z \leq \frac{\tilde{C}\sigma}{\rho^3} \|\psi, \beta\|_z = \frac{d}{\rho^3} \|\psi, \beta\|_z$$

(7.50)

for all $(\psi, \beta) \in \mathcal{X}$. By Lemma 7.2 and (7.50)

$$\langle \mathcal{S}'(0,0)(\psi, \beta), (\psi, \beta) \rangle = \langle \mathcal{S}'(0,0)(\psi, \beta), (\psi, \beta) \rangle + \langle \mathcal{S}'_N(0,0)(\psi, \beta), (\psi, \beta) \rangle$$

$$\geq \frac{2d}{\rho^3} \|\psi, \beta\|_z^2 - \frac{d}{\rho^3} \|\psi, \beta\|_z^2 \geq \frac{d}{\rho^3} \|\psi, \beta\|_z^2$$

for all $(\psi, \beta) \in \mathcal{X}$. □

A consequence of the positivity of $\mathcal{S}'(0,0)$ is its invertibility.

**Lemma 7.5**

1. There exists $\bar{d} > 0$ such that when $|\gamma| \rho^3 < \sigma$ where $\sigma$ is given in Lemma 7.4, $\|\mathcal{S}'(0,0)(\psi, \beta)\|_z \geq \frac{\bar{d}}{\rho^3} \|\psi, \beta\|_z$ holds for all $(\psi, \beta) \in \mathcal{X}$.

2. The linear map $\mathcal{S}'(0,0)$ is one-to-one and onto from $\mathcal{X}$ to $\mathcal{Z}$, and the operator norm of $(\mathcal{S}'(0,0))^{-1}$ satisfies $\|\mathcal{S}'(0,0))^{-1}\| \leq \frac{\sigma^2}{\rho^3}$.

**Proof.** By Lemma 7.4 it is easy to see that if $|\gamma| \rho^3 < \gamma$, then for all $(\psi, \beta) \in \mathcal{X}$

$$\|\psi, \beta\|_z \leq \frac{\rho^3}{d} \|\mathcal{S}'(0,0)(\psi, \beta)\|_z$$

(7.51)

The first part of Lemma 7.5 asserts that the $\mathcal{Z}$-norm of $(\psi, \beta)$ on the left side of (7.51) can be strengthened to the stronger $\mathcal{X}$-norm, if $d$ is replaced by a possibly smaller $\bar{d}$.

If part 1 is false, then there exist $\gamma_0$, $\rho_0$, and $(\psi_0, \beta_0) \in \mathcal{X}$ such that $|\gamma_0| \rho_0^3 < \sigma$, $\|\psi_0, \beta_0\|_X = 1$ and with $\rho = \rho_0$ and $\gamma = \gamma_0$ in $\mathcal{S}$,

$$\|\rho^3 \mathcal{S}'(0,0)(\psi_0, \beta_0)\|_z \to 0, \text{ as } n \to \infty.$$  

(7.52)

By (7.51),

$$\|\psi_0, \beta_0\|_z \to 0.$$  

(7.53)

Moreover due to the compactness of the embedding $H^2(-1,1) \to C^1[-1,1]$ and $\|\psi_0, \beta_0\|_X = 1$, $\|\psi_{n,i}\|_{C^1} \to 0$ and in particular

$$\psi_{n,i}(\pm 1) = 0, \text{ } i = 1, 2, 0.$$  

(7.54)

Since $\mathcal{S}'(0,0) = \mathcal{S}'_L(0,0) + \mathcal{S}'_N(0,0)$, and (7.50) and (7.53) imply that

$$\|\rho^3 \mathcal{S}'_N(0,0)(\psi_0, \beta_0)\|_z \to 0,$$  

(7.55)
we derive from (7.52) and (7.55) that
\[ \| \rho_n^3 S_L'(0, 0)(\psi_n, \beta_n) \|_Z \to 0. \] (7.56)
By (7.5) write
\[ \rho_n^3 S_L'(0, 0)(\psi_n, \beta_n) = \Pi \left( \begin{array}{c} -\rho_n^3 t_{11} \psi_{n,1}' \\ -\rho_n^3 t_{11} \psi_{n,2}' \\ -\rho_n^3 \psi_{n,0}'_0 \\ 0 \end{array} \right) + \Pi \left( \begin{array}{c} 0 \\ \rho_n^3 F_{00} \psi_{n,1} \\ \rho_n^3 F_{00} \psi_{n,2} \\ \rho_n^3 t_{11}(\psi_{n,1}'_1 + \psi_{n,2}'_1) + \rho_n^3 (4l_{ss} + 2l_{00}) \beta_n \end{array} \right). \] (7.57)
Here \( \rho_n^3 t_{11}, \rho_n^3 F_{00}, \rho_n^3 t_{ss}, \rho_n^3 t_{11} \), and \( \rho_n^3 t_{00} \) are all constants independent of \( \rho_n \). By (7.53) and (7.54) we find that
\[ \| \Pi \left( \begin{array}{c} -\rho_n^3 t_{11} \psi_{n,1}' \\ -\rho_n^3 t_{11} \psi_{n,2}' \\ -\rho_n^3 \psi_{n,0}'_0 \\ 0 \end{array} \right) \|_Z \to 0. \] (7.58)
Then (7.56), (7.57) and (7.58) give that
\[ \| \Pi \left( \begin{array}{c} -\rho_n^3 t_{11} \psi_{n,1}' \\ -\rho_n^3 t_{11} \psi_{n,2}' \\ -\rho_n^3 \psi_{n,0}'_0 \\ 0 \end{array} \right) \|_Z \to 0. \] (7.59)
By the definition of \( \Pi \), (6.21),
\[ \Pi \left( \begin{array}{c} -\rho_n^3 t_{11} \psi_{n,1}' \\ -\rho_n^3 t_{11} \psi_{n,2}' \\ -\rho_n^3 \psi_{n,0}'_0 \\ 0 \end{array} \right) = \left( \begin{array}{c} -\rho_n^3 t_{11} \psi_{n,1}' \\ -\rho_n^3 t_{11} \psi_{n,2}' \\ -\rho_n^3 \psi_{n,0}'_0 \\ 0 \end{array} \right) + \left( \begin{array}{c} \rho_n^3 t_{11} \psi_{n,1}' \frac{1}{1} + \rho_n^3 t_{11} \psi_{n,2}' \frac{1}{1} + \rho_n^3 \psi_{n,0}'_0 \frac{1}{1} \\ \rho_n^3 t_{11} \psi_{n,1}' \frac{1}{1} + \rho_n^3 t_{11} \psi_{n,2}' \frac{1}{1} - \rho_n^3 \psi_{n,0}'_0 \frac{1}{1} \\ \rho_n^3 \psi_{n,1}' \frac{1}{1} - \rho_n^3 \psi_{n,2}' \frac{1}{1} + \rho_n^3 \psi_{n,0}'_0 \frac{1}{1} \\ 0 \end{array} \right). \] (7.60)
Moreover (7.54) implies that
\[ \left( \begin{array}{c} \rho_n^3 t_{11} \psi_{n,1}' \frac{1}{1} + \rho_n^3 t_{11} \psi_{n,2}' \frac{1}{1} + \rho_n^3 \psi_{n,0}'_0 \frac{1}{1} \\ \rho_n^3 t_{11} \psi_{n,1}' \frac{1}{1} + \rho_n^3 t_{11} \psi_{n,2}' \frac{1}{1} - \rho_n^3 \psi_{n,0}'_0 \frac{1}{1} \\ \rho_n^3 \psi_{n,1}' \frac{1}{1} - \rho_n^3 \psi_{n,2}' \frac{1}{1} + \rho_n^3 \psi_{n,0}'_0 \frac{1}{1} \\ 0 \end{array} \right) \rightarrow \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \in \mathbb{R}^4. \] (7.61)
Therefore by (7.59), (7.60) and (7.61),
\[ \| \psi_{n,i}' \|_{L^2} \to 0, \quad i = 1, 2, 0. \] (7.62)
From (7.53) and (7.62) we deduce that \( \|(\psi_n, \beta_n)\|_X \to 0 \), a contradiction to our assumption at the beginning that \( \|(\psi_n, \beta_n)\|_X = 1 \).

For part 2, it suffices to show that \( S'(0, 0) \) is onto. First note that by the standard theory of second order linear differential equations, \( S'(0, 0) \) is an unbounded self-adjoint operator on \( Z \) with the domain \( X \subset Z \). Second if \( (\tilde{\psi}, \tilde{\beta}) \in Z \) is perpendicular to the range of \( S'(0, 0) \), i.e., \( \langle S'(0, 0)(\psi, \beta), (\tilde{\psi}, \tilde{\beta}) \rangle = 0 \) for all \( (\psi, \beta) \in X \), then the self-adjointness of \( S'(0, 0) \) implies that \( (\tilde{\psi}, \tilde{\beta}) \in \bar{X} \) and \( S'(0, 0)(\tilde{\psi}, \tilde{\beta}) = 0 \). By (7.51), \( (\tilde{\psi}, \tilde{\beta}) \) is zero. Hence the range of \( S'(0, 0) \) is dense in \( Z \). Finally (7.51) implies that the range of \( S'(0, 0) \) is a closed subset of \( Z \). Therefore \( S'(0, 0) \) is onto.

## 8 Fixed point

We now solve the equation
\[ S(\phi, \alpha) = (0, 0) \] (8.1)
for \( (\phi, \alpha) \) near \( (0, 0) \) in each restricted class of perturbed double bubbles associated with the \((\xi, \theta)\)-frame. We first need an estimate on the second derivative of \( S \).
Lemma 8.1 There exists $\hat{C} > 0$ such that for all $(\phi, \alpha) \in \mathcal{D}(S)$,

$$\|S''((\psi, \beta), (\tilde{\psi}, \tilde{\beta}))\|_Z \leq \hat{C}(\rho^{-5} + |\gamma|\rho^{-2})\| (\psi, \beta)\|_X \|(\tilde{\psi}, \tilde{\beta})\|_X$$

holds for all $(\psi, \beta), (\tilde{\psi}, \tilde{\beta}) \in X$.

We skip the proof of this lemma, which is straight estimation as in the proof of [29, Lemma 3.2].

Lemma 8.2 There exists $\sigma > 0$ such that (8.1) admits a solution $(\phi^*, \alpha^*)$ satisfying $\|(\phi^*, \alpha^*)\|_X \leq \frac{2\hat{C}|\gamma|\rho^3}{d}$, provided $|\gamma|\rho^3 < \sigma$.

Proof. For $(\phi, \alpha) \in \mathcal{D}(S)$ write

$$S(\phi, \alpha) = S(0, 0) + S'(0, 0)(\phi, \alpha) + R(\phi, \alpha) \quad (8.2)$$

where $R(\phi, \alpha)$ is a higher order term defined by (8.2). Define an operator $T$ from $\mathcal{D}(S) \subset X$ into $X$ by

$$T(\phi, \alpha) = -(S'(0, 0))^{-1}(S(0, 0) + R(\phi, \alpha)) \quad (8.3)$$

and re-write the equation $S(\phi, \alpha) = 0$ as a fixed point problem $T(\phi, \alpha) = (\phi, \alpha)$.

Let $c \in (0, \bar{c})$, where $\bar{c}$ is given in (6.24), and define a closed ball $W = \{(\phi, \alpha) \in X : \|(\phi, \alpha)\|_X \leq c\rho^2\} \subset \mathcal{D}(S)$. For $(\phi, \alpha) \in W$,

$$\|R(\phi, \alpha)\|_Z \leq \frac{1}{2} \sup_{\tau \in (0, 1)} \|S''((\phi, \alpha), (\phi, \alpha))\|_Z \leq \frac{\hat{C}(\rho^{-5} + |\gamma|\rho^{-2})}{2} \|(\phi, \alpha)\|_X^2 \quad (8.4)$$

by Lemma 8.1. Then by Lemmas 6.2 and 7.5

$$\|T(\phi, \alpha)\|_X \leq \|(S'(0, 0))^{-1}\|(\|S(0, 0)\|_Z + \|R(\phi, \alpha)\|_Z) \leq \frac{\rho^3}{d} \left( \hat{C}|\gamma|\rho^2 + \frac{\hat{C}(\rho^{-5} + |\gamma|\rho^{-2})(c\rho^2)^2}{2} \right) \leq \frac{\hat{C}\sigma}{d} \rho^2 + \frac{\hat{C} + \hat{C}}{2d} c^2 \rho^2. \quad (8.5)$$

Let $(\psi, \beta) \in W$. Then

$$\|T(\phi, \alpha) - T(\psi, \beta)\|_X \leq \|(S'(0, 0))^{-1}\| \|R(\phi, \alpha) - R(\psi, \beta)\|_Z \leq \frac{\rho^3}{d} \|S(\phi, \alpha) - S(\psi, \beta) - S'(0, 0)((\phi, \alpha) - (\psi, \beta))\|_Z \leq \frac{\rho^3}{d} \|S(\phi, \alpha) - S(\psi, \beta) - S'(\psi, \beta)((\phi, \alpha) - (\psi, \beta))\|_Z + \frac{\rho^3}{d} \|(S'(\psi, \beta) - S'(0, 0))((\phi, \alpha) - (\psi, \beta))\|_Z \leq \frac{\rho^3}{2d} \sup_{\tau \in (0, 1)} \|S''((\psi, \beta) + \tau((\phi, \alpha) - (\psi, \beta)))\| \|\phi, \alpha) - (\psi, \beta)\|_X \| \|\psi, \beta\|_X \right)\|_X \| \|\phi, \alpha) - (\psi, \beta)\|_X \leq \frac{\rho^3\hat{C}(\rho^{-5} + |\gamma|\rho^{-2})}{d} \left( \frac{1}{2} c\rho^2 + c\rho^2 \right) \|\phi, \alpha) - (\psi, \beta)\|_X \leq \frac{3\hat{C}(1 + \sigma)c}{2d} \|\phi, \alpha) - (\psi, \beta)\|_X. \quad (8.6)$$

Take

$$c = \min \left\{ \frac{d}{6\hat{C}^2}, \bar{c} \right\}. \quad (8.7)$$
Let $\sigma$ be small enough so that Lemma 7.5 holds, and moreover

$$\sigma \leq \min \left\{ 1, \frac{\hat{d}c}{2C} \right\}. \quad (8.8)$$

It follows from (8.5) and (8.6) that

$$\|T(\phi, \alpha)\|_x \leq c\rho^2 \quad \text{and} \quad \|T(\phi, \alpha) - T(\psi, \beta)\|_x \leq \frac{1}{2}\|\phi, \alpha - (\psi, \beta)\|_x \quad (8.9)$$

for all $(\phi, \alpha), (\psi, \beta) \in \mathcal{W}$. The contraction mapping theorem shows that $T$ has a fixed point in $\mathcal{W}$. This fixed point is denoted by $(\phi^*, \alpha^*)$.

To prove the estimate of $(\phi^*, \alpha^*)$, revisit the equation $(\phi, \alpha) = T(\phi, \alpha)$, satisfied by $(\phi^*, \alpha^*)$, and derive from (8.4) that

$$\| (\phi^*, \alpha^*) \|_x \leq \| (S'(0, 0))^{-1}\|\|S(0, 0)\|_x + \|R(\phi^*, \alpha^*)\|_x$$

$$\leq \frac{\rho^3}{d} \left( \tilde{C}|\gamma|\rho^2 + \tilde{C}(\rho^{-5} + |\gamma|\rho^{-2})\| (\phi^*, \alpha^*)\|_x^2 \right).$$

Rewrite the above as

$$\left(1 - \frac{\tilde{C}(\rho^{-2} + |\gamma|\rho)}{2d}\| (\phi^*, \alpha^*)\|_x \right)\| (\phi^*, \alpha^*)\|_x \leq \frac{\tilde{C}|\gamma|\rho^{5}}{d}. \quad (8.10)$$

In (8.10) estimate

$$\frac{\tilde{C}(\rho^{-2} + |\gamma|\rho)}{2d}\| (\phi^*, \alpha^*)\|_x \leq \frac{\tilde{C}(\rho^{-2} + |\gamma|\rho)(cp^2)}{2d} \leq \frac{\tilde{C}(1 + \sigma)}{2d} \leq \frac{1}{6} \quad (8.11)$$

by (8.7) and (8.8). The estimate of $(\phi^*, \alpha^*)$ follows from (8.10). □

**Lemma 8.3** The solution $(\phi^*, \alpha^*)$ from Lemma 8.2 solves the following equations:

$$\begin{align*}
\kappa_1 + \gamma_{11}I_{\Omega_1} + \gamma_{12}I_{\Omega_2} &= \lambda_1 \quad \text{on} \quad \partial \Omega_1 \setminus \partial \Omega_2 \\
\kappa_2 + \gamma_{21}I_{\Omega_1} + \gamma_{22}I_{\Omega_2} &= \lambda_2 \quad \text{on} \quad \partial \Omega_2 \setminus \partial \Omega_1 \\
\kappa_0 + (\gamma_{11} - \gamma_{12})I_{\Omega_1} + (\gamma_{21} - \gamma_{22})I_{\Omega_2} &= \lambda_1 - \lambda_2 \quad \text{on} \quad \partial \Omega_1 \cap \partial \Omega_2 \\
(T_1 + T_2 + T_0)(t) \cdot (0, t)|_{t=1} &= 0.
\end{align*}$$

**Proof.** The first three equations of the lemma clearly follow from (6.21), (6.36) and the first three equations of $S(\phi, \alpha) = (0, 0)$. The fourth equation of $S(\phi, \alpha) = (0, 0)$ states that

$$\kappa_\alpha(\phi, \alpha) + h(\phi, \alpha) = 0. \quad (8.12)$$

The terms in (6.29) that make up $\kappa_\alpha$ are simplified as follows. For $i = 0$,

$$\int_{-1}^{1} \frac{\partial L_0(\phi_0^*, \alpha)}{\partial \alpha} dt = \int_{-1}^{1} \left[ \frac{\phi_0'}{\sqrt{\left(\phi_0''\right)^2 + a^2}} - \frac{\phi_0'}{a^3A} + \frac{a}{\sqrt{\left(\phi_0''\right)^2 + a^2}} \left( \frac{1}{aA} \right) \right] dt$$

$$= \int_{-1}^{1} \left[ \frac{\phi_0'}{\sqrt{\left(\phi_0''\right)^2 + a^2}} - \frac{\phi_0'}{a^3A} + \frac{a}{\sqrt{\left(\phi_0''\right)^2 + a^2}} \left( \frac{t}{aA} \right) \right] dt - 1$$

$$= \int_{-1}^{1} \frac{\partial}{\partial t} \left( \frac{\phi_0'}{\sqrt{\left(\phi_0''\right)^2 + a^2}} \right) \left( - \frac{\phi_0'}{a^3A} + \frac{a}{\sqrt{\left(\phi_0''\right)^2 + a^2}} \left( \frac{t}{aA} \right) \right) dt$$

$$= T_0 \cdot \left( 0, \frac{t}{aA} \right) \bigg|_{-1}^{1} + \int_{-1}^{1} \kappa_0(\phi_0, \alpha) \epsilon_0(\phi_0, \alpha) dt; \quad (8.13)$$

similarly for $i = 1, 2,$

$$\left. \frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \phi_i'} \right|_{-1}^{1} + \int_{-1}^{1} \frac{\partial L_i(\phi_i', \phi_i, \alpha)}{\partial \alpha} dt = T_i \cdot \left( 0, \frac{t}{aA} \right) \bigg|_{-1}^{1} + \int_{-1}^{1} \kappa_i(\phi_i, \alpha) \epsilon_i(\phi_i, \alpha) dt. \quad (8.14)$$

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They turn (8.12) to
\[ \int_{-1}^{1} \left\{ (\kappa_1 + \gamma_1 I_{01} + \gamma_2 I_{02})e_1 + (\kappa_2 + \gamma_2 I_{01} + \gamma_2 I_{02})e_2 + [\kappa_0 + (\gamma_1 - \gamma_2) I_{01} + (\gamma_2 - \gamma_2) I_{02}] e_0 \right\} dt \]
\[ + T_1 \cdot \left( 0, \frac{t}{aA} \right) \bigg|_{-1}^{1} + T_2 \cdot \left( 0, \frac{t}{aA} \right) \bigg|_{-1}^{1} + T_0 \cdot \left( 0, \frac{t}{aA} \right) \bigg|_{-1}^{1} = 0. \quad (8.15) \]

The first three equations in the lemma show that the integral term in (8.15) vanishes. Hence the last equation of the lemma holds.

The first three equations satisfied by \((\phi^*, \alpha^*)\) in the lemma are just the equations (2.3)-(2.5). However the fourth equation in the lemma does not imply the 120 degree angle condition (2.6). For most \((\xi, \theta) \in D_3 \times S^1\) at which the reference frame is set and the restricted class of perturbed double bubbles specified, the corresponding fixed point \((\phi^*, \alpha^*)\) from Lemma 8.2 does not satisfy (2.6). In the next section we will find a particular \((\xi, \theta)\) whose corresponding \((\phi^*, \alpha^*)\) does satisfy (2.6).

The first part of the next lemma reveals that the solution found in Lemma 8.2 is a local minimizer of \(\mathcal{J}\) under the \(Y\)-norm. The second part of the lemma will be used later when we study the dependence of \((\phi^*, \alpha^*)\) on \((\xi, \theta)\).

**Lemma 8.4** 1. There exist \(\tilde{d} > 0\) and \(\sigma > 0\) such that if \(|\gamma| \rho^3 < \sigma\), then the solution \((\phi^*, \alpha^*)\) found in Lemma 8.2 satisfies \(\langle S'((\phi^*, \alpha^*)(\psi, \beta), (\psi, \beta)\rangle \geq \frac{\tilde{d}}{\rho^3} \|\psi, \beta\|_Y^2\) for all \((\psi, \beta) \in X\).

2. There exist \(d > 0\) and \(\sigma > 0\) such that if \(|\gamma| \rho^3 < \sigma\), then the solution \((\phi^*, \alpha^*)\) satisfies \(\|S'((\phi^*, \alpha^*)(\psi, \beta)\|_Z \geq \frac{d}{\rho^3} \|\psi, \beta\|_Y\) for all \((\psi, \beta) \in X\).

**Proof.** There exists \(\tilde{d} \in (0, 1)\) such that \(\langle S'((\phi^*, \alpha^*)(\psi, \beta), (\psi, \beta)\rangle = \langle S'(0, 0)(\psi, \beta), (\psi, \beta)\rangle + \langle S''(\tilde{d}((\phi^*, \alpha^*)), (\phi^*, \alpha^*), (\psi, \beta)\rangle, (\psi, \beta)\rangle\).

Similar to Lemma 8.1, we can show that for all \((\phi, \alpha) \in D(S),\)
\[ \|S''(\phi, \alpha)((\phi^*, \alpha^*), (\psi, \beta))\|_X \leq \tilde{C}(\rho^{-5} + |\gamma| \rho^{-2}) \|\phi^*, \alpha^*\|_X \|\psi, \beta\|_Y^2. \quad (8.16) \]

See [29, Lemma 4.1] for the proof of a similar formula. Consequently by Lemmas 7.4 and 8.2
\[ \langle S'((\phi^*, \alpha^*)(\psi, \beta), (\psi, \beta)\rangle \geq \frac{d}{\rho^3} \|\psi, \beta\|_Y^2 - \tilde{C}(\rho^{-5} + |\gamma| \rho^{-2}) \frac{2\tilde{C}|\gamma| \rho^5}{\rho} \|\psi, \beta\|_Y^2 \]
\[ \geq \frac{1}{\rho^3} \left( d - \frac{2\tilde{C}(\rho^5 + |\gamma| \rho^{-2})}{\rho} \right) \|\psi, \beta\|_Y^2 \geq \frac{d}{2\rho^3} \|\psi, \beta\|_Y^2 \]
if \(\sigma\) is sufficiently small. The first part follows if \(\tilde{d} = \frac{d}{2}\).

By Lemmas 7.5, 8.1 and 8.2,
\[ \|S'((\phi^*, \alpha^*)(\psi, \beta)\|_Z \geq \|S'(0, 0)(\psi, \beta)\|_Z - \sup_{\tau \in (0, 1)} \|S''(\tau((\phi^*, \alpha^*)), (\psi, \beta))\|_Z \]
\[ \geq \frac{\tilde{d}}{\rho^3} \|\psi, \beta\|_X - \tilde{C}(\rho^{-5} + |\gamma| \rho^{-2}) \|\phi^*, \alpha^*\|_X \|\psi, \beta\|_X \]
\[ \geq \left( \frac{\tilde{d}}{\rho^3} - \tilde{C}(\rho^{-5} + |\gamma| \rho^{-2}) \frac{2\tilde{C}|\gamma| \rho^5}{\rho} \right) \|\psi, \beta\|_Z \]
\[ \geq \frac{1}{\rho^3} \left( d - \frac{2\tilde{C}(\rho^5 + |\gamma| \rho^{-2})}{\rho} \right) \|\psi, \beta\|_Z \geq \frac{\tilde{d}}{2\rho^3} \|\psi, \beta\|_Z \]
if \(\sigma\) is sufficiently small. Part 2 follows if \(\tilde{d} = \frac{d}{2}\).

**Lemma 8.5** If \(\sigma\) is small, then \(\|\mathcal{J}((\phi^*, \alpha^*) - \mathcal{J}(0, 0))\| \leq |\gamma| \rho^3 \left( \frac{\tilde{C}^2}{d} |\gamma| \rho^3 + \frac{10\tilde{C}^3}{3d^3} (|\gamma| \rho^3)^2 + \frac{10\tilde{C}^3}{3d^3} (|\gamma| \rho^3)^3 \right)\).
Lemma 9.1

\[ \mathcal{J}(\phi^*, \alpha^*) = J(0, 0) + \langle S(0, 0), (\phi^*, \alpha^*) \rangle + \frac{1}{2} \langle S'(0, 0)(\phi^*, \alpha^*), (\phi^*, \alpha^*) \rangle + \frac{1}{6} \langle S''(\tilde{\tau}(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\phi^*, \alpha^*)) \rangle \]

for some \( \tilde{\tau} \in (0, 1) \). Also expanding \( S(\phi^*, \alpha^*) \) gives

\[ \|S(\phi^*, \alpha^*) - S(0, 0) - S'(0, 0)(\phi^*, \alpha^*)\|_z \leq \sup_{\tau \in (0,1)} \frac{1}{2} \|S''(\tau(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\phi^*, \alpha^*))\|_z. \]

(8.17)

Since \( S(\phi^*, \alpha^*) = 0 \), the above shows that

\[ \|S(0, 0) + S'(0, 0)(\phi^*, \alpha^*)\|_z \leq \sup_{\tau \in (0,1)} \frac{1}{2} \|S''(\tau(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\phi^*, \alpha^*))\|_z, \]

which implies that

\[ \|S(0, 0), (\phi^*, \alpha^*) + \langle S'(0, 0)(\phi^*, \alpha^*), (\phi^*, \alpha^*) \rangle \|_z \leq \left( \frac{1}{2} \sup_{\tau \in (0,1)} \|S''(\tau(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\phi^*, \alpha^*))\|_z \right) \|((\phi^*, \alpha^*))\|_x. \]

(8.19)

It follows from (8.17) and (8.19) that

\[ \|J(\phi^*, \alpha^*) - J(0, 0) - \frac{1}{2} \langle S(0, 0), (\phi^*, \alpha^*) \rangle \|_x \leq \left( \frac{5}{12} \sup_{\tau \in (0,1)} \|S''(\tau(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\phi^*, \alpha^*))\|_z \right) \|((\phi^*, \alpha^*))\|_x. \]

(8.20)

Lemmas 6.2, 8.1 and 8.2 show that

\[ \mathcal{J}(\phi^*, \alpha^*) - J(0, 0) \|_x \leq \frac{1}{2} \|\langle S(0, 0), (\phi^*, \alpha^*) \rangle + \frac{5}{12} \sup_{\tau \in (0,1)} \|S''(\tau(\phi^*, \alpha^*))((\phi^*, \alpha^*), (\phi^*, \alpha^*))\|_z \|((\phi^*, \alpha^*))\|_x \]

\[ \leq \frac{1}{2} \left( \frac{C^2}{d^2} \right) \|\gamma \|_\rho^3 + \frac{10\tilde{C}C^3}{3d^3} \|\gamma \|_\rho^3 + \frac{10\tilde{C}C^3}{3d^3} \|\gamma \|_\rho^3 \right)^3 \]

(8.21)

which proves the lemma. \( \square \)

9 Minimization

The solution \( (\phi^*, \alpha^*) \) of (8.1) found in Lemma 8.2 depends on \( \xi \) and \( \theta \). To emphasize this dependence, write \( \phi^* = \phi^*(\xi, \theta) \) and \( \alpha^* = \alpha^*(\xi, \theta) \). The perfect double bubble \( B(\rho, \xi, \theta) \) whose internal representation is \( (0,0) \) also depends on \( \xi \) and \( \theta \). Now let \( \xi \) vary in \( D_3 \) and \( \theta \) vary in \( S^1 \) and set

\[ J(\xi, \theta) = \mathcal{J}(\phi^*(\xi, \theta), \alpha^*(\xi, \theta)) \quad \text{and} \quad \tilde{J}(\xi, \theta) = J(B(\rho, \xi, \theta)). \]

(9.1)

Both \( J \) and \( \tilde{J} \) are treated as functions of \( (\xi, \theta) \in D_3 \times S^1 \).

Lemma 9.1 When \( \delta \) and \( \sigma \) are sufficiently small, the function \( J \) defined on \( D_3 \times S^1 \) attains every minimum in \( D_3 \times S^1 \), the interior of \( \overline{D_3} \times S^1 \).

Proof. Let \( (\xi, \theta) \in \partial D_3 \times S^1 \) and \( (\tilde{\xi}, \tilde{\theta}) \in \overline{D_3} \times S^1 \), with \( \tilde{\xi} \) being a minimum of \( R(z, z) \) in \( D \), i.e. \( R(\tilde{\xi}, \tilde{\xi}) = \min_{z \in D} R(z, z) \). Recall that by (3.5) every minimum of \( R(z, z) \) in \( D \) must be in \( D_3 \). By Lemma 8.5,

\[ J(\xi, \theta) - J(\tilde{\xi}, \tilde{\theta}) \geq \tilde{J}(\xi, \theta) - J(\tilde{\xi}, \tilde{\theta}) - 2|\gamma|\rho^4 \left( \frac{C^2\sigma}{d} + \frac{10\tilde{C}C^3\sigma^2}{3d^3} + \frac{10\tilde{C}C^3\sigma^3}{3d^3} \right). \]

(9.2)

Lemma 3.1 shows that

\[ \tilde{J}(\xi, \theta) - J(\tilde{\xi}, \tilde{\theta}) \geq \frac{\sum_{i,j=1}^2 \gamma_{ij}m^2}{8} (R(\xi, \xi) - R(\tilde{\xi}, \tilde{\xi})) - 3|\gamma|\rho^5 m^2 \max_{x,y \in \overline{T_\delta}} |\nabla R(x, y)| \]

(9.3)
The condition (2.9) gives that
\[
\sum_{i,j=1}^{2} \gamma_{ij} \geq \frac{b|\gamma|}{4}.
\] (9.4)

Then (9.2), (9.3) and (9.4) show that
\[
J(\xi, \theta) - J(\hat{\xi}, \hat{\theta})
\geq \frac{b|\gamma| \rho^2 m^2}{32} (R(\xi, \xi) - R(\hat{\xi}, \hat{\xi})) - \left[ 2|\gamma| \rho^4 \left( \frac{C^2 \sigma}{d} + \frac{10C^2 \sigma^2}{3d^3} + \frac{10C^2 \sigma^2}{3d^3} \right) + 3|\gamma| \rho^5 m^2 \max_{x,y \in \mathcal{D}_\pi} |\nabla R(x, y)| \right]
\geq |\gamma| \rho^4 \left\{ \frac{2C^2 \sigma}{d} + \frac{2C^2 \sigma^2}{3d^3} + \frac{2C^2 \sigma^2}{3d^3} + 3\delta m^2 \max_{x,y \in \mathcal{D}_\pi} |\nabla R(x, y)| \right\}.
\] (9.5)

Because of (3.5), if \( \sigma \) and \( \delta \) are sufficiently small, then
\[
J(\xi, \theta) - J(\hat{\xi}, \hat{\theta}) > 0.
\] (9.6)

Therefore any minimum of \( J \) on \( \mathcal{D}_\delta \times S^1 \) must be in \( D_\delta \times S^1 \), the interior of \( \mathcal{D}_\delta \times S^1 \). \( \Box \)

Note that this is the first time after (3.6) that \( \delta \) is required to be small. It is also the first time that the condition (2.9) is used. Only from this moment on, \( \delta \) and \( \sigma \) become dependent on \( b \).

The dependence of \( (\phi^*, \alpha^*) = (\phi^*(t, \xi, \theta), \alpha^*(\xi, \theta)) \) on \( \xi = (\xi^1, \xi^2) \), and \( \theta \) is investigated in the next lemma.

**Lemma 9.2** When \( \sigma \) is sufficiently small, \( \| \frac{\partial (\phi^*, \alpha^*)}{\partial \xi^l} \|_X = O(|\gamma| \rho^5) \), \( l = 1, 2 \), and \( \| \frac{\partial (\phi^*, \alpha^*)}{\partial \theta} \|_X = O(|\gamma| \rho^6) \) uniformly with respect to all \( (\xi, \theta) \in \mathcal{D}_\delta \times S^1 \).

**Proof.** The equation (8.1) is now written as
\[
S(\phi, \alpha, \xi, \theta) = 0,
\] (9.7)
with the operator \( S \) acting as
\[
S: (\phi, \alpha) \times (\xi, \theta) \to S(\phi, \alpha, \xi, \theta).
\] (9.8)

Estimate \( \frac{DS(\phi, \alpha, \xi, \theta)}{D\xi^l} \) and \( \frac{DS(\phi, \alpha, \xi, \theta)}{D\theta} \) the Fréchet derivatives of \( S \) with respect to \( \xi^l \) and \( \theta \) respectively. In \( S \), only the parts involving \( I_{\Omega_l} \) depend on \( \xi \) and \( \theta \). And in
\[
I_{\Omega_l} = \int_{\Omega_l} G(e^{i\theta} r(t) + \xi, e^{i\theta} y + \xi) \, dy = \int_{\Omega_l} \frac{1}{2\pi} \log \frac{1}{|r_j(t) - y|} \, dy + \int_{\Omega_l} R(e^{i\theta} r_j(t) + \xi, e^{i\theta} y + \xi) \, dy,
\]
\( \xi \) and \( \theta \) appears in the regular part \( R \) of \( G \). Then clearly
\[
\frac{\partial I_{\Omega_l}}{\partial \xi^l} = O(\rho^3) \quad \text{and} \quad \frac{\partial I_{\Omega_l}}{\partial \theta} = O(\rho^2)
\] (9.9)
hold uniformly with respect to \( t, \xi, \text{ and } \theta \). Consequently
\[
\left\| \frac{DS(\phi, \alpha, \xi, \theta)}{D\xi^l} \right\| = O(|\gamma| \rho^2) \quad \text{and} \quad \left\| \frac{DS(\phi, \alpha, \xi, \theta)}{D\theta} \right\| = O(|\gamma| \rho^3).
\] (9.10)

On the other hand Lemma 8.4 part 2 shows that at \( (\phi^*(t, \xi, \theta), \alpha^*(\xi, \theta)) \), the solution found in Lemma 8.2,
\[
\left\| \left( \frac{DS(\phi^*, \alpha^*, \xi, \theta)}{D(\phi, \alpha)} \right)^{-1} \right\| \leq \frac{\rho^3}{d}
\] (9.11)
if \( \sigma \) is small. Note that \( \frac{DS(\phi^*, \alpha^*, \xi, \theta)}{D(\phi, \alpha)} \) here is the same as \( S'(\phi^*, \alpha^*) \) in Lemma 8.4. The implicit function theorem reveals that when \( \sigma \) is small enough,
\[
\left\| \frac{D(\phi^*, \alpha^*)}{D\xi^l} \right\| = O(|\gamma| \rho^5) \quad \text{and} \quad \left\| \frac{D(\phi^*, \alpha^*)}{D\theta} \right\| = O(|\gamma| \rho^6).
\] (9.12)
One views \( \epsilon \) approximately horizontal deformation whose infinitesimal element is \( \vec{r} \) in (9.1) is minimized at \((\bar{t}, \bar{\xi}, \bar{\theta})\). On the other hand Lemma 4.3, which holds for both restricted deformations and non-restricted deformations, shows that

\[
\partial J(\xi, \theta) = \frac{\partial J(\xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})}
\]

and if \((\xi, \theta) = (0, \epsilon, 0)\), then it is nearly a vertical deformation whose infinitesimal element is

\[
\vec{r} = \epsilon \vec{r}^*(t, \xi, \theta) + \xi.
\]

One views \( \tilde{r}_1(t, \xi, \theta) \) as a three parameter family of deformations of \( r_1^*(t, \bar{0}, \bar{0}) \). If \((\xi, \theta) = (\epsilon, 0, 0)\), then it is an approximately horizontal deformation whose infinitesimal element is

\[
X^H(t) = \frac{\partial \tilde{r}_1(t, \xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = (1, 0) + \frac{\partial r_1^*(t, \xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})};
\]

if \((\xi, \theta) = (0, \epsilon, 0)\), then it is nearly a vertical deformation whose infinitesimal element is

\[
X^V(t) = \frac{\partial \tilde{r}_1(t, \xi, \theta)}{\partial \theta} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = (0, 1) + \frac{\partial r_1^*(t, \xi, \theta)}{\partial \theta} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})};
\]

and if \((\xi, \theta) = (0, 0, \epsilon)\), then it is almost a rotational deformation whose infinitesimal element is

\[
X^R(t) = \frac{\partial \tilde{r}_1(t, \xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = i \vec{r}^*_1(t, \bar{0}, \bar{0}) + \frac{\partial r_1^*(t, \xi, \theta)}{\partial \theta} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})}.
\]

Note that these three deformations are no longer in the restricted class.

By Lemma 9.2, since \((\bar{0}, \bar{0})\) is an interior minimum of \( J \),

\[
\frac{\partial J(\xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = \frac{\partial J(\xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = \frac{\partial J(\xi, \theta)}{\partial \theta} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = 0.
\]

On the other hand Lemma 4.3, which holds for both restricted deformations and non-restricted deformations, shows that \( \frac{\partial J(\xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = \frac{\partial J(\xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} \) are equal to

\[
(T_1 + T_2 + T_0) \cdot X \bigg|_{-1} = \int_{\partial \Omega_1 \setminus \partial \Omega_2} (k + \gamma_{11} I_{\Omega_1} + \gamma_{12} I_{\Omega_2})N_1 \cdot X \, ds - \int_{\partial \Omega_2 \setminus \partial \Omega_1} (k + \gamma_{21} I_{\Omega_1} + \gamma_{22} I_{\Omega_2})N_2 \cdot X \, ds
\]

\[
- \int_{\partial \Omega_1 \cap \partial \Omega_2} (k + (\gamma_{11} - \gamma_{12}) I_{\Omega_1} + (\gamma_{21} - \gamma_{22}) I_{\Omega_2})N_0 \cdot X \, ds
\]

with \( X \) being \( X^H, X^V, \) and \( X^R \) respectively. In (9.19) \( T_i \) and \( N_i \) are the tangent and normal vectors of the curves \( r_1^*(t, \bar{0}, \bar{0}) \). But these curves satisfy the first three equations of Lemma 8.3. Hence the integrals in (9.19) vanish and the following equations hold.

\[
0 = \frac{\partial J(\xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = (T_1 + T_2 + T_0) \cdot X^H \bigg|_{-1}
\]

\[
0 = \frac{\partial J(\xi, \theta)}{\partial \xi} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = (T_1 + T_2 + T_0) \cdot X^V \bigg|_{-1}
\]

\[
0 = \frac{\partial J(\xi, \theta)}{\partial \theta} \bigg|_{(\xi, \theta) = (\bar{t}, \bar{\xi}, \bar{\theta})} = (T_1 + T_2 + T_0) \cdot X^R \bigg|_{-1}.
\]

The last equation in Lemma 8.3 says that

\[
(T_1 + T_2 + T_0) \cdot X^S \bigg|_{-1} = 0 \text{ where } X^S(1) = (0, 1), X^S(-1) = (0, -1).
\]
Unlike $X^H$, $X^V$, and $X^R$, this $X^S$ is the infinitesimal element of a restricted deformation. It stretches the middle curve $r_0'(0)$ connecting the two triple junction points of $(\phi^*(\cdot, 0), \alpha^*(0, 0))$. The equations (9.20)-(9.23) form a four by four linear homogeneous system for the two components of the vector $(T_1 + T_2 + T_0)(1)$ and the two components of the vector $(T_1 + T_2 + T_0)(-1)$. The coefficients of the matrix are the components of $X^H(\pm 1)$, $X^V(\pm 1)$, $X^R(\pm 1)$, and $X^S(\pm 1)$. To estimate these coefficients, note that by Lemma 9.2 and the transformations (6.3) and (6.4),

$$
\frac{\partial r_i^n(\pm 1, \xi, \theta)}{\partial \xi_1} \bigg|_{(\vec{a}, 0)} = O(\gamma|\rho|^4), \quad \frac{\partial r_i^n(\pm 1, \xi, \theta)}{\partial \xi_2} \bigg|_{(\vec{a}, 0)} = O(\gamma|\rho|^4), \quad \frac{\partial r_i^n(\pm 1, \xi, \theta)}{\partial \theta} \bigg|_{(\vec{a}, 0)} = O(\gamma|\rho|^5). \tag{9.24}
$$

It follows that

$$
|X^H(\pm 1) - (1, 0)| = O(\gamma|\rho|^4), \quad |X^V(\pm 1) - (0, 1)| = O(\gamma|\rho|^4), \quad |X^R(\pm 1) - (\mp a, 0)| = O(\gamma|\rho|^5). \tag{9.25}
$$

By (9.15)-(9.17) and (9.25) the linear system is written as

$$
\begin{bmatrix}
1 + O(\gamma|\rho|^4) & O(\gamma|\rho|^4) & -1 + O(\gamma|\rho|^4) & O(\gamma|\rho|^4)
\end{bmatrix}
\begin{bmatrix}
\frac{1}{1} & \frac{1}{1} & \frac{1}{1} & \frac{1}{1}
\end{bmatrix}
= \begin{bmatrix}
0
0
0
0
\end{bmatrix}. \tag{9.26}
$$

Since the matrix on the left side is non-singular when $\delta$ and $\varepsilon$ are small,

$$
(T_1 + T_2 + T_0)(1) = (T_1 + T_2 + T_0)(-1) = 0. \tag{9.27}
$$

In (2.6) the $\nu_i$’s are the unit inward tangential vectors at the triple junction points, so $\nu_i = -T_i$ at the point $P^+$ and $\nu_i = T_i$ at $P^-$. Hence (9.27) implies (2.6).

According to Lemma 8.2 the solution $(\phi^*(\cdot, 0), \alpha^*(0, 0))$ is found in the space $\mathcal{X}^*$, so the curves $\phi_i^*(\cdot, 0, 0)$ are in $H^2(1, 1)$. The standard boot-strapping argument applied to the second order integro-differential equations (2.3)-(2.5) shows that the $\phi_i^*(\cdot, 0, 0)$’s are all $C^\infty$. By the transformation (6.3) we conclude that the two bubbles of the solution are enclosed by continuous curves that are $C^\infty$ except at the triple junction points.

Our assertion that the solution $(\phi^*(\cdot, 0, 0), \alpha^*(0, 0))$ is stable is interpreted by its local minimization property. Recall that the solution $(\phi^*(\cdot, 0, 0), \alpha^*(0, 0))$ is found in two steps. First for each $(\xi, \theta) \in \overline{D_3} \times S^1$, a fixed point $(\phi^*(\cdot, \xi, \theta), \alpha^*(\xi, \theta))$ is constructed in a restricted class of perturbed double bubbles. This fixed point is shown to be a local minimizer of $J$ in the restricted class in Lemma 8.4 part 1. In the second step $J$ is minimized among the $(\phi^*(\cdot, \xi, \theta), \alpha^*(\xi, \theta))$’s where $(\xi, \theta)$ ranges over $\overline{D_3} \times S^1$, and $(\phi^*(\cdot, 0, 0), \alpha^*(0, 0))$ emerges as a minimum. As a minimum of local minimizers from restricted classes, $(\phi^*(\cdot, 0, 0), \alpha^*(0, 0))$ is a local minimizer of $J$ in a neighborhood of arbitrarily perturbed double bubbles; hence our claim that $(\phi^*(\cdot, 0, 0), \alpha^*(0, 0))$ is stable.

How much does this change of a perturbed double bubble resemble an exact double bubble? One may consider the ratio of $\|\phi^*(\cdot, 0, 0), \alpha^*(0, 0)\|_\mathcal{X}$ and the area $m|\rho|^2$. Both $\phi^*$ and $\alpha^*$ have the “dimension” of area, as seen from (6.3) and (6.4), so the ratio is a good “dimensionless” quantity. By Lemma 8.2,

$$
\frac{\|\phi^*(\cdot, 0, 0), \alpha^*(0, 0)\|_\mathcal{X}}{m|\rho|^2} \leq \frac{2\overline{C}|\rho|^3}{dm} \leq \frac{2\overline{C}\sigma}{dm}. \tag{9.28}
$$

Therefore the smaller $|\gamma|/|\rho|^3$ is, the closer the solution is to an exact double bubble. The bound $\sigma$ on $|\gamma|/|\rho|^3$ is also a bound on the deviation of the solution from a standard double bubble. □

### 10 Discussion

Lemma 7.2 is interesting in its own right. It addresses an issue in the two component, equal area isoperimetric problem. It shows that in the restricted class the perimeter functional $J_L$ has a positive second variation at the standard double bubble. In other words, the smallest eigenvalue of the linearized problem $\mathcal{S}_L^*(0, 0)$ is strictly positive.

Where the double bubble solution is located in the domain $D$ can be ascertained from the proof of Lemma 9.1. Denote the minimum of $J$ on $\overline{D_3}$ by $(\xi(p, \gamma), \theta(p, \gamma))$ to emphasize its dependence on $p$ and $\gamma$, (in the last section for simplicity we assumed that this point is $(0, 0)$). If $(\xi(p, \gamma), \theta(p, \gamma)) \to (\xi^0, \theta^0)$ as $p \to 0$ and $|\gamma|/|\rho|^3 \to 0$ possibly along a subsequence, then

$$
R(\xi^0, \xi^0) = \min_{z \in \overline{D_3}} R(z, z); \tag{10.1}
$$
namely that if $\rho$ and $|\gamma|\rho^3$ are small, then the approximate double bubble solution is situated close to a minimum point of the function $R(z, z)$. However the direction of this perturbed double bubble cannot be determined from the argument in Lemma 9.1. A much more delicate study is needed to determine what $\theta^0$ is. A somewhat similar question for binary systems is the determination of the direction of the oval shaped solution to (1.2) in [31].

The binary system (1.1) or the ternary system (1.3) does not allow different constituents to mix. This is a simplification in the strong segregation limit. The original density functional theories in [20, 19] do not have this limitation. There one uses density fields, i.e. functions on $D$, instead of subsets of $D$, to describe the concentrations of the constituents. In the binary case there is a function $u$ defined on $D$ which represents the concentration of one constituent; the function $1-u$ gives the concentration of the other constituent. The free energy of the binary system takes the form

$$I_B(u) = \int_D \left[ \frac{\epsilon^2}{2} |\nabla u|^2 + W_B(u) + \frac{\epsilon}{2} (-\Delta)^{-1/2} (u - \omega) \right] dx. \quad (10.2)$$

In (10.2) $\epsilon$ is a positive parameter; $\gamma$ and $\omega$ are the same as the ones in (1.1); $W_B(u)$ is a double-well potential with two minimum points at $0$ and $1$, such as $W_B(u) = u^2(1-u)^2$; $u$ satisfies the constraint $\int_D u(x) dx = \omega |D|$. For the ternary system one has two density functions $u_1$ and $u_2$ on $D$ that, together with $1-u_1-u_2$, give the concentrations of the three constituents respectively. The free energy is given by

$$I_T(u) = \int_D \left[ \frac{\epsilon^2}{2} (|u_1|^2 + |u_2|^2) + W_T(u) + \frac{\gamma}{2} \sum_{i,j=1}^2 \left((-\Delta)^{-1/2}(u_i - \omega_i)\right)\left((-\Delta)^{-1/2}(u_j - \omega_j)\right) \right] dx \quad (10.3)$$

where $u = (u_1, u_2)$, $W_T$ is a triple-well potential with three minimum points at $(1,0), (0,1)$ and $(0,0)$, like $W_T(u) = ((u_1-1)^2 + u_2^2)(u_1^2 + (u_2-1)^2)(u_1^2 + u_2^2)$, and $u_1$ and $u_2$ satisfy the constraints $\int_D u_1(x) dx = \omega_1|D|$ and $\int_D u_2(x) dx = \omega_2|D|$. One can interpret $\mathcal{J}_B$ and $\mathcal{J}_T$ of (1.1) and (1.3) as limits of $\epsilon^{-1} I_B$ and $\epsilon^{-1} I_T$ as $\epsilon \to 0$. The convergence may be formulated under the framework of the $\Gamma$-convergence theory developed in [8, 17, 16, 4]; see [23, 26] for more details. There are also activator-inhibitor type reaction diffusion PDE systems, such as the Gierer-Meinhardt system [11], that can be reduced to problems like $\mathcal{J}_B$; see [31].

Conversely by a result of Kohn and Sternberg [15] one can show that if there is an isolated local minimizer of $\mathcal{J}_B$ (or $\mathcal{J}_T$ respectively), then for sufficient small $\epsilon$ there is a local minimizer of $I_B$ (or $I_T$) near the local minimizer of $\mathcal{J}_B$ (or $\mathcal{J}_T$). Unfortunately the local minimality concept in this result is defined with respect to the rather weak $L^1(D)$ norm; namely the distance between two subsets $E$ and $F$ of $D$ is $||\chi_E - \chi_F||_{L^1}$. Nevertheless there is a theorem by Acerbi, Fusco and Morini [1] regarding $\mathcal{J}_B$, which states that if a critical point of $\mathcal{J}_B$ has a positive second variation (see [7] for the formula of the second variation), then it is always a local minimizer under the $L^1$ norm. Hopefully a similar property holds for $\mathcal{J}_T$, so one can construct local minimizers of $I_T$ from critical points of $\mathcal{J}_T$ that have positive second variations. According to Lemma 8.4 the critical point found in Theorem 2.1 already has positive second variation with respect to restricted deformations. If the point $(\xi^0, \theta^0) \in D_1 \times S^1$ given before (10.1) has some “non-degeneracy” property, then we believe that the critical point should also have positive second variation with respect to deformations outside the restricted class.

References


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