PROFILE OF THE LEAST ENERGY SOLUTION OF A SINGULAR PERTURBED NEUMANN PROBLEM WITH MIXED POWERS

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ABSTRACT. We consider the problem $\varepsilon^2\Delta u - u^q + u^p = 0$ in Ω, $u > 0$ in Ω, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ where Ω is a smooth bounded domain in $\mathbb{R}^N$, $1 < q < p < \frac{N+2}{N-2}$ if $N \geq 2$ and ε is a small positive parameter. We determine the location and shape of the least energy solution when $\varepsilon \to 0$.

1. INTRODUCTION

There has been considerable interest in understanding the behavoir of positive solutions of the elliptic problem

$$(1.1)\qquad \begin{cases} \varepsilon^2\Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 \text{ in } \Omega, \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\varepsilon > 0$ is a parameter, f is a changing sign superlinear nonlinearity and Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $F(u) = \int_0^u f(t)dt$. We consider the problems in the zero mass case i.e. when $f(0) = 0$ and $f'(0) = 0$. It is easy to check that the problem (1.1) admits solutions on Ω if $f'(0) < 0$, while there may be no nontrivial solutions for small $\varepsilon > 0$ if $f'(0) > 0$. Thus problem (1.1) can be viewed as *borderline* problems. Berestycki and Lions in [2] proved the existence of ground state solutions if $f(u)$ behaves like $|u|^p$ for large u and $|u|^q$ for small u where p and q are respectively supercritical and subcritical. This type of equations arises in the Yang-Mills theory, in various mathematical models derived from population theory, chemical reactor theory and are much harder to handle; see Gidas [10] and Gidas-Ni-Nirenberg [11]. In this paper we consider the following singular perturbed problem,

$$(1.2)\qquad \begin{cases} \varepsilon^2\Delta u - u^q + u^p = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $\varepsilon > 0$ is a small number and ν denotes the unit normal to $\partial\Omega$. Here $1 < q < p < \frac{N+2}{N-2}$ and $N \geq 2$.

This problem with the Dirichlet boundary condition was first studied by Dancer-Santra [6] and they have proved that there exists $q_\star = \frac{N}{N-2}$ called the *zero mass*

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exponent such that when $q \in (\frac{N}{N-2}, \frac{N+2}{N-2})$, the least energy solution concentrates at a harmonic center of Ω. Moreover, q_* is critical to (1.2) in the determination of concentration of the least energy solution. Furthermore, Dancer-Santra-Wei [7] proved that $q \in (1, \frac{N}{N-2})$, the least energy solution concentrates at the global minimum of $\mathcal{R}_q$ (re-normalized energy) where

(1.3)
$$\mathcal{R}_q(\xi) := \lim_{\delta\to 0}\left\{\int_{\Omega\setminus B_\delta(\xi)} \frac{1}{2}|\nabla \mathcal{G}_q(x,\xi)|^2 + \frac{1}{q+1}\mathcal{G}_q^{q+1}(x,\xi) - \frac{(q-1)}{2(q+1)(2+2\alpha-N)}\delta^{-2-2\alpha}\omega_q^{q+1}\right\}$$

and $\mathcal{G}_q(\cdot,\xi)$ is the unique positive weakly singular solution to the problem

$$\begin{cases} \Delta_x \mathcal{G}_q(x,\xi) - \mathcal{G}_q(x,\xi)^q = 0 & \text{in } \Omega\setminus\{\xi\},\\ \mathcal{G}_q(x,\xi) \sim \dfrac{\omega_q}{|x-\xi|^\alpha} & \text{for } x\sim\xi \\ \mathcal{G}_q(x,\xi) = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

and when $q = q_\star$, u_ε concentrates at the global minima of $\Psi_{q_\star}$, where $\Psi_{q_\star}$ is defined by

$$\begin{aligned} \Psi_{q_\star}(\xi): &= \int_\Omega |\nabla \mathcal{H}_{q_\star}(x,\xi)|^2\,dx \\ &+ (N-2)^2\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N-2}}dx \\ &+ \frac{1}{2}(N-2)^2\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N-1}}dx \\ &+ \frac{(N-1)(N-2)}{2}\int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x-\xi|^{2(N-1)}|\log|x-\xi||^{N}}dx \end{aligned}$$

where $\mathcal{H}_{q_\star}(\cdot,\xi)$ is the solution to the problem

$$\begin{cases} \Delta_x \mathcal{H}_{q_\star}(x,\xi) = 0 & \text{in } \Omega,\\ \mathcal{H}_{q_\star}(x,\xi) = \dfrac{1}{|x-\xi|^{N-2}|\log|x-\xi||^{\frac{N-2}{2}}} & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

and

$$\omega_q^{q-1} = \begin{cases} \frac{2}{q-1}\left[\frac{2}{q-1} - (N-2)\right] & \text{if } q < q_\star \\ \left(\frac{N-2}{\sqrt{2}}\right)^{N-2} & \text{if } q = q_\star. \end{cases} \tag{1.6}$$

In this paper, we consider the analogue Neumann problem (1.2). As in the Dirichlet problem, there are *zero mass exponents* for the Neumann problem. We now derive the zero mass exponent, which will be crucial in determining the points of concentration.

As in [16], we first define the least energy solution. Let the associated functional to the problem (1.2) be

$$I_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 - \frac{1}{p+1}(u^+)^{p+1} + \frac{1}{q+1}(u^+)^{q+1}\right)dx.$$

Easy to check that $I_\varepsilon(u)$ satisfies Palais-Smale condition and all the conditions of the mountain pass theorem and hence there exists a mountain pass solution $u_\varepsilon > 0$

and a mountain pass critical value characterized by

$$0 < c_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon(\gamma(t))$$

where

$$\Gamma_\varepsilon = \{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\},$$

where $I_\varepsilon(e) < 0$ and $e(x) = k$ is a constant function on Ω, k chosen sufficiently large. Note that as 0 is a strict local minima of I_ε, $c_\varepsilon > 0$, $\forall\varepsilon > 0$. Let

$$\mathcal{N}_\varepsilon(\Omega) = \left\{u \in H^1(\Omega) : \varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega (u^+)^{q+1} = \int_\Omega (u^+)^{p+1}\right\}.$$

The problem is now to obtain the asymptotic behavior of c_ε as $\varepsilon \to 0$. To this end, we start with the entire problem

$$\begin{cases} \Delta U - U^q + U^p = 0 & \text{in } \mathbb{R}^N, \\ U > 0 & \text{in } \mathbb{R}^N, \\ U \to 0 & \text{as } |x| \to \infty, \\ U \in C^2\left(\mathbb{R}^N\right). \end{cases} \tag{1.7}$$

By Li-Ni [15] and Kwong–Zhang [14], (1.7) has a unique radial solution U such that $U \in \mathcal{D}^{1,2}\left(\mathbb{R}^N\right) \cap \mathrm{L}^{q+1}\left(\mathbb{R}^N\right)$ where $\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u : |\nabla u| \in L^2(\mathbb{R}^N)$ and $u \in L^{2^*}(\mathbb{R}^N)\}$ when $N \geq 3$. Moreover, U behaves at infinity as

$$U(r) \sim \begin{cases} \dfrac{1}{r^{\frac{2}{q-1}}} & \text{if } 1 < q < \dfrac{N}{N-2}, \\ \dfrac{1}{r^{N-2}} & \text{if } \dfrac{N}{N-2} < q < \dfrac{N+2}{N-2}. \\ \dfrac{1}{r^{N-2}(\log r)^{\frac{N-2}{2}}} & \text{if } q = \dfrac{N}{N-2}. \end{cases} \tag{1.8}$$

When $q = 1$, Ni–Takagi [16] showed that for sufficiently small ε, the least energy solution is a single boundary spike and has only one local maximum $P_\varepsilon \in \partial\Omega$. Moreover, in [17], they prove that $H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$ where $H(P)$ is the mean curvature of $\partial\Omega$ at P. A simplified proof was given by Del Pino–Felmer in [8], for a wide class of nonlinearities.

We first point out a useful lemma whose proof follows from the computations in Ni-Takagi [16].

Lemma 1.1. *Let $A(x)$ be a radial function with $A(x) \sim \frac{C}{|x|^\nu}$ as $|x| \to +\infty$ and $\gamma > N+1$. Then for $P \in \partial\Omega$, we have the following asymptotic expansion*

$$\int_\Omega A\left(\frac{x-P}{\varepsilon}\right) dx = \varepsilon^N \left[\frac{c}{2} - \varepsilon K H(P) + o(\varepsilon)\right] \tag{1.9}$$

where $H(P)$ is the mean curvature of the boundary at the point P

$$c = \int_{\mathbb{R}^N} A(x)dx$$

and

$$K = \frac{1}{2}\int_{\partial\mathbb{R}^N_+} |y|^2 A(y,0)dy.$$

Now we take

$$G(x)=\frac{1}{2}|\nabla U|^2+\frac{1}{q+1}U^{q+1}-\frac{1}{p+1}U^{p+1} \tag{1.10}$$

We claim that $K>0$. Note that from algebraic decay of U we obtain

$$\begin{aligned} K=\frac{1}{4}\int_{\partial\mathbb{R}^N_+}[(U')^2-F(U)]|y|^2dy' &= \frac{N-1}{4}\int_{\mathbb{R}^N_+}[(U')^2-F(U)]y_N dy' \\ &= \frac{N-1}{N+1}\int_{\mathbb{R}^N_+}(U'(|y|))^2 y_N dy. \end{aligned} \tag{1.11}$$

This proves the claim.

Observe that the restriction $\gamma>N+1$ is necessary otherwise K is not defined.

Then the lowest decay rate in (1.10) is given by the gradient term since $2(\alpha+1)\le\alpha(q+1)$. Note that the equality holds for $\alpha=\frac{2}{q-1}$.

So if $2(\alpha+1)>N+1$, we obtain an estimate depending only on the mean curvature. As a result if $2(\alpha+1)>N+1$, we obtain an estimate on the least energy (as in [16]) depending only on the mean curvature. So if $\alpha>\frac{N-1}{2}$, we have

$$c_\varepsilon=\varepsilon^N\left[\frac{c}{2}-\varepsilon KH(P_\varepsilon)+o(\varepsilon)\right] \tag{1.12}$$

where P_ε is the unique local maximum point of u_ε and $H(P_\varepsilon)$ is the boundary mean curvature function at $P_\varepsilon\in\partial\Omega$.

Following the same argument in Ni-Takagi [16], we can then prove that $H(P_\epsilon)\to\max_{P\in\partial\Omega}H(P)$ as $\epsilon\to 0$.

Observe that $\alpha>\frac{N-1}{2}$ is satisfied if and only if either $N\ge 4$, or $N=3, q<3$, or $N=2, q<5$.

The most interesting cases are

1) $N=3, q\ge 3, (\alpha=1)$. Note that when $N=3$ and $q=3$, we are in the situation of a zero mass exponent.

2) $N=2, q\ge 5, (\alpha=\frac{2}{q-1})$.

The main objective of this paper is to locate the maximum point P_ε in the remaining cases. It turns out that as in the Dirichlet problem, the location of the spikes is determined in a nonlocal way.

Let $P\in\partial\Omega$. We define a diffeomorphism straightening of the boundary in a neighborhood of P. After rotation and translation of the coordinate system we may assume that the inward normal to $\partial\Omega$ at P points in the direction of the positive x_N axis and that $P=0$.

Let $x'=(x_1,x_2,\cdots,x_{N-1})$ and $B'_\delta=\{x'\in\mathbb{R}^{N-1}:|x'|<\delta_0\}$ and $\Omega_1=\Omega\cap B(P,\delta_0)$, where $B(P,\delta_0)=\{x\in\mathbb{R}^N:|x-P|<\delta_0\}$. Since $\partial\Omega$ is smooth, we can choose a $\delta_0>0$ such that $\partial\Omega\cap B(P,\delta_0)$ can be represented by the graph of a smooth function $f=f_P:B(\delta'_0)\to\mathbb{R}$ where

$$f_P(0)=0, \nabla f_P(0)=0 \text{ and } \partial\Omega\cap B(P,\delta_0)=\{(x',x_N)\in B(P,\delta):x_N-P_N>f_P(x'-P')\}$$

$$f_P(x'-P')=\frac{1}{2}\sum_{i=1}^{N-1}k_i(x_i-P_i)^2+\mathcal{O}(|x'-P'|^3)$$

where $k_i(i=1,\cdots,N-1)$ are the principal curvatures at P. Note that the first condition implies that $\{x_N=0\}$ is a tangent plane of $\partial\Omega$ at P.

We deform the boundary near P. For $x \in \Omega_1 = \Omega \cap B(P, \delta_0)$, set

$$\varepsilon y' = x' - P', \varepsilon y_N = x_N - P_N - f(x' - P'). \tag{1.13}$$

This transformation we denote by $y = T_\varepsilon(x)$. Note that the Jacobian of T_ε equals ε^N. Its inverse is called $x = T_\varepsilon^{-1}(y)$. Moreover,

$$x' = P' + \varepsilon y', x_N = P_N + \varepsilon y_N + f(\varepsilon(y' - P')). \tag{1.14}$$

The Laplace operator and the boundary operator reduces to

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla_{x'} f|^2}}(\nabla_{x'} f, -1) \tag{1.15}$$

$$\frac{\partial}{\partial \nu} = \frac{1}{\sqrt{1 + |\nabla_{x'} f|^2}}\left\{\sum_{j=1}^{N-1} f_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_N}\right\}\Bigg|_{x_N - P_N = f(x' - P')} \tag{1.16}$$

and the Laplace operator becomes

$$\varepsilon^2 \Delta_x = \Delta_y + |\nabla_{x'} f|^2 \frac{\partial^2}{\partial^2 y_N} - 2\sum_{i=1}^{N-1} f_i \frac{\partial^2}{\partial y_i \partial y_N} - \varepsilon \Delta_{x'} f \frac{\partial}{\partial y_N}. \tag{1.17}$$

Throughout this paper, we use the following notation:

$$y = (y', y_N), y' = (y_1, y_2, \cdots, y_{N-1}) \text{ and } \mathbb{R}_+^{N-1} = \{y \in \mathbb{R}^N : y_N > 0\}.$$

When $N = 2$, we define a space

$$\mathcal{D} = \{u \in W_{loc}^{1,2}(\mathbb{R}^2) : |\nabla u| \le \frac{C}{|x|^{\alpha+1}}; |u(x)| \le \frac{C}{|x|^\alpha} \text{ whenever } |x| \gg 1\},$$

where $C > 0$ is independent of u. Then

$$I_\infty(U) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla U|^2 - \frac{1}{p+1}U^{p+1} + \frac{1}{q+1}U^{q+1}\right) dx \tag{1.18}$$

is well defined on $\mathcal{D}$. Note that when $N \ge 3$, $I_\infty(U)$ is well defined in $\mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$. In this paper, we show that when $\alpha < \frac{1}{2}$ and $N = 2$, the asymptotic behavior of the least energy solution of the Neumann problem (1.2) is not determined by the mean curvature of $\partial\Omega$, instead it is determined by a nonlinear singular problem. For any $P \in \partial\Omega$, we define the renormalized energy in $\mathbb{R}^2$ by

$$\begin{aligned}\Phi_q(P) : \quad &= \quad \lim_{\delta \to 0} \left[\frac{1}{2}\int_{\Omega \setminus \Omega \cap B_\delta(P)} |\nabla G_q(x, P)|^2 dx + \frac{1}{q+1}\int_{\Omega \setminus \Omega \cap B_\delta(P)} |G_q(x, P)|^{q+1} dx\right. \\ (1.19) \quad &- \quad \left.\frac{q-1}{4(q+1)\alpha}\delta^{-(2\alpha+2)}\omega_q^{q+1}\right].\end{aligned}$$

where G_q is the unique (up to a modulo constant) positive solution

$$\begin{cases} \Delta_x G_q(x, P) - G_q(x, P)^q = 0 & \text{in } \overline{\Omega} \setminus \{P\}, \\ \dfrac{\partial G_q(x, P)}{\partial \nu} = 0 & \text{on } \partial\Omega \setminus \{P\} \\ G_q(x, P) \sim \dfrac{\omega_q}{|x - P|^\alpha} & \text{when } \ x \sim P. \end{cases} \tag{1.20}$$

Now we state the main results of the paper

Theorem 1.1. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the least energy positive solution of (1.2) $u_\varepsilon \in H^1(\Omega)$ has a unique point of maximum $P_\varepsilon \in \partial\Omega$.*
(a) When $N = 2$ and $q > 5$, u_ε concentrates at the global minimum of Φ_q, where Φ_q satisfies (1.3) and

$$I_\varepsilon(u_\varepsilon) = \frac{\varepsilon^2}{2} I_\infty + \varepsilon^{2+2\alpha}\Phi_q(P_\varepsilon) + o(\varepsilon^{2+2\alpha})$$

where Φ_q satisfies (1.19).
(b)When $N = 2$ and $q = 5$, u_ε concentrates at a local maxima of H, where H is the boundary curvature function and

$$I_\varepsilon(u_\varepsilon) = \frac{\varepsilon^2}{2} I_\infty - \frac{(1-\sigma_0)}{8}\varepsilon^3\left(\log\frac{1}{\varepsilon}\right)H(P_\varepsilon) + o\left(\varepsilon^3\left(\log\frac{1}{\varepsilon}\right)\right)$$

for some $\sigma_0 < 1$.

Theorem 1.2. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the least energy positive solution of (1.2) $u_\varepsilon \in H^1(\Omega)$ has a unique point of maximum $P_\varepsilon \in \partial\Omega$.*
(a)When $N = 3$ and $q > 3$, u_ε concentrates at a local maximum of H, where H is the boundary curvature function and

$$I_\varepsilon(u_\varepsilon) = \frac{\varepsilon^3}{2} I_\infty - \gamma_3^2\varepsilon^4\left(\log\frac{1}{\varepsilon}\right)H(P_\varepsilon) + o\left(\varepsilon^4\left(\log\frac{1}{\varepsilon}\right)\right).$$

where $\gamma_3 = \lim_{|x|\to+\infty}|x|U(x)$.
(b) When $N = 3$ and $q = 3$,(corresponds to the zero mass exponent) u_ε concentrates at a local maximum of H, where H is the boundary curvature function

$$I_\varepsilon(u_\varepsilon) = \frac{\varepsilon^3}{2} I_\infty - \varepsilon^4\left(\log\left(\log\frac{1}{\varepsilon}\right)\right)\frac{H(P_\varepsilon)}{4} + o\left(\varepsilon^4\left(\log\left(\log\frac{1}{\varepsilon}\right)\right)\right).$$

By concentration, we mean u_ε converge uniformly to zero in compact subsets of $\Omega \setminus \{P\}$ while there exists a $c > 0$ such that $u_\varepsilon(P_\varepsilon) \geq c$ as $\varepsilon \to 0$.

Renormalized energy is a well-known concept in theoretical physics for instance see Bethuel-Brezis-Hélein [1] is independent of the core radius and is a function of the singularity position which characterizes the energy content of a dislocated body. They established that a family of global minimizers of

$$K_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2;\ u \in H^1(\Omega, \mathbb{C}) \tag{1.21}$$

with Dirichlet constraint $u = g$ on $\partial\Omega$ where g is a smooth function with values in $\mathbb{S}^1$. When $n := deg(g; \partial\Omega) > 0$, it was found that u_ε has exactly n zeros (called vortices) of local degree one, which approach, up to subsequence, n distinct points ξ_j for which

$$u_\varepsilon(x) \to e^{i\varphi(x,\xi)}\prod_{i=1}^{n}\frac{x-\xi}{|x-\xi|} = w(x,\xi).$$

Besides, ξ globally minimizes a re-normalized energy, $W(\xi)$, characterized as the limit

$$W(\xi) = \lim_{\rho\to 0}\left[\int_{\Omega\setminus\cup j=1^n B_\rho(\xi_j)}|\nabla_x w|^2 - n\pi\log\frac{1}{\rho}\right]. \tag{1.22}$$

for which explicit expression in terms of Greens functions can be found in Bethuel-Brezis-Hélein [1]. The asymptotic expansion of $W(\xi)$, of (1.22) shows that the renormalized energy is is the remaining energy after the removal of the singular core energy $n\pi \log \frac{1}{\rho}$ has been removed, see Kleman [13].

2. Preliminaries

We recall some well-known results to (1.2).

Lemma 2.1. *(a) For all* $\varepsilon > 0$

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) = \inf_{u\in\mathcal{N}_\varepsilon(\Omega)} I_\varepsilon(u) = \inf_{u\in H^1(\Omega), u\not\equiv 0} \max_{t\geq 0} I_\varepsilon(tu).$$

Proof. For the sake of completeness we prove this well known lemma. Let $\varepsilon > 0$ be fixed. First note that

$$\inf_{\gamma \in \Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) \leq \inf_{u\in H^1(\Omega)} \max_{t\geq 0} I_\varepsilon(tu) \tag{2.1}$$

We first claim that $\inf_{u\in\mathcal{N}_\varepsilon(\Omega)} I_\varepsilon(u) = \inf_{u\in H^1(\Omega)} \max_{t\geq 0} I_\varepsilon(tu)$. Define $\beta(t) = I_\varepsilon(tu)$. Due to the nature of the nonlinearity we have $\beta(0) = 0, \beta(t) > 0$ for small $t > 0$ and $\beta(t) < 0$ for $t > 0$ sufficiently large. Hence $\max_{t\in[0,+\infty)} \beta(t)$ is achieved. Also note that $\beta'(t) = 0$ implies $\varepsilon^2\|u\|^2_{H^1(\Omega)} = g(t)$ where

$$g(t) = t^{p-1}\int_\Omega (u^+)^{p+1} - t^{q-1}\int_\Omega (u^+)^{q+1}.$$

It is easy to see that g is an increasing function of t whenever $g(t) > 0$. Thus there exists a unique t such that $\|u\|_{H^1(\Omega)} = g(t)$. Hence there exist a unique point $\theta(u)$ such that $\beta'(\theta(u)u) = 0$ and $\theta(u)u \in \mathcal{N}_\varepsilon(\Omega)$. This implies that $\mathcal{N}_\varepsilon(\Omega)$ is radially homeomorphic to $H^1(\Omega)\setminus\{0\}$ if we prove that $\theta : H^1(\Omega)\setminus\{0\} \to \mathbb{R}^+$ is continuous. In order to do so let $u_n \to u$ in $H^1(\Omega)\setminus\{0\}$. Then $u_n \to u$ in $H^1(\Omega)$ and $u_n \to u$ in $L^r(\Omega)$ for all $r \leq \frac{N+2}{N-2}$ and

$$\int_\Omega \varepsilon^2|\nabla u_n|^2 = \theta^{p-1}(u_n)\int_\Omega (u_n^+)^{p+1} - \theta^{q-1}(u_n)\int_\Omega (u_n^+)^{q+1} \tag{2.2}$$

which proves there exist constants $m > 0$ and $M > 0$ independent of n such that $m \leq \theta(u_n) \leq M$. By passing to the limit in (2.2) the whole sequence $\{\theta(u_n)\}$ converges as u_n is convergent and hence $\theta(u) = \theta_0$ where $\theta_0 u \in \mathcal{N}_\varepsilon$ which proves our claim.

Next we claim that $\inf_{\gamma \in \Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) = \inf_{u\in\mathcal{N}_\varepsilon(\Omega)} I_\varepsilon(u)$. It is easy to see that $\inf_{\gamma \in \Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) \geq \inf_{u\in\mathcal{N}_\varepsilon(\Omega)} I_\varepsilon(u)$ by (2.1). It is enough to prove that any $\gamma \in \Gamma_\varepsilon$ intersects $\mathcal{N}_\varepsilon$. Note that $I_\varepsilon(u) > 0$ for $\|u\|_{H^1(\Omega)}$ sufficiently small and $I_\varepsilon(\gamma(1)) < 0$ which implies the required result. □

Lemma 2.2. *When $N = 2$, then I_∞ satisfies the Palais Smale condition on $\mathcal{D}$ and hence the functional I_∞ satisfies all the conditions of mountain pass theorem on $\mathcal{D}$.*

Proof. Define a norm on $\mathcal{D}$ as

$$\|u\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^2} |\nabla u|^2\right)^{1/2} + \left(\int_{\mathbb{R}^2} |u|^{q+1}\right)^{1/q+1} \quad \forall u \in \mathcal{D}.$$

Note that $(\mathcal{D}, \|u\|_{\mathcal{D}})$ is a Banach space. We claim that $\mathcal{D} \hookrightarrow L^{p+1}(\mathbb{R}^2)$ is a continuous embedding provided $1 < p < \infty$. Define $I_\infty : \mathcal{D} \to \mathbb{R}$ as

$$I_\infty(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} + \frac{1}{q+1}|u|^{q+1}\right).$$

Now we need to show that I_∞ satisfies Palais Smale condition on $\mathcal{D}$. Let u_n be a sequence in $\mathcal{D}$ such that $I_\infty(u_n) \le C$ and $I'_\infty(u_n)u_n = o(1)\|u_n\|_{\mathcal{D}}$. Then we obtain that u_n satisfies

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^2} |\nabla u_n|^2 + \left(\frac{1}{q+1} - \frac{1}{p+1}\right) \int_{\mathbb{R}^2} |u_n|^{q+1} = C + o(1)\|u_n\|_{\mathcal{D}}$$

Hence there exists $C_1 > 0$ such that

$$C_1 \left(\int_{\mathbb{R}^2} |\nabla u_n|^2 + \int_{\mathbb{R}^2} |u_n|^{q+1} \right) = C + o(1)\|u_n\|_{\mathcal{D}}$$

which implies that

$$\left(\int_{\mathbb{R}^2} |\nabla u_n|^2 \right) \le C + o(1)\|u_n\|_{\mathcal{D}}$$
$$\left(\int_{\mathbb{R}^2} |u_n|^{q+1} \right) \le C + o(1)\|u_n\|_{\mathcal{D}}.$$

Hence

$$\|u_n\|_{\mathcal{D}} \le \min\{(C + o(1)\|u_n\|_{\mathcal{D}})^{1/2}, (C + o(1)\|u_n\|_{\mathcal{D}})^{1/q+1}\}$$

which implies that u_n is bounded in $\mathcal{D}$.

This implies

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 \le C$$

and

$$\int_{\mathbb{R}^2} |u_n|^{q+1} \le C.$$

Hence by reflexivity, we obtain $\nabla u_n \rightharpoonup \nabla u$ in L^2 and $u_n \rightharpoonup u$ in L^{q+1}. Also by Rellich Lemma u_n converges strongly in compact subset of L^2 and L^{q+1}. Hence there exists a subsequence of u_n such that $u_n \to u$ a.e. But $|u_n| \le \frac{C}{|x|^\alpha}$ and $|\nabla u_n| \le \frac{C}{|x|^{\alpha+1}}$ for $|x| \gg 1$. By using the decay estimates we can show that u_n converges strongly u in $\mathcal{D}$.

Let $\mathcal{D}_r$ be the subspace of $\mathcal{D}$ consisting of radially symmetric functions. Then $\mathcal{D}_r \hookrightarrow L^{p+1}(\mathbb{R}^2)$ is a compact embedding provided $2 < p + 1 < \infty$.

Suppose T is a bounded set in $\mathcal{D}_r$. Then $|u(r)| \le \epsilon$ if $u \in T$ and $r \ge R$. Hence

$$\begin{aligned}\int_R^\infty |u(r)|^{p+1} r &= \int_R^\infty |u(r)|^{p-q} |u(r)|^{q+1} r \\ &\le \epsilon \int_R^\infty |u|^{q+1} r \le \epsilon \|u\|_{L^{q+1}}\end{aligned}$$

Now we know that bounded sets in $\mathcal{D}_r$ will converge strongly in $L^{p+1}(\mathbb{R}^2)$ on compact subsets and hence we can use the usual diagonalization argument to obtain a strongly convergent subsequence in $L^{p+1}(\mathbb{R}^2)$ from a sequence in T. As a matter of fact I_∞ satisfies all the conditions of the mountain pass theorem in $\mathcal{D}_r$. Hence there exists a $c > 0$ such that

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\infty(\gamma(t)) = \inf_{u \in \mathcal{D}_r} \max_{t \ge 0} I_\infty(tu)$$

where

$$\Gamma = \{\gamma \in C([0,1];\mathcal{D}_r); \gamma(0) = 0, I_\infty(\gamma(1)) \le 0\}$$

Hence there exists a positive radial solution of (1.7) obtained by the mountain pass theorem. Hence by Lemma 2.1, U is a mountain pass solution of (1.7). □

Since

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon(\Omega)} I_\varepsilon(u) = I_\varepsilon(u_\varepsilon)$$

we have

$$c_\varepsilon = I_\varepsilon(u_\varepsilon) = \varepsilon^2 \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |\nabla u_\varepsilon|^2 + \left(\frac{1}{q+1} - \frac{1}{p+1}\right) \int_\Omega u_\varepsilon^{q+1} \tag{2.3}$$

which implies that $\varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2$, $\int_\Omega u_\varepsilon^{p+1}$ and $\int_\Omega u_\varepsilon^{q+1}$ are uniformly bounded. Let P_ε be a local maxima of (1.2), then $u_\varepsilon(P_\varepsilon) \ge 1$. By Gidas-Spruck [12], we obtain $\|u_\varepsilon\|_{L^\infty(\overline{\Omega})} \le C$. Hence $\|u_\varepsilon\|_{C^{2,\beta}_{loc}(\overline{\Omega})} \le C$ for some $0 < \beta < 1$, as a result $u_\varepsilon(P_\varepsilon + \varepsilon x) \to U(x)$ uniformly in $\Omega_{\varepsilon,P} = \{x / P_\varepsilon + \varepsilon x \in \Omega\}$ where U satisfies (1.7).
Moreover, if $\alpha := \max\{\frac{2}{q-1}, N-2\}$, by Dancer–Santra [6],

$$\lim_{|x| \to \infty} |x|^\alpha U(x) = \omega_q > 0, \text{ if } q \neq q_\star. \tag{2.4}$$

It is easy to check that if

$$q < q_\star \tag{2.5}$$

then $\alpha > N-2$ and

$$U(x) = \frac{\omega_q}{|x|^\alpha} + \mathcal{O}\left(\frac{1}{|x|^{(p-q)\alpha+\alpha}}\right) \text{ as } |x| \to \infty, \tag{2.6}$$

where $\alpha = -\frac{N-2}{2} + \frac{\sqrt{(N-2)^2 + 4\omega_q^2}}{2}$. Moreover,

$$\lim_{r \to \infty} r^{\alpha(q+1)} U_r^2(r) = \omega_q^{q+1}.$$

3. Linear Theory in $\mathbb{R}^2$

Consider the operator $L = \Delta + f'(U)$.

Lemma 3.1. *Let ψ be a bounded solution of*

$$L(\psi) = 0.$$

Then $\psi \in span\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\}$.

Proof. Let us write

$$\psi = \sum_{k=1}^{\infty} \phi_k(r) S_k(\theta)$$

where $r = |x|$, $\theta = \frac{x}{|x|} \in \mathbb{S}^1$; and $-\Delta_{\mathbb{S}^1} S_k = \lambda S_k$ where $\lambda_k = k^2; k \in \mathbb{Z}^+ \cup \{0\}$ and whose multiplicity is given by $M_k - M_{k-2}$ where $M_k = \frac{(k+1)!}{k!}$ for $k \ge 2$. Note that $\lambda_0 = 0$ has algebraic multiplicity one and $\lambda_1 = 1$ has algebraic multiplicity 2. Then ϕ_k satisfy an infinite system of ODE given by,

$$\phi_k'' + \frac{1}{r}\phi_k' + \left(pU^{p-1} - qU^{q-1} - \frac{\lambda_k}{r^2}\right)\phi_k = 0, \ r \in (0,\infty). \tag{3.1}$$

Also note that (3.1) has two linearly independent solutions $z_{1,k}$ and $z_{2,k}$. Let

$$A_k(\phi) = \phi'' + \frac{1}{r}\phi' + \left(pU^{p-1} - qU^{q-1} - \frac{\lambda_k}{r^2}\right)\phi$$

Also recall that if one solution $z_{1,k}$ to (3.1) is known, a second linearly independent solution can be found in any interval where $z_{1,k}$ does not vanish as

$$z_{2,k}(r) = z_{1,k}(r) \int z_{1,k}^{-2} r^{-1} dr$$

where $\int$ denotes antiderivatives. One can obtain the asymptotic behavior of any solution z as $r \to \infty$ by examining the indicial roots of the associated Euler equation. The limiting equation becomes

$$r^2\phi'' + r\phi' - (q\alpha^2 + \lambda_k)\phi = 0 \tag{3.2}$$

whose indicial roots are given by

$$\mu_k^{\pm} = \begin{cases} \sqrt{(q\alpha^2 + \lambda_k)} & \text{if } k \neq 0 \\ \sqrt{q}\alpha & \text{if } k = 0 \end{cases}$$

In this way we see that the asymptotic behavior is ruled by $z(r) \sim r^{-\mu}$ as $r \to +\infty$; where μ satisfies the problem

$$\mu^2 - (q\omega_q^{q-1} + \lambda_k) = 0 \text{ if } \alpha = \frac{2}{q-1}. \tag{3.3}$$

□

Claim 1 If $k = 0$, equation (3.1) has no nontrivial solution in $\mathcal{D}$.

Since (3.1) is a second order differential equation it has two solutions g_1 and g_2. The other solution g_1 satisfies

$$(rg_{1,r})_r = -f'(U(r))rg_1(r). \tag{3.4}$$

Note that we can choose $R > 0$ such that for $r \geq R$ we obtain $f'(U(r)) \leq 0$. If we choose $g_1(R) = 1$ and $g_1'(R) > 0$ we obtain (3.4) that $rg_{1,r}$ is increasing for all $r \geq R$ and hence there exist a constant $c > 0$ such that $rg_{1,r} \geq c$. Hence by integration we can show $g_1(r) \to +\infty$ as $r \to \infty$. As a result, g_1 does not belong to $\mathcal{D}$. We consider the solution $g_2(0) = 1$ we can show exactly as in [14] that g_2 satisfies $\lim_{r\to+\infty} g_2(r) = K \neq 0$. Hence, $g_2(r) \notin \mathcal{D}$. Furthermore, note that the operator is not non-degenerate in the space of bounded functions.

Claim 2 If $k = 1$, then all solutions of equation (3.1) are constant multiples of U'.

In this case $\lambda_1 = 1$ and hence we have $z_{1,1}(r) = -U'(r)$ is a solution to the problem (3.1) and is positive $(0, +\infty)$. Hence we define

$$z_{1,2}(r) = z_{1,1}(r) \int_1^r z_{1,1}(s)^{-2} s^{-1} ds$$

Let us check how $z_{1,2}(r)$ behaves at infinity.
Again when $\alpha = \frac{2}{q-1}$, then $|U_r| \sim r^{-\alpha q+1}$ as $r \to \infty$ and hence $z_{1,2}(r) \sim r^{\alpha q-1}$ and as $\alpha q = 2 + \alpha > 2$, $z_{1,2} \notin \mathcal{D}$. Hence any family of solutions of (3.1) is given by $\phi_1 = cU'(r)$ for some $c \in \mathbb{R}$.

Claim 3 If $k \geq 2$, equation (3.1) admits only trivial solution in $\mathcal{D}$. We will show that if $A_k(\phi_k) = 0$, then $\phi_k = 0$. Note that $-U'$ is a positive solution of A_1. Let us study the first eigenvalue of the problem

$$\begin{cases} A_1(\phi) = \lambda\phi & \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} \phi^2 = 1 \end{cases} \tag{3.5}$$

We know $U_{rr} \sim \frac{1}{r^{\alpha q}}$ as $r \to \infty$. Note that if $\lambda_1 > 0$, then $\int_{\mathbb{R}^2} \phi_1 U' = 0$ and hence there exists a point in $\mathbb{R}^2$ such that ϕ_1 changes sign. But ϕ_1 is the first eigenfunction corresponding to λ_1 and hence it has a definite sign. Hence $\lambda_1 \leq 0$. Thus A_1 is an operator having no positive eigenvalues. Hence for $k \geq 2$, $c_k = k^2 - 1 > 0$. Now

$$A_k = A_1 - \frac{k^2 - 1}{r^2} I$$

where I is the identity. Hence $0 = -\int_{\mathbb{R}^2} A_k(\phi_k)\phi_k \geq c_k \int_{\mathbb{R}^N} \frac{\phi_k^2}{r^2}$ and as $\phi_k \in C(\mathbb{R}^2)$, we have $\phi_k \equiv 0$.

Remark 3.1. *Hence deduce that for any $\phi \in Ker(-\Delta - pU^{p-1} + qU^{q-1})$, then $\phi = U'(r)S_1$ where S_1 satisfies*

$$-\Delta_{\mathbb{S}^1} S_1 = \lambda_1 S_1.$$

Now $Ker(-\Delta_{\mathbb{S}^1} - \lambda_1 I)$ is 2 dimensional and hence $Ker(-\Delta_{\mathbb{S}^1} - \lambda_1 I) = span\{S_{1,1}, S_{1,2}\} \simeq span\ \mathbb{R}^2$. Hence

$$Ker(\Delta + f'(U)) = span\{U'(r)S_{1,1}, U'(r)S_{1,2}\} = span\left\{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\right\}.$$

This implies that $Ker(\Delta + f'(U)) = \{\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}\}$ in $\mathcal{D}$.

Corollary 3.1. *If we restrict $Ker(\Delta + f'(U))$ to $\mathcal{D}(\mathbb{R}^2_+) = \mathcal{D} \cap \{\frac{\partial u}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+\}$ then $Ker(\Delta + f'(U)) \cap \mathcal{D}(\mathbb{R}^2_+) = \{\frac{\partial U}{\partial y_1}\}$.*

Remark 3.2. *When $N \geq 3$, $Ker(\Delta + f'(U)) \cap \mathcal{D}^{1,2}(\mathbb{R}^N_+) = \{\frac{\partial U}{\partial x_1}, \cdots \frac{\partial U}{\partial x_{N-1}}\}$ where $\mathcal{D}^{1,2}(\mathbb{R}^N_+) = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N_+), \frac{\partial u}{\partial y_N} = 0 \text{ on } \mathbb{R}^N_+\}$.*

For any $P \in \mathbb{R}^N$ and for any $\varepsilon > 0$ set

$$U_{\varepsilon,P}(x) := U\left(\frac{x - P}{\varepsilon}\right) \quad x \in \mathbb{R}^N.$$

It is clear that $U_{\varepsilon,P}$ solves

$$\varepsilon^2 \Delta U_{\varepsilon,P} - U^q_{\varepsilon,P} + U^p_{\varepsilon,P} = 0 \text{ in } \mathbb{R}^N. \tag{3.6}$$

4. Profile of Spike $N = 2$ and $q > 5$.

Lemma 4.1. *Then (1.20) admits a solution. Furthermore,*

$$G_q(x, P) = \frac{\omega_q}{|x - P|^\alpha} + \mathcal{O}\left(\frac{1}{|x - P|^{\alpha - 1}}\right). \tag{4.1}$$

Proof. In order to prove existence of solution of (1.20) we consider

$$\begin{cases} \Delta\phi_0 - \phi_0 = 0 \text{ in } \Omega \\ \dfrac{\partial\phi_0}{\partial\nu} = \left|\dfrac{\partial U_0}{\partial\nu}\right| \text{ on } \partial\Omega \end{cases} \tag{4.2}$$

where $U_0 = \omega_q|x-P|^{-\frac{2}{q-1}}$ and $P \in \partial\Omega$. Note that this problem has L^∞ solution since it is easy to check that $|\frac{\partial U_0}{\partial\nu}| \leq \frac{1}{|x-P|^\alpha}$ and the solution $|\phi_0| \leq C_1|x-P|^{1-\alpha} + C_2$. Secondly, we use $U_0 \pm C\phi_0$ as sub-super solution to the problem

$$\begin{cases} \Delta G_\varepsilon - G_\varepsilon^q = 0 \text{ in } \Omega_\varepsilon = \Omega \setminus B_\varepsilon(P) \\ \partial_\nu G_\varepsilon = 0 \text{ on } \partial\Omega \cap \partial\Omega_\varepsilon \\ G_\varepsilon = \omega_q\varepsilon^{-\alpha} \text{ in } \partial B_\varepsilon(P) \end{cases} \tag{4.3}$$

Then we can show that

$$U_0 - C\phi_0 \leq G_\varepsilon \leq U_0 + C\phi_0$$

for C large independent of ε. Taking $\varepsilon \to 0$ we obtain

$$U_0 - C\phi_0 \leq G_q \leq U_0 + C\phi_0.$$

This proves the existence of G_q, as well as the asymptotic behavior. Note that this solution is unique up to a constant. □

We define

$$f_q(x,P) = G_q(x,P) - \frac{\omega_q}{|x-P|^\alpha}.$$

Lemma 4.2. *Then close to $P \in \partial\Omega$ the following happens*

$$|\nabla f_q(x,P)| = \mathcal{O}(|x-P|^{-\alpha}) \tag{4.4}$$

and

$$|\Delta f_q(x,P)| = \mathcal{O}(|x-P|^{-(\alpha+1)}) \tag{4.5}$$

near P.

Proof. Without loss of generality, we consider $P = 0$. Then

$$\Delta f - \frac{q\alpha^2}{|x|^2} f = \mathcal{O}(|x|^{-(\alpha+1)}). \tag{4.6}$$

It is easy to check that there exists a $R > 0$ such that

$$|f(x)| \leq C|x|^\nu \text{ in } B_R(0) \cap \Omega.$$

Let $x \in B(\frac{R}{2})$ and $r = \frac{|x|}{2}$. For any $y \in B_1$ we define $\tilde{f}(y) = f(x+ry)$. Then from (4.6) we have

$$\Delta\tilde{f} = r^2\Delta f = q\alpha^2\tilde{f} + \mathcal{O}(|x+ry|^{1-\alpha}).$$

Hence by elliptic estimates

$$\begin{aligned} |\nabla\tilde{f}(0)| &\leq C(\|\tilde{f}\|_{L^\infty(B_1(0))} + \|\Delta\tilde{f}\|_{L^\infty(B_1(0))}) \\ &\leq C\|\tilde{f}\|_{L^\infty(B_1(0))} \\ &\leq C\|f\|_{L^\infty(B_1(x))}. \end{aligned}$$

As a result, $|\nabla f(x)| \leq C|x|^{-\alpha}$. Similarly

$$|\Delta\tilde{f}(0)| \leq C\|\tilde{f}\|_{L^\infty(B_1(0))}$$

and hence we have

$$|\Delta f(x)| \leq C|x|^{-(\alpha+1)}.$$

□

5. Construction of the Projection

Consider the problem

$$\begin{cases} \Delta\varphi - \dfrac{q\alpha^2}{|x|^2}\varphi = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\varphi}{\partial y_2} = \dfrac{1}{|x|^\alpha} & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}. \end{cases} \tag{5.1}$$

Let $\varphi = \frac{1}{|x|^\alpha} y_2 + \hat{\varphi}$ be a solution of (5.1). Then $\hat{\varphi}$ satisfies

$$\begin{cases} \Delta\hat{\varphi} - \dfrac{q\alpha^2}{|x|^2}\hat{\varphi} + \Delta(\dfrac{1}{|x|^\alpha} y_2) - \dfrac{q\alpha^2}{|x|^2}\dfrac{y_2}{|x|^\alpha} = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\hat{\varphi}}{\partial y_2} = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{5.2}$$

Consider $\hat{\varphi} = r^\beta Q(\theta)$ with $\beta = 1-\alpha$ and $Q(\theta) = Q(-\theta)$. Then we have

$$\Delta(r^\beta Q(\theta)) - \frac{q\alpha^2}{r^2} r^\beta Q(\theta) = [(\beta^2 - q\alpha^2)Q(\theta) + Q_{\theta\theta}] r^{\beta-2}. \tag{5.3}$$

As a result we have

$$\begin{aligned} Q_{\theta\theta} + (\beta^2 - q\alpha^2)Q(\theta) &= -[(\sin\theta)_{\theta\theta} + (\beta - q\alpha^2)\sin\theta] \\ &= (q\alpha^2 - \beta^2 + 1)\sin\theta. \end{aligned} \tag{5.4}$$

Now we need to solve

$$\begin{cases} Q_{\theta\theta} + (\beta^2 - q\alpha^2)Q(\theta) = |\sin\theta|(q\alpha^2 - \beta^2 + 1) & \text{in } (0,\pi), \\ Q'(0) = Q'(\pi) = 0 & . \end{cases} \tag{5.5}$$

This problem can be uniquely solved as long as

$$\beta^2 - q\alpha^2 \neq n^2$$

that is

$$(1-\alpha)^2 - q\alpha^2 \neq 1.$$

We denote this solution as $q_0(\theta)$. Thus we can write

$$\varphi_1 = r^{1-\alpha}[\sin\theta + q_0(\theta)]. \tag{5.6}$$

Next we solve

$$\begin{cases} \Delta\varphi_0 - qU^{q-1}\varphi_0 + pU^{p-1}\varphi_0 = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\varphi_0}{\partial y_2} = \dfrac{1}{|x|^\alpha} & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}. \end{cases} \tag{5.7}$$

Let $\varphi_0 = \varphi_1 + \hat{\varphi}_0$ be a solution of (5.7). Then $\hat{\varphi}_0$ satisfies

$$\begin{cases} \Delta\hat{\varphi}_0 - qU^{q-1}\hat{\varphi}_0 + pU^{p-1}\hat{\varphi}_0 + \mathcal{O}\left(\dfrac{1}{|x|^{2+\sigma+\alpha-1}}\right) = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\hat{\varphi}_0}{\partial y_2} = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{5.8}$$

which can be uniquely solved if $\hat{\varphi}_0$ is even in y_1 and by super-solution method we obtain for $|x| \gg 1$

$$\hat{\varphi}_0(x) = \mathcal{O}\bigg(\frac{1}{|x|^{\alpha-1+\sigma}}\bigg).$$

Choose a $\eta = \eta_\delta \in C_0^\infty(\mathbb{R}^2)$ such that $0 \leq \eta \leq 1$

$$\eta_\delta(x) = \begin{cases} 1 & \text{in } |x-P| \leq \delta, \\ 0 & \text{in } |x-P| > 2\delta. \end{cases} \tag{5.9}$$

We define a *nonlinear projection* in the following way: $PU_{\varepsilon,P} \in \mathrm{H}^1(\Omega)$ is defined as

$$PU_{\varepsilon,P} = \eta(U_{\varepsilon,P} + \varepsilon\varphi_0(T_\varepsilon(x))) + (1-\eta)\varepsilon^\alpha G_q(x,P). \tag{5.10}$$

Then we have

$$PU_{\varepsilon,P} = (U_{\varepsilon,P} + \varepsilon\varphi_0(T_\varepsilon(x))) + (1-\eta)[\varepsilon^\alpha G_q(x,P) - (U_{\varepsilon,P} + \varepsilon\varphi_0)].$$

Lemma 5.1. *For any $P \in \partial\Omega$, the following expansion holds*

$$I_\varepsilon\left(PU_{\varepsilon,P}\right) = \frac{\varepsilon^2}{2} I_\infty(U) + \varepsilon^{2\alpha+2}\Phi_q(P) + o\left(\varepsilon^{(2\alpha+2)}\right) \tag{5.11}$$

where

$$I_\infty(U) := \int_{\mathbb{R}^2}\left[\frac{p-1}{2(p+1)}U^{p+1}(x) - \frac{q-1}{2(q+1)}U^{q+1}(x)\right]dx. \tag{5.12}$$

Proof. Set $F(s) := \frac{1}{p+1}(s^+)^{p+1} - \frac{1}{q+1}(s^+)^{q+1}$. Here $\alpha = \frac{2}{q-1}$. We compute the energy as follows.

$$J_\varepsilon\left(PU_{\varepsilon,P}\right) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla\left(PU_{\varepsilon,P}(x)\right)|^2 dx + \frac{1}{q+1}\int_\Omega \left(PU_{\varepsilon,P}(x)\right)^{q+1} dx - \frac{1}{p+1}\int_\Omega \left(PU_{\varepsilon,P}(x)\right)^{p+1} dx.$$

Using the definition of

$$\begin{aligned}
\int_\Omega \left(PU_{\varepsilon,P}(x)\right)^{q+1} dx &= \int_{B_\delta(P)\cap\Omega} (U_{\varepsilon,P} + \varepsilon\varphi_0(T_\varepsilon(x)))^{q+1} + \varepsilon^{\alpha(q+1)}\int_{\Omega\setminus(B_{2\delta}(P)\cap\Omega)} G_q^{q+1}(x,P) \\
&+ \int_{\Omega\cap\{\delta<|x-P|<2\delta\}} (\varepsilon^\alpha G_q + (U_{\varepsilon,P} + \varepsilon\varphi_0 - \varepsilon^\alpha G_q)\eta)^{q+1} \\
&= \int_{\Omega\cap B_\delta(P)} U_{\varepsilon,P}(x)^{q+1} + \varepsilon^{\alpha(q+1)}\int_{\Omega\setminus(B_\delta(P)\cap\Omega)} G^{q+1}(x,P) \\
&+ \int_{\delta<|x-P|<2\delta} [(\varepsilon^\alpha G_q + (U_{\varepsilon,P} + \varepsilon\varphi_0 - \varepsilon^\alpha G_q)\eta)^{q+1} - (\varepsilon^\alpha G_q)^{q+1}]dx \\
&= I_1 + I_2 + I_3
\end{aligned}$$

We have

$$
\begin{aligned}
I_1 &= \int_{B_\delta(P)\cap\Omega} (U_{\varepsilon,P} + \varepsilon\varphi_0(T_\varepsilon(x)))^{q+1} \\
&= \int_{B_\delta(P)\cap\Omega} U_{\varepsilon,P}^{q+1} + \varepsilon\mathcal{O}\Big(\int_{B_\delta(P)\cap\Omega} U_{\varepsilon,P}^{q}\varphi_0(T_\varepsilon(x))\Big) \\
&= \int_{B_\delta^+(P)} U_{\varepsilon,P}^{q+1} - \int_{B_\delta^+(P)\setminus\Omega} U_{\varepsilon,P}^{q+1} + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon^2\int_{\mathbb{R}^2_+} U^{q+1}dx - \int_{\mathbb{R}^2_+\setminus B_\delta^+(P)} U_{\varepsilon,P}^{q+1}dx - \int_{B_\delta^+(P)\setminus\Omega} U_{\varepsilon,P}^{q+1} + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon^2\int_{\mathbb{R}^2_+} U^{q+1}dx - \frac{\omega_q^{q+1}}{2\alpha}\varepsilon^{2\alpha+2}\delta^{-2\alpha-2} - \int_{B_\delta^+(P)\setminus\Omega} U_{\varepsilon,P}^{q+1} + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon^2\int_{\mathbb{R}^2_+} U^{q+1}dx - \frac{\omega_q^{q+1}}{2\alpha}\varepsilon^{2\alpha+2}\delta^{-2\alpha-2} - \varepsilon^2\int_{B^+_{\frac{\delta}{\varepsilon}}(P)\setminus\Omega_\varepsilon} U^{q+1} + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Now we estimate

$$
\begin{aligned}
\varepsilon^2\int_{B^+_{\frac{\delta}{\varepsilon}}(P)\setminus\Omega_\varepsilon} U^{q+1} &= \varepsilon^2\int_0^{\frac{\delta}{\varepsilon}}\int_0^{\frac{f(\varepsilon y_1)}{\varepsilon}} U^{q+1}(y_1,y_2)dy_2dy_1 \\
&= \varepsilon^2\int_0^{\frac{\delta}{\varepsilon}}\int_0^{\frac{f(\varepsilon y_1)}{\varepsilon}} [U^{q+1}(y_1,0) + \mathcal{O}(|y_2|U^{q+1}(y',0))]dy_2dy_1 \\
&= \frac{\varepsilon^3 H(P)}{2}\int_0^{\frac{\delta}{\varepsilon}} [U^{q+1}(y_1,0)y_1^2dy_1 + \mathcal{O}(\varepsilon^2)] = o(\varepsilon^{2\alpha+2}) \qquad (5.13)
\end{aligned}
$$

by choosing δ sufficiently close to ε.
Using the fact that $\alpha(q+1) = \alpha+2$, we have

$$
\begin{aligned}
I_3 &= \int_{\Omega\cap\{\delta<|x-P|<2\delta\}} [(\varepsilon^\alpha G_q + (U_{\varepsilon,P} + \varepsilon\varphi_0 - \varepsilon^\alpha G_q)\eta)^{q+1} - (\varepsilon^\alpha G_q)^{q+1}]dx \\
&= \mathcal{O}(1)\varepsilon^{2+\alpha}\int_{\Omega\cap\{\delta<|x-\xi|<2\delta\}} G_q^q(x,\xi)(U_{\varepsilon,P} + \varepsilon\varphi_0 - \varepsilon^\alpha G_q)dx \\
&= \mathcal{O}(1)\varepsilon^{2+2\alpha}\int_{\Omega\cap\{\delta<|x-\xi|<2\delta\}} G_q^q(x,\xi)\Big\{\frac{\varepsilon^{\alpha(p-q)}}{|x-\xi|^{\alpha(p-q)+\alpha}} + |x-\xi|^{1-\alpha}\Big\}dx \\
&= o(\varepsilon^{2+2\alpha}).
\end{aligned}
$$

First note that

$$
\nabla PU_{\varepsilon,P}(x) = \begin{cases} \nabla U_{\varepsilon,P} + \varepsilon\nabla\varphi_0 & \text{in } |x-P|\le\delta, \\ \varepsilon^\alpha\nabla G_q & \text{in } |x-P|>2\delta. \end{cases} \qquad (5.14)
$$

and in the annulus $\delta<|x-P|<2\delta$ we have

$$
\nabla PU_{\varepsilon,P}(x) = \varepsilon^\alpha\nabla G_q(x,P) + \nabla\eta(\varepsilon^\alpha G_q(x,P) - U_{\varepsilon,P} - \varepsilon\varphi_0) + \eta\nabla(\varepsilon^\alpha G_q(x,P) - U_{\varepsilon,P} - \varepsilon\varphi_0).
$$

Hence we obtain

$$
\begin{aligned}
\int_{\Omega} |\nabla PU_{\varepsilon,P}|^2 &= \int_{\Omega\cap B_\delta(P)} |\nabla U_{\varepsilon,P} + \varepsilon\nabla\varphi_0|^2 + \varepsilon^{2\alpha}\int_{\Omega\setminus\Omega\cap B_\delta(P)} |\nabla G_q(x,P)|^2 \\
&+ \int_{\Omega\cap\{\delta<|x-P|<2\delta\}} |\nabla\eta|^2|\varepsilon^{\alpha}G_q(x,P) - U_{\varepsilon,P} - \varepsilon\varphi_0|^2 \\
&+ 2\int_{\Omega\cap\{\delta<|x-P|<2\delta\}} |\eta|^2|\nabla(\varepsilon^{\alpha}G_q(x,P) - U_{\varepsilon,P} - \varepsilon\varphi_0)|^2 \\
&+ 2\varepsilon^{\alpha}\int_{\Omega\cap\{\delta<|x-P|<2\delta\}} \eta\nabla G_q\nabla(\varepsilon^{\alpha}G_q(x,P) - U_{\varepsilon,P} - \varepsilon\varphi_0) \\
&+ 2\varepsilon^{\alpha}\int_{\Omega\cap\{\delta<|x-P|<2\delta\}} \nabla\eta\nabla G_q(\varepsilon^{\alpha}G_q(x,P) - U_{\varepsilon,P} - \varepsilon\varphi_0) \\
&+ 2\int_{\Omega\cap\{\delta<|x-P|<2\delta\}} \eta\nabla\eta\nabla(\varepsilon^{\alpha}G_q - U_{\varepsilon,P} - \varepsilon\varphi_0)(\varepsilon^{\alpha}G_q - U_{\varepsilon,P} - \varepsilon\varphi_0).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\varepsilon^2\int_{\Omega} |\nabla\left(PU_{\varepsilon,P}(x)\right)|^2 dx &= \varepsilon^2\int_{\mathbb{R}^2_+} |\nabla U|^2 + \varepsilon^{2+2\alpha}\left[\int_{\Omega\setminus\Omega\cap B_\delta(P)} |\nabla G_q(x,P)|^2 - \omega_q^{q+1}\delta^{-2\alpha-2}\right] \\
&+ o(\varepsilon^{2\alpha+2})
\end{aligned}
$$

and similarly we have

$$
\int_{\Omega} \left(PU_{\varepsilon,P}(x)\right)^{p+1} dx = \varepsilon^N\int_{\mathbb{R}^2_+} U^{p+1} + o(\varepsilon^{2\alpha+2}).
$$

Hence we have

$$
I_\varepsilon\left(PU_{\varepsilon,P}\right) = \frac{\varepsilon^2}{2}I_\infty + \varepsilon^{2\alpha+2}\Phi_q(P) + o(1)\varepsilon^{2\alpha+2}. \tag{5.15}
$$

□

Let

$$
E_\varepsilon[u] = \varepsilon^2\Delta u + f(u).
$$

Now we estimate the error due to $PU_{\varepsilon,P}(x)$.

Lemma 5.2. *For $\delta > 0$, sufficiently small, there exists $\sigma' > 0$ such that*

$$
E_\varepsilon[PU_{\varepsilon,P}(x)] = \begin{cases} \varepsilon^2\mathcal{O}(f''(U_{\varepsilon,P})\varphi_0^2(T_\varepsilon(x))) & \text{in } |x-P| < \delta, \\ \mathcal{O}\left(\varepsilon^{2+\alpha}\delta^{1-\alpha}\dfrac{1}{|x-P|^2}\right) & \text{in } \delta < |x-P| < 2\delta \\ \varepsilon^{\alpha p}G_q^p & \text{in } |x-P| > 2\delta. \end{cases} \tag{5.16}
$$

Proof. First it is easy check that

$$
E_\varepsilon[PU_{\varepsilon,P}(x)] = \varepsilon^{\alpha p}G_q^p \text{ in } \ |x-P| > 2\delta \tag{5.17}
$$

First we estimate the error in the $|x-P|<\delta$. As $q>5$ we have

$$\begin{aligned} E_\varepsilon[PU_{\varepsilon,P}(x)] &= \Big\{\varepsilon^2\Delta U_{\varepsilon,P}+f(U_{\varepsilon,P})\Big\} \\ &+ \varepsilon\Big\{\varepsilon^2\Delta\varphi_0+f'(U_{\varepsilon,P})\varphi_0\Big\} \\ &+ \Big\{f(U_{\varepsilon,P}+\varepsilon\varphi_0)-f(U_{\varepsilon,P})-\varepsilon f'(U_{\varepsilon,P})\varphi_0\Big\} \\ &= \varepsilon^2\mathcal{O}(f''(U_{\varepsilon,P})\varphi_0^2(T_\varepsilon(x))). \end{aligned}$$

So we need to calculate the error when $\delta<|x-P|<2\delta$. We write

$$PU_{\varepsilon,P}(x)=U_{\varepsilon,P}(x)+(1-\eta)(\varepsilon^\alpha G_q(x,P)-U_{\varepsilon,P}(x)-\varepsilon\varphi_0).$$

Hence we have

$$\begin{aligned} \Delta PU_{\varepsilon,P}(x) &= \Delta U_{\varepsilon,P}(x)+\Delta(1-\eta)(\varepsilon^\alpha G_q(x,P)-U_{\varepsilon,P}(x)-\varepsilon\varphi_0) \\ &= \Delta U_{\varepsilon,P}(x)+(1-\eta)\Delta(\varepsilon^\alpha G_q(x,P)-U_{\varepsilon,P}(x)-\varepsilon\varphi_0) \\ &- 2\nabla\eta\nabla(\varepsilon^\alpha G_q(x,P)-U_{\varepsilon,P}(x)-\varepsilon\varphi_0)+\Delta\eta(\varepsilon^\alpha G_q(x,P)-U_{\varepsilon,P}(x)-\varepsilon\varphi_0). \end{aligned}$$

As a result, we have

$$\begin{aligned} \varepsilon^2\Delta PU_{\varepsilon,P}(x) &= \varepsilon^2\Delta U_{\varepsilon,P}(x)+\mathcal{O}\bigg(\varepsilon^{2+\alpha}|x-P|^{-(\alpha+1)}+\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha(p-q)+\alpha+2}} \\ &+ \varepsilon^{2+\alpha}|x-P|^{-\alpha}+\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha(p-q)+\alpha+1}} \\ &+ \varepsilon^{2+\alpha}|x-P|^{1-\alpha}+\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha(p-q)+\alpha}}\bigg); \end{aligned}$$

$$\begin{aligned} (PU_{\varepsilon,P}(x))^q &= (U_{\varepsilon,P}(x))^q+\mathcal{O}(U_{\varepsilon,P}^{q-1}(\varepsilon^\alpha G_q-U_{\varepsilon,P}-\varepsilon\varphi_0)) \\ &= U_{\varepsilon,P}^q+\mathcal{O}\bigg(\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha p}}+\varepsilon^{2+\alpha}|x-P|^{-(\alpha+1)}\bigg); \end{aligned}$$

and

$$\begin{aligned} (PU_{\varepsilon,P}(x))^p &= (U_{\varepsilon,P}(x))^p+\mathcal{O}(U_{\varepsilon,P}^{p-1}(\varepsilon^\alpha G_q-U_{\varepsilon,P}-\varepsilon\varphi_0)) \\ &= U_{\varepsilon,P}^p+\mathcal{O}\bigg(\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha p}}+\varepsilon^{2+\alpha}|x-P|^{-(\alpha+1)}\bigg). \end{aligned}$$

Summing up all the terms and using the fact (3.6) we obtain

$$\begin{aligned} E_\varepsilon[PU_{\varepsilon,P}(x)] &= \mathcal{O}\bigg(\varepsilon^{2+\alpha}|x-P|^{-(\alpha+1)}+\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha(p-q)+\alpha+2}} \\ &+ \varepsilon^{2+\alpha}|x-P|^{-\alpha}+\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha(p-q)+\alpha+1}} \\ &+ \varepsilon^{2+\alpha}|x-P|^{1-\alpha}+\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha(p-q)+\alpha}}\bigg)+\mathcal{O}\bigg(\frac{\varepsilon^{\alpha(p-q)+\alpha+2}}{|x-P|^{\alpha p}}+\varepsilon^{2+\alpha}|x-P|^{-\alpha-1}\bigg). \end{aligned}$$

As a result, we can choose $\sigma'\in(0,1)$ sufficiently small such that

$$E_\varepsilon[PU_{\varepsilon,P}(x)] = \mathcal{O}\bigg(\frac{\varepsilon^{2+\alpha}\delta^{1-\alpha}}{|x-P|^2}\bigg). \tag{5.18}$$

□

6. Refinement of the projection

Now we refine the projection $PU_{\varepsilon,P}$. We define a projection of the form

$$V_{\varepsilon,P} = PU_{\varepsilon,P} + \varepsilon^{\alpha}\delta^{1-\alpha} v_1 \tag{6.1}$$

where

$$\begin{cases} \Delta v_1 + qU^{q-1} v_1 = 0 & \text{in } \Omega, \\ \dfrac{\partial v_1}{\partial \nu} = -\dfrac{1}{\varepsilon^{\alpha}\delta^{1-\alpha}} \dfrac{\partial PU_{\varepsilon,P_\varepsilon}}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Note that v_1 is bounded and is chosen in such a way that $\frac{\partial V_{\varepsilon,P}}{\partial \nu} = 0$ on $\partial\Omega$.

Lemma 6.1. *For any $P \in \partial\Omega$, the following expansion holds*

$$I_\varepsilon\left(V_{\varepsilon,P}\right) = I_\varepsilon\left(PU_{\varepsilon,P}\right) + o\left(\varepsilon^{(2\alpha+2)}\right). \tag{6.3}$$

Proof. By definition we have

$$\begin{aligned}
I_\varepsilon\left(V_{\varepsilon,P}\right) &= I_\varepsilon\left(PU_{\varepsilon,P}\right) + \frac{\varepsilon^{2+2\alpha}\delta^{2(1-\alpha)}}{2}\int_\Omega |\nabla v_1|^2 \\
&+ \varepsilon^{2+\alpha}\delta^{(1-\alpha)}\int_\Omega \nabla PU_{\varepsilon,P}\nabla v_1 \\
&- \int_\Omega \{F(PU_{\varepsilon,P} + \varepsilon^{\alpha}\delta^{1-\alpha}v_1) - F(PU_{\varepsilon,P})\} \\
&= I_\varepsilon\left(PU_{\varepsilon,P}\right) + \frac{\varepsilon^{2+2\alpha}\delta^{2(1-\alpha)}}{2}\int_\Omega |\nabla v_1|^2 \\
&+ \varepsilon^{\alpha}\delta^{(1-\alpha)}\int_\Omega \{\varepsilon^2\nabla PU_{\varepsilon,P}\nabla v_1 + f(PU_{\varepsilon,P})v_1\} \\
&- \int_\Omega \{F(PU_{\varepsilon,P} + \varepsilon^{\alpha}\delta^{1-\alpha}v_1) - F(PU_{\varepsilon,P}) - \varepsilon^{\alpha}\delta^{1-\alpha}f(PU_{\varepsilon,P})v_1\} \\
&= I_\varepsilon\left(PU_{\varepsilon,P}\right) + \frac{\varepsilon^{2+2\alpha}\delta^{2(1-\alpha)}}{2}\int_\Omega |\nabla v_1|^2 \\
&- \varepsilon^{\alpha}\delta^{(1-\alpha)}\int_\Omega \{\varepsilon^2\Delta PU_{\varepsilon,P} + f(PU_{\varepsilon,P})\}v_1 + \varepsilon^{2+\alpha}\delta^{(1-\alpha)}\int_{\partial\Omega}\frac{\partial PU_{\varepsilon,P}}{\partial\nu}v_1 \\
&- \int_\Omega \{F(PU_{\varepsilon,P} + \varepsilon^{\alpha}\delta^{1-\alpha}v_1) - F(PU_{\varepsilon,P}) - \varepsilon^{\alpha}\delta^{1-\alpha}f(PU_{\varepsilon,P})v_1\} \\
&= I_\varepsilon\left(PU_{\varepsilon,P}\right) + \frac{\varepsilon^{2+2\alpha}\delta^{2(1-\alpha)}}{2}\int_\Omega |\nabla v_1|^2 \\
&- \varepsilon^{\alpha}\delta^{(1-\alpha)}\int_\Omega E_\varepsilon(PU_{\varepsilon,P})v_1 + \varepsilon^{2+\alpha}\delta^{(1-\alpha)}\int_{\partial\Omega}\frac{\partial PU_{\varepsilon,P}}{\partial\nu}v_1 \\
&- \int_\Omega \{F(PU_{\varepsilon,P} + \varepsilon^{\alpha}\delta^{1-\alpha}v_1) - F(PU_{\varepsilon,P}) - \varepsilon^{\alpha}\delta^{1-\alpha}f(PU_{\varepsilon,P})v_1\}
\end{aligned}$$

It is easy to check that

$$\frac{\varepsilon^{2+2\alpha}\delta^{2(1-\alpha)}}{2}\int_\Omega |\nabla v_1|^2 = o(\varepsilon^{2+2\alpha})$$

$$\varepsilon^{2+\alpha}\delta^{(1-\alpha)}\int_{\partial\Omega}\frac{\partial PU_{\varepsilon,P}}{\partial\nu}v_1 = o(\varepsilon^{2+2\alpha}).$$

Now we estimate

$$\begin{aligned}\int_\Omega E_\varepsilon(PU_{\varepsilon,P})v_1\,dx &= \int_{\Omega\cap B_\delta(P)} E_\varepsilon(PU_{\varepsilon,P})v_1 + \int_{\Omega\cap(B_{2\delta}(P)\setminus B_\delta(P))} E_\varepsilon(PU_{\varepsilon,P})v_1\\ &+ \int_{\Omega\setminus B_{2\delta}(P)} E_\varepsilon(PU_{\varepsilon,P})v_1\,dx\\ &= I_1+I_2+I_3.\end{aligned}$$

Now we estimate I_1. Then we have

$$\begin{aligned}\int_{\Omega\cap B_\delta(P)} E_\varepsilon(PU_{\varepsilon,P})v_1 &= \int_{\Omega\cap B_{\varepsilon R}(P)} E_\varepsilon(PU_{\varepsilon,P})v_1 + \int_{\Omega\cap(B_\delta\setminus B_{\varepsilon R}(P))} E_\varepsilon(PU_{\varepsilon,P})v_1 = \mathcal{O}(\varepsilon^4)\\ &+ O(\varepsilon^{2+\alpha}\delta^{2-\alpha})\end{aligned}$$

¿From I_2 we have

$$I_2 = \mathcal{O}(\varepsilon^{2+\alpha}\delta^{1-\alpha}\log\delta).$$

Furthermore, we obtain

$$I_3 = o(\varepsilon^{2+\alpha}).$$

As $q>5$, we obtain

$$\begin{aligned}&\int_\Omega\{F(PU_{\varepsilon,P}+\varepsilon^\alpha\delta^{1-\alpha}v_1) - F(PU_{\varepsilon,P}) - \varepsilon^\alpha\delta^{1-\alpha}f(PU_{\varepsilon,P})v_1\}\\ =\ &\varepsilon^{2\alpha}\delta^{2-2\alpha}\mathcal{O}(\int_\Omega f'(PU_{\varepsilon,P})v_1^2) = \mathcal{O}(\varepsilon^{2+2\alpha}\delta^{2-2\alpha}).\end{aligned}$$

Using the above facts, we obtain

$$I_\varepsilon(V_{\varepsilon,P}) = I_\varepsilon(PU_{\varepsilon,P}) + o(\varepsilon^{2+2\alpha}).$$

□

Lemma 6.2. *The error due to the refined projection is given by*

(6.4) $$E_\varepsilon[V_{\varepsilon,P}(x)] = E_\varepsilon[PU_{\varepsilon,P}(x)] + \varepsilon^{2+\alpha}\delta^{1-\alpha}\Delta v_1 + \varepsilon^\alpha\delta^{1-\alpha}\mathcal{O}(f'(PU_{\varepsilon,P})v_1).$$

Proof. We have

$$\begin{aligned}E_\varepsilon[V_{\varepsilon,P}(x)] &= E_\varepsilon[PU_{\varepsilon,P}(x)] + \varepsilon^{2+\alpha}\delta^{1-\alpha}\Delta v_1\\ &+ \{f(PU_{\varepsilon,P}(x)+\varepsilon^\alpha\delta^{1-\alpha}v_1) - f(PU_{\varepsilon,P}(x))\}.\end{aligned}$$

When $|x-P|<\delta$ we have

$$\begin{aligned}E_\varepsilon[V_{\varepsilon,P}(x)] &= \varepsilon^2\mathcal{O}(f''(U_{\varepsilon,P}+\varepsilon\varphi_0)\varphi_0^2) + \varepsilon^{2+\alpha}\delta^{1-\alpha}\Delta v_1\\ &+ \varepsilon^\alpha\delta^{1-\alpha}\mathcal{O}(f'(U_{\varepsilon,P}+\varepsilon\varphi_0)v_1).\end{aligned}$$

In the neck region, $\delta<|x-P|<2\delta$ we have

$$\begin{aligned}E_\varepsilon[V_{\varepsilon,P}(x)] &= \varepsilon^{2+\alpha}\delta^{1-\alpha}\mathcal{O}(\frac{1}{|x-P|^2}) + \varepsilon^{2+\alpha}\delta^{1-\alpha}\Delta v_1\\ &+ \varepsilon^\alpha\delta^{1-\alpha}\mathcal{O}(f'(U_{\varepsilon,P}+\varepsilon\varphi_0)v_1).\end{aligned}$$

□

Lemma 6.3. *Moreover, if $P \in \partial\Omega$, then*

$$c_\varepsilon \le \frac{\varepsilon^2}{2} I_\infty + \varepsilon^{2\alpha+2}\Phi_q(P) + o(\varepsilon^{2\alpha+2}).$$

Proof. For $t > 0$ let $\beta(t) = I_\varepsilon(tV_{\varepsilon,P})$, then by Lemma 2.1 we have

$$c_\varepsilon \le \max_{t>0} \beta(t)$$

and hence there exists a unique $t_\varepsilon > 0$ such that

$$\beta(t_\varepsilon) = \max_{t>0} \beta(t) \text{ and } \beta'(t_\varepsilon) = 0.$$

We claim that $t_\varepsilon = 1 + \mathcal{O}(\varepsilon^{\alpha+\sigma'})$ for some $\sigma' > 0$ sufficiently small. We have

$$\begin{aligned} \langle I'_\varepsilon(V_{\varepsilon,P}), V_{\varepsilon,P}\rangle &= \int_\Omega \Big(\varepsilon^2|\nabla V_{\varepsilon,P}|^2 - (V_{\varepsilon,P})_+^{p+1} + (V_{\varepsilon,P})_+^{q+1}\Big) \\ &= \int_\Omega E_\varepsilon[V_{\varepsilon,P}]V_{\varepsilon,P} = \mathcal{O}(\varepsilon^{2\alpha+2+\sigma'}). \end{aligned} \tag{6.5}$$

Since $\langle I'_\varepsilon(t_\varepsilon V_{\varepsilon,P}), V_{\varepsilon,P}\rangle = 0$ and $\langle I'_\varepsilon(V_{\varepsilon,P}), V_{\varepsilon,P}\rangle = \mathcal{O}(1)\varepsilon^{2+2\alpha}$, we have

$$\langle I'_\varepsilon(t_\varepsilon V_{\varepsilon,P}) - I'_\varepsilon(V_{\varepsilon,P}), V_{\varepsilon,P}\rangle = \mathcal{O}(1)\varepsilon^{2(\alpha+1)+\sigma'}$$

which implies

$$(t_\varepsilon^2 - 1)\int_\Omega \varepsilon^2|\nabla V_{\varepsilon,P}|^2 - (t_\varepsilon^{p+1}-1)\int_\Omega (V_{\varepsilon,P})_+^{p+1} + (t_\varepsilon^{q+1}-1)\int_\Omega (V_{\varepsilon,P})_+^{q+1} = \mathcal{O}(1)\varepsilon^{2+2\alpha+\sigma'}$$

and letting $\tilde{V}_{\varepsilon,P}(x) = V_{\varepsilon,P}(\varepsilon x + P)$ in Ω_ε we have

$$(t_\varepsilon^2 - 1)\int_{\Omega_\varepsilon} |\nabla \tilde{V}_{\varepsilon,P}|^2 - (t_\varepsilon^{p+1}-1)\int_{\Omega_\varepsilon} (\tilde{V}_{\varepsilon,P})_+^{p+1} + (t_\varepsilon^{q+1}-1)\int_{\Omega_\varepsilon} (\tilde{V}_{\varepsilon,P})_+^{q+1} = \mathcal{O}(1)\varepsilon^{\sigma'+\alpha}$$

which implies that $t_\varepsilon - 1 = \mathcal{O}(1)\varepsilon^{\alpha+\sigma'}$. Furthermore,

$$\begin{aligned} J''_\varepsilon(V_{\varepsilon,P})\langle V_{\varepsilon,P}, V_{\varepsilon,P}\rangle &= \int_{\Omega_\varepsilon} \Big(\varepsilon^2|\nabla V_{\varepsilon,P}|^2 - p(V_{\varepsilon,P})_+^{p+1} + q(V_{\varepsilon,P})_+^{q+1}\Big) \\ &= \varepsilon^N \int_{\mathbb{R}^N} \Big(-(p-1)U^{p+1} + (q-1)U^{q+1}\Big) + O(1)\varepsilon^{\alpha(q+1)} \\ &= \varepsilon^2\Big(-(p-q)\int_{\mathbb{R}^2} U^{p+1} - (q-1)\int_{\mathbb{R}^2} |\nabla U|^2 + o(1)\Big) \\ &= \mathcal{O}(\varepsilon^2). \end{aligned} \tag{6.6}$$

As a result, we obtain

$$\begin{aligned} I_\varepsilon(u_\varepsilon) &\le \max_{t>0} I_\varepsilon(tV_{\varepsilon,P}) = J_\varepsilon(t_\varepsilon V_{\varepsilon,P}) \\ &= I_\varepsilon(V_{\varepsilon,P}) + (t_\varepsilon - 1)\langle I'_\varepsilon(V_{\varepsilon,P}), V_{\varepsilon,P}\rangle + (t_\varepsilon - 1)^2\mathcal{O}(\varepsilon^2) \\ &\le J_\varepsilon(V_{\varepsilon,P}) + o(1)\varepsilon^{2+2\alpha} \\ &= \frac{\varepsilon^2}{2} I_\infty + \varepsilon^{2+2\alpha}\Phi_q(P) + o\left(\varepsilon^{2+2\alpha}\right). \end{aligned}$$

□

Lemma 6.4. *For sufficiently small $\varepsilon > 0$, u_ε has a unique maximum $P_\varepsilon \in \partial\Omega$.*

Proof. First note by an application of mountain pass theorem, $\varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2 \leq C$ and hence by Moser iteration, $u_\varepsilon(x)$ is uniformly bounded. Thus applying Schauder estimates we obtain a $C > 0$ such that $\|\varepsilon D u_\varepsilon\|_{L^\infty} \leq C$. Let $P_\varepsilon \in \overline{\Omega}$ be a local maxima of u_ε. If $P_\varepsilon \in \Omega$, then $u_\varepsilon(P_\varepsilon) \geq 1$. If $P_\varepsilon \in \partial\Omega$ then there exists a point S_ε such that $u_\varepsilon(S_\varepsilon) \geq 1$, otherwise by the boundary Hopf lemma we must have $\frac{\partial u_\varepsilon(P_\varepsilon)}{\varepsilon} > 0$, a contradiction. Suppose $\frac{d(P_\varepsilon, \partial\Omega)}{\varepsilon} \to +\infty$, as $\varepsilon \to 0$, then by the change of variable $v_\varepsilon(x) = u_\varepsilon(P_\varepsilon + \varepsilon x)$ and v_ε satisfies

$$
\begin{cases}
\Delta v_\varepsilon - v_\varepsilon^q + v_\varepsilon^p = 0 & \text{in } \Omega_{\varepsilon, P_\varepsilon} \\
v_\varepsilon(x) > 0 & \text{in } \Omega_{\varepsilon, P_\varepsilon} \\
\dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega_{\varepsilon, P_\varepsilon}
\end{cases}
\tag{6.7}
$$

where $\Omega_{\varepsilon, P_\varepsilon} = \frac{1}{\varepsilon}(\Omega - P_\varepsilon)$ and $v_\varepsilon \to v$ in C^2_{loc} where

$$
\begin{cases}
\Delta v - v^q + v^p = 0 & \text{in } \mathbb{R}^2 \\
v(x) > 0 & \text{in } \mathbb{R}^2 \\
u(x) \to 0 \text{ as } |x| \to \infty
\end{cases}
\tag{6.8}
$$

Using this we can show that $c_\varepsilon = \varepsilon^2(I_\infty + o(1))$, a contradiction to Lemma 6.3. As a result, $\frac{d(P_\varepsilon, \partial\Omega)}{\varepsilon}$ is uniformly bounded. If possible, let $P_{\varepsilon,1}$ and $P_{\varepsilon,2}$ are two distinct local maxima of u_ε. Then $u_\varepsilon(P_{\varepsilon,1}) \geq 1$ and $u_\varepsilon(P_{\varepsilon,2}) \geq 1$. Suppose $Q_\varepsilon = \frac{P_{\varepsilon,1} - P_{\varepsilon,2}}{\varepsilon}$. Suppose along a subsequence $|Q_\varepsilon| \to \delta_0 \in [0, +\infty)$. Let $Q = \lim_{\varepsilon \to 0} \frac{P_{\varepsilon,1} - P_{\varepsilon,2}}{\varepsilon}$. Then if $\delta_0 > 0$, then define $v_\varepsilon(y) = u_\varepsilon(\varepsilon y + P_{\varepsilon,2})$ then it follows that, $v_\varepsilon \to U$ in $C^2_{loc}(\mathbb{R}^N)$ and satisfies

$$
\begin{cases}
-\Delta U = U^p - U^q & \text{in } \mathbb{R}^2 \\
U'(0) = U'(\delta_0) = 0 & \\
U \to 0 & \text{as } |x| \to \infty
\end{cases}
$$

which is a contradiction as $U'(r) < 0$ for $r \in (0, +\infty)$. Now suppose $\delta_0 = 0$. Then $v_\varepsilon \to U$ in $C^2_{loc}(\mathbb{R}^2)$ and U has a unique critical point at 0 (since $U(0) > 1$ and U is a radial). Thus v_ε has a critical point in a neighborhood of zero which is a contradiction. Hence $|Q_\varepsilon| \to +\infty$ as $\varepsilon \to 0$.

We claim that u_ε has exactly one maximum for sufficiently small $\varepsilon > 0$. First note that as u_ε is a mountain pass solution and hence it has Morse index at most one. By the above result $\frac{|P_{1,\varepsilon} - P_{2,\varepsilon}|}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Now by Section 2, the principal eigenvalue $\lambda_1 > 0$ such that $\Delta\psi + f'(U)\psi = -\lambda_1\psi$ and is easy to check that $\psi_1 \in \mathcal{D}(\mathbb{R}^2)$ hence $\int_{\mathbb{R}^2} |\nabla\psi|^2 - f'(U)\psi^2 < 0$. Now using an appropriate cut-off function, we can obtain the same property for ψ with compact support. Now define a two dimensional subspace spanned by $\psi_1(x) = \psi(\frac{x - P_{1,\varepsilon}}{\varepsilon})$ and $\psi_2(x) = \psi(\frac{x - P_{2,\varepsilon}}{\varepsilon})$ where $x \in \Omega$. Note that the support $supp\, \psi_1 \cap supp\, \psi_2 = \emptyset$ as $\frac{|P_{1,\varepsilon} - P_{2,\varepsilon}|}{\varepsilon} \to +\infty$. Hence we obtain a two dimensional space on which $\varepsilon^2 \int_\Omega |\nabla\psi_i|^2 - f'(u_\varepsilon)\psi_i^2 = \int_{\mathbb{R}^N} |\nabla\psi_i|^2 - f'(U)\psi_i^2 < 0$ for $i = 1, 2$. As $u_\varepsilon \to U$ in $C^2_{loc}(\mathbb{R}^2)$ and ψ_i has compact support. Hence u_ε has Morse index at least two, a contradiction.

The proof of $P_\varepsilon \in \partial\Omega$ follows exactly as Ni-Takagi [16]. □

7. LOWER BOUND

First we prove that

Lemma 7.1. *There exists constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1\varepsilon^\alpha G_q(x,P_\varepsilon) \le u_\varepsilon(x) \le C_2\varepsilon^\alpha G_q(x,P_\varepsilon) \text{ in } \Omega\setminus\Omega\cap B_{\varepsilon R}(P_\varepsilon) \tag{7.1}$$

for some $R > 0$ sufficiently large.

Proof. In $\Omega\setminus B_{\varepsilon R}(x_\varepsilon)$, u_ε and $\varepsilon^\alpha G_q(.,P_\varepsilon)$ are bounded. We have $\varepsilon^2\Delta u_\varepsilon - u_\varepsilon^q = -u_\varepsilon^p \le 0$ and $\Delta G_q - G_q^q = 0$. Note $u_\varepsilon(P_\varepsilon) = \|u_\varepsilon\|_\infty \ge 1$. Since by Hopf maximum principle we can choose $0 < \eta < 1$ such that

$$\frac{\partial u_\varepsilon}{\partial\nu} \le \varepsilon^\alpha\eta\frac{\partial G_q(x,P_\varepsilon)}{\partial\nu} \text{ on } \partial(\Omega\setminus\Omega\cap B_{\varepsilon R}(P_\varepsilon)).$$

Then we have

$$\Delta(\eta G_q) - (\eta G_q)^q = \eta\Delta G_q - \eta^q G_q^q = (\eta-\eta^q)G_q^q \ge 0. \tag{7.2}$$

Hence

$$\varepsilon^2\Delta(u_\varepsilon - \eta\varepsilon^\alpha G_q) - u_\varepsilon^q + (\eta\varepsilon^\alpha G_q)^q \le 0$$

which implies that

$$\varepsilon^2\Delta(u_\varepsilon - \eta\varepsilon^\alpha G_q) - \frac{u_\varepsilon^q - (\eta\varepsilon^\alpha G_q)^q}{u_\varepsilon - \eta\varepsilon^\alpha G_q}(u_\varepsilon - \eta\varepsilon^\alpha G_q) \le 0.$$

Hence by the maximum principle we have $u_\varepsilon \ge \eta\varepsilon^\alpha G_q$ in $\Omega\setminus B_{\varepsilon R}(P_\varepsilon)$.
For the upper bound, let $0 < \theta < 1$ such that $u_\varepsilon < \theta$ in $\Omega\setminus B_{\varepsilon R}(P_\varepsilon)$ and $\eta_1 \gg 1$ such that

$$\frac{\partial u_\varepsilon}{\partial\nu} \ge \varepsilon^\alpha\eta_1\frac{\partial G_q(x,P_\varepsilon)}{\partial\nu} \text{ on } \partial(\Omega\setminus\Omega\cap B_{\varepsilon R}(P_\varepsilon)).$$

then we have

$$\Delta(\eta_1 G_q) - (\eta_1 G_q)^q = \eta_1\Delta G_q - \eta_1^q G_q^q = (\eta_1-\eta_1^q)G_q^q. \tag{7.3}$$

Then u_ε satisfies

$$\varepsilon^2\Delta u_\varepsilon - u_\varepsilon^q \ge -\theta^p \text{ in } \Omega\setminus B_{\varepsilon R}(P_\varepsilon).$$

As a result, we obtain

$$\varepsilon^2\Delta(u_\varepsilon - \eta_1\varepsilon^\alpha G_q) - \frac{u_\varepsilon^q - (\eta_1\varepsilon^\alpha G_q)^q}{u_\varepsilon - \eta_1\varepsilon^\alpha G_q}(u_\varepsilon - \eta_1\varepsilon^\alpha G_q) \ge -\theta^p - (\eta_1-\eta_1^q)G_q^q \ge 0.$$

Hence we obtain by the maximum principle in $\Omega\setminus B_{\varepsilon R}(P_\varepsilon)$

$$u_\varepsilon(x) \le C_2\varepsilon^\alpha G_q(x,P_\varepsilon).$$

□

In order to obtain the lower bound we define

$$u_\varepsilon = V_{\varepsilon,P_\varepsilon} + \varepsilon^\alpha\psi_\varepsilon \tag{7.4}$$

If we plug this in equation (1.2), then $\psi_\varepsilon \in H^1(\Omega)$ satisfies

$$\begin{cases} \varepsilon^2\Delta\psi_\varepsilon + f'(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon = -\varepsilon^{-\alpha}E_\varepsilon[V_{\varepsilon,P_\varepsilon}] + N_\varepsilon[\psi_\varepsilon] & \text{in } \Omega,\\ \dfrac{\partial\psi_\varepsilon}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.5}$$

where

$$N_\varepsilon[\psi_\varepsilon] = \varepsilon^{-\alpha}\{f(V_{\varepsilon,P_\varepsilon} + \varepsilon^\alpha\psi_\varepsilon) - f(V_{\varepsilon,P_\varepsilon}) - \varepsilon^\alpha f'(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon\}.$$

Lemma 7.2. *For sufficiently small $\varepsilon > 0$, there exists $C > 0$ such that*

$$\|\psi_\varepsilon\|_{L^\infty(\overline{\Omega})} \leq C. \tag{7.6}$$

Proof. We claim that ψ_ε is uniformly bounded. If possible, let there exists a sequence ε_k such that $\|\psi_{\varepsilon,k}\|_\infty \to \infty$. Let $|\psi_\varepsilon|$ have its maximum at a point $k_\varepsilon \in \overline{\Omega}$. As $\frac{\partial \psi_\varepsilon}{\partial \nu} = 0$ by Hopf's lemma $k_\varepsilon \in int(\Omega)$.

We claim that $\frac{|k_\varepsilon - P_\varepsilon|}{\varepsilon} < C$.
Suppose this is not true then $\frac{|k_\varepsilon - P_\varepsilon|}{\varepsilon} \to +\infty$. Then we have three cases; $|P_\varepsilon - k_\varepsilon| \leq \delta$, $\delta < |P_\varepsilon - k_\varepsilon| \leq 2\delta$ or $|P_\varepsilon - k_\varepsilon| \geq 2\delta$.

Case 1 When $|P_\varepsilon - k_\varepsilon| \geq 2\delta$, and as a result $-\Delta\psi_\varepsilon(k_\varepsilon) \geq 0$ and there exists a $c > 0$ such that $\psi_\varepsilon(k_\varepsilon) \geq c$. We have from (7.5)

$$0 \leq -\varepsilon^{2+\alpha}\Delta\psi_\varepsilon(k_\varepsilon) = \{f(V_{\varepsilon,P_\varepsilon}(k_\varepsilon) + \varepsilon^\alpha\psi_\varepsilon(k_\varepsilon)) - f(V_{\varepsilon,P_\varepsilon})\} - E_\varepsilon[V_{\varepsilon,x_\varepsilon}]$$

which reduces to

$$(G_q(k_\varepsilon, P_\varepsilon) + \delta^{1-\alpha}v_1(k_\varepsilon) + c)^q \leq G_q^q(k_\varepsilon, P_\varepsilon) + o(1)$$

and hence a contradiction.

Case 2 When $|P_\varepsilon - k_\varepsilon| < \delta$. Then $\varepsilon R < |P_\varepsilon - k_\varepsilon| < \delta$

$$\{f(V_{\varepsilon,P_\varepsilon}(k_\varepsilon) + \varepsilon^\alpha\psi_\varepsilon(k_\varepsilon)) - f(V_{\varepsilon,P_\varepsilon})\} - E_\varepsilon[V_{\varepsilon,P_\varepsilon}] \geq 0.$$

This implies that

$$\left(\frac{1}{|k_\varepsilon - P_\varepsilon|^\alpha} + c + o(1)\right) \leq \left(\frac{1}{|k_\varepsilon - P_\varepsilon|^\alpha}\right)$$

which is a contradiction. The other case is much easier to handle.

Thus we consider $\psi_\varepsilon(x) = \psi_\varepsilon(k_\varepsilon + \varepsilon x)$

$$\Psi_\varepsilon = \frac{\psi_\varepsilon}{\|\psi_\varepsilon\|_\infty}.$$

By the Schauder estimates, we obtain $\|\Psi_\varepsilon\|_{C^{1,\theta}_{loc}}$ is bounded for some $\theta \in (0,1]$ and hence by the Arzela-Ascoli's theorem there exists $\Psi_0 \in C^1$ such that $\|\Psi_\varepsilon - \Psi_0\|_{C^1_{loc}} \to 0$ as $\varepsilon \to 0$. Using the fact that $\frac{d(k_\varepsilon, \partial\Omega)}{\varepsilon} \leq C$, ψ_0 satisfies

$$\begin{cases} \Delta\Psi_0 + f'(U)\Psi_0 = 0 & \text{in } \mathbb{R}^2_+ \\ |\Psi_0| \leq 1 & \\ \dfrac{\partial\Psi_0}{\partial y_2} = 0 & \text{in } \partial\mathbb{R}^2_+ \end{cases} \tag{7.7}$$

Now we show that $\Psi_0 \in \mathcal{D}$.

We obtain a contradiction by showing that $\nabla\Psi_0(0) = 0$. Using the fact that $\nabla u_\varepsilon(P_\varepsilon) = 0$ and

$$\nabla\Psi_\varepsilon(0) = \frac{\nabla u_\varepsilon(P_\varepsilon) - \nabla V_{\varepsilon,P_\varepsilon}(P_\varepsilon)}{\varepsilon^\alpha\|\psi_\varepsilon\|_\infty}$$

we obtain $\nabla\Psi_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. This implies that $\nabla\Psi_0(0) = 0$ by pointwise convergence and hence $\nabla(a_1\frac{\partial U}{\partial x_1})(0) = 0$ and this implies that $a_1 = 0$.

□

Lemma 7.3. *We have,*

$$c_\varepsilon = \frac{\varepsilon^2}{2} I_\infty(U) + \varepsilon^{2\alpha+2}\Phi_q(P_\varepsilon) + o(\varepsilon^{2(\alpha+1)}). \tag{7.8}$$

Proof. We want to write $u_\varepsilon = V_{\varepsilon,P_\varepsilon} + \varepsilon^\alpha \psi_\varepsilon$. So we have

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= J_\varepsilon(V_{\varepsilon,P_\varepsilon}) \\ &+ \varepsilon^\alpha \int_\Omega (\varepsilon^2 \nabla V_{\varepsilon,P_\varepsilon} \nabla \psi_\varepsilon - f(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon) dx \\ &+ \frac{\varepsilon^{2\alpha}}{2}\left(\int_\Omega \varepsilon^2 |\nabla \psi_\varepsilon|^2 dx - f'(V_{\varepsilon,x_\varepsilon})\psi_\varepsilon^2 \right) \\ &- \int_\Omega \left[F(V_{\varepsilon,P_\varepsilon} + \varepsilon^\alpha \psi_\varepsilon) - F(V_{\varepsilon,P_\varepsilon}) - \varepsilon^\alpha f(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon - \frac{\varepsilon^{2\alpha}}{2} f'(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon^2 \right]. \end{aligned}$$

which can be expressed as

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &= J_\varepsilon(V_{\varepsilon,P_\varepsilon}) \\ &+ \varepsilon^\alpha \int_\Omega E_\varepsilon[V_{\varepsilon,P_\varepsilon}]\psi_\varepsilon dx \\ &+ \frac{\varepsilon^{2\alpha}}{2}\left(\varepsilon^2 \int_\Omega |\nabla \psi_\varepsilon|^2 dx - f'(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon^2 \right) \\ &- \int_\Omega \left[F(V_{\varepsilon,P_\varepsilon} + \varepsilon^\alpha \psi_\varepsilon) - F(V_{\varepsilon,P_\varepsilon}) - \varepsilon^\alpha f(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon - \frac{\varepsilon^{2\alpha}}{2} f'(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon^2 \right]. \end{aligned}$$

Now we estimate the following terms

$$\begin{aligned} \int_\Omega E_\varepsilon[V_{\varepsilon,P_\varepsilon}]\psi_\varepsilon dx &= \int_{|x-P_\varepsilon|<\varepsilon R} E_\varepsilon[V_{\varepsilon,P_\varepsilon}]\psi_\varepsilon + \int_{\varepsilon R<|x-P_\varepsilon|<2\delta} E_\varepsilon[V_{\varepsilon,P_\varepsilon}]\psi_\varepsilon \\ &+ \int_{\delta<|x-P_\varepsilon|<2\delta} E_\varepsilon[V_{\varepsilon,P_\varepsilon}]\psi_\varepsilon + \int_{|x-P_\varepsilon|>2\delta} E_\varepsilon[V_{\varepsilon,P_\varepsilon}]\psi_\varepsilon \\ &\le C\varepsilon^4 + C\varepsilon^{2+\alpha}\delta^{1-\alpha}|\log\delta| \\ &+ C\varepsilon^{2+\alpha+\sigma'} \int_{\delta<|x-P_\varepsilon|<2\delta} \frac{1}{|x-P_\varepsilon|^2} + \varepsilon^{\alpha p} \int_{|x-P_\varepsilon|>2\delta} G_q^p \psi_\varepsilon \\ &\le o(1)\varepsilon^{\alpha+2}. \end{aligned}$$

¿From (7.5)

$$\int_\Omega \{\varepsilon^2 |\nabla\psi_\varepsilon|^2 dx - f'(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon^2\} = \varepsilon^{-\alpha}\int_\Omega E_\varepsilon[V_{\varepsilon,P_\varepsilon}]\psi_\varepsilon - \int_\Omega N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon.$$

As a result, we only estimate

$$\begin{aligned} \int_\Omega N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon &= \int_{|x-P_\varepsilon|\le\varepsilon R} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon + \int_{\varepsilon R<|x-P_\varepsilon|\le\delta} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon \\ &+ \int_{\delta<|x-P_\varepsilon|<2\delta} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon + \int_{|x-P_\varepsilon|\ge 2\delta} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon \\ &= I_1 + I_2 + \int_{\delta<|x-P_\varepsilon|<2\delta} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon + \int_{|x-P_\varepsilon|\ge 2\delta} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon. \end{aligned}$$

We compute I_1. As $q > 5$, we obtain

$$I_1 = \varepsilon^\alpha \mathcal{O}\Big(\int_{B_{\varepsilon R}(P_\varepsilon)} (U_{\varepsilon,P_\varepsilon} + \varepsilon\varphi_0)^{q-2}\psi_\varepsilon^3 \Big) = \mathcal{O}(\varepsilon^{\alpha+2}).$$

We calculate I_2.

$$\begin{aligned} I_2 &= \varepsilon^\alpha \mathcal{O}\Big(\int_{B_\delta(P_\varepsilon)\setminus B_{\varepsilon R}(P_\varepsilon)} (U_{\varepsilon,P_\varepsilon} + \varepsilon\varphi_0)^{q-2}\psi_\varepsilon^3 \Big) \\ &= \varepsilon^\alpha \mathcal{O}\Big(\int_{B_\delta(P_\varepsilon)\setminus B_{\varepsilon R}(P_\varepsilon)} \frac{\varepsilon^{2-\alpha}}{|x-P_\varepsilon|^{2-\alpha}} \Big) = \mathcal{O}(\varepsilon^2\delta^\alpha). \end{aligned}$$

Estimating in the neck region

$$\int_{\delta<|x-P_\varepsilon|<2\delta} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon = \mathcal{O}\Big(\varepsilon^\alpha \int_{\delta<|x-P_\varepsilon|<2\delta} V_{\varepsilon,P_\varepsilon}^{q-2}\psi_\varepsilon^3 \Big).$$

In the neck region we have

$$V_{\varepsilon,P_\varepsilon} = U_{\varepsilon,P_\varepsilon} + (1-\eta)(\varepsilon^\alpha G_q - U_{\varepsilon,P_\varepsilon} - \varepsilon\varphi_0).$$

In order to estimate

$$\begin{aligned} \varepsilon^\alpha \int_{\delta<|x-P_\varepsilon|<2\delta} V_{\varepsilon,P_\varepsilon}^{q-2}\psi_\varepsilon^3 &= \varepsilon^2 \int_{\delta<|x-P_\varepsilon|<2\delta} \frac{1}{|x-P_\varepsilon|^{\alpha(q-2)}}\psi_\varepsilon^3 \\ &\le C\varepsilon^2 \int_{\delta<|x-P_\varepsilon|<2\delta} \frac{1}{|x-P_\varepsilon|^{2-\alpha}} \\ &= \mathcal{O}(\varepsilon^2\delta^\alpha). \end{aligned}$$

Whenever $|x-P_\varepsilon| > 2\delta$, we have

$$\int_{|x-P_\varepsilon|\ge 2\delta} N_\varepsilon[\psi_\varepsilon]\psi_\varepsilon = o(\varepsilon^{\alpha q}).$$

Similarly, we show that

$$\begin{aligned} &\int_\Omega \Big[F(V_{\varepsilon,P_\varepsilon} + \varepsilon^\alpha\psi_\varepsilon) - F(V_{\varepsilon,P_\varepsilon}) - \varepsilon^\alpha f(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon - \frac{\varepsilon^{2\alpha}}{2} f'(V_{\varepsilon,P_\varepsilon})\psi_\varepsilon^2 \Big] \\ = \ & o(\varepsilon^{2+2\alpha}). \end{aligned}$$

The estimate follows exactly as the previous estimate. This completes the proof. □

Remark 7.1. *As a result of Lemma 6.3 and Lemma 7.3, we obtain* $\Phi_q(P_\varepsilon) \to \min_{P\in\partial\Omega} \Phi_q(P)$. *Hence Theorem 1.1 is proved.*

8. Profile of Spikes $N = 2$ and $q = 5$

In this case $\alpha = \frac{1}{2}$. The proof of Theorem 1.1 remains almost the same. So we calculate only estimate (8.1) as K is not integrable. So we have

$$\begin{aligned} \varepsilon^2 \int_{B^+_{\frac{\delta}{\varepsilon}}(P)\setminus\Omega_\varepsilon} U^6 &= \varepsilon^2 \int_0^{\frac{\delta}{\varepsilon}} \int_0^{\frac{f(\varepsilon y_1)}{\varepsilon}} U^6(y_1,y_2)dy_2dy_1 \\ &= \varepsilon^2 \int_0^{\frac{\delta}{\varepsilon}} \int_0^{\frac{f(\varepsilon y_1)}{\varepsilon}} [U^6(y_1,0) + \mathcal{O}(|y_2|U^6(y',0))]dy_2dy_1 \\ &= \frac{\varepsilon^3 H(P)}{2} \int_0^{\frac{\delta}{\varepsilon}} [U^6(y_1,0)y_1^2dy_1 + \mathcal{O}(\varepsilon^2)U^6(y_1,0)y_1^3]dy_1. \qquad (8.1) \end{aligned}$$

As $U^6(y_1,0) \sim \frac{\omega_q^6}{y_1^3}$, we estimate the first term in (8.2) in the following way,

$$\begin{aligned} \frac{\varepsilon^3 H(P)}{2}\int_0^{\frac{\delta}{\varepsilon}} U^6(y_1,0)y_1^2 dy_1 &= \frac{\varepsilon^3 H(P)}{2}\int_0^{R} U^6(y_1,0)y_1^2 dy_1 + \frac{\varepsilon^3 H(P)}{2}\int_R^{\frac{\delta}{\varepsilon}} U^6(y_1,0)y_1^2 dy_1 \\ &= \mathcal{O}(\varepsilon^3) + \frac{\omega_q^6 H(P)}{2}\varepsilon^3 \int_R^{\frac{\delta}{\varepsilon}} \frac{1}{y_1} dy_1 \\ &= \frac{\omega_q^6 H(P)\varepsilon^3}{2}\log\frac{\delta}{\varepsilon} + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{8.2}$$

Moreover, it is also easy to check that

$$\varepsilon^2 \int_\Omega |\nabla U_{\varepsilon,P}|^2 = -\frac{\omega_q^4 H(P)\varepsilon^3}{2}\log\frac{\delta}{\varepsilon} + \mathcal{O}(\varepsilon^3) \tag{8.3}$$

As $\delta = \varepsilon^{\sigma_0}$ we have from (8.2) and (8.3)

$$I_\varepsilon(u_\varepsilon) = \frac{\varepsilon^2}{2} I_\infty - \frac{1-\sigma_0}{8}\varepsilon^3\Big(\log\frac{1}{\varepsilon}\Big) H(P_\varepsilon) + o\Big(\varepsilon^4\Big(\log\frac{1}{\varepsilon}\Big)\Big). \tag{8.4}$$

as $\omega_q = \frac{1}{\sqrt{2}}$.

9. Profile of Spikes $N = 3$ and $q > 3$

When $q > 3$, $U(r) \sim \frac{\gamma_3}{r}$ as $r \to +\infty$. The projection $PU_{\varepsilon,P} = \eta U_{\varepsilon,P}$ where η is the same cut-off function defined in (5.9). In this case we perform the reduction in $\mathcal{D}^{1,2}(\mathbb{R}^3_+)$. Note that in this case K is not integrable. Therefore from Lemma 1.1 we estimate the terms involved in K. Note that in this case $\varepsilon^2|\nabla U_{\varepsilon,P}|^2$ is the lowest order term in the energy expansion and hence

$$\begin{aligned} \varepsilon^2 \int_\Omega |\nabla U_{\varepsilon,P}|^2 &= \varepsilon^2 \int_{\partial\Omega} U_{\varepsilon,P}\frac{\partial U_{\varepsilon,P}}{\partial \nu} + \int_\Omega U_{\varepsilon,P} f(U_{\varepsilon,P}) \\ &= \varepsilon^2 \int_{\partial\Omega\cap B_\delta(P)} U_{\varepsilon,P}\frac{\partial U_{\varepsilon,P}}{\partial \nu} + \mathcal{O}(\varepsilon^4) \end{aligned} \tag{9.1}$$

Now from (1.16) we have

$$\frac{\partial U_\varepsilon}{\partial \nu} = \frac{1}{\varepsilon}(1+|\nabla_{x'} f|^2)^{-\frac{1}{2}}\Big[\sum_{i=1}^{2}\frac{\partial f}{\partial y_i}\frac{\partial U_{\varepsilon,P}}{\partial z_i} - \frac{\partial U_{\varepsilon,P}}{\partial z_N}\Big].$$

Thus we have

$$
\begin{aligned}
\varepsilon^2 \int_{\partial\Omega\cap B_\delta(P)} U_{\varepsilon,P}\frac{\partial U_{\varepsilon,P}}{\partial\nu} &= \varepsilon\int_{B^2_\delta(P)}\left[\sum_{i=1}^{2}\frac{\partial f}{\partial y_i}\frac{\partial U_{\varepsilon,P}}{\partial z_i} - \frac{\partial U_{\varepsilon,P}}{\partial z_N}\right]dy' \\
&= \varepsilon^3\int_{B^2_{\frac{\delta}{\varepsilon}}(0)} U\left(y', \frac{f(\varepsilon y')}{\varepsilon}\right)\left[\sum_{i=1}^{2}(\varepsilon k_i y_i + (\varepsilon^2|y'|^2))\right. \\
&\times \left.\frac{\partial U(y',\frac{f(\varepsilon y')}{\varepsilon})}{\partial y_i} - \frac{\partial U(y',\frac{f(\varepsilon y')}{\varepsilon})}{\partial y_N}\right] \\
&= \varepsilon^3\left[\int_{B^2_{\frac{\delta}{\varepsilon}}(0)} U(y',0)\frac{\partial U(y',0)}{\partial r}\sum_{i=1}^{2}k_i y_i^2|y'|^{-1}\varepsilon\right. \\
&- \left.\int_{B^2_{\frac{\delta}{\varepsilon}}(0)} U(y',0)\frac{\partial^2 U(y',0)}{\partial y_N^2}\sum_{i=1}^{2}k_i y_i^2\varepsilon + \mathcal{O}(\varepsilon^2)\right] \\
&= \varepsilon^4\frac{H(P)}{2}\int_{B^2_{\frac{\delta}{\varepsilon}}(0)} U(y',0)\frac{\partial U(y',0)}{\partial r}|y'|dy' \\
&+ o\left(\varepsilon^4\left(\log\frac{1}{\varepsilon}\right)\right) \\
&= -\varepsilon^4\left(\log\frac{1}{\varepsilon}\right)\frac{H(P)}{2}\gamma_3^2 + o\left(\varepsilon^4\left(\log\frac{1}{\varepsilon}\right)\right)
\end{aligned}
$$

using the fact that

$$
\frac{\partial U(y',0)}{\partial r}|y'|^{-1} = \frac{\partial^2 U(y',0)}{\partial y_N^2}.
$$

10. Profile of Spikes $N = 3$ and $q = 3$

When $q = 3$, by Lemma 1.1 of [7], we have $U(r) \sim \frac{1}{\sqrt{2}}\frac{1}{r\sqrt{\log r}}$ as $r \to \infty$ and $|U_r|^2 \sim \frac{1}{4}\frac{1}{r^4\log r}$. Note that in this $\varepsilon^2|\nabla U_{\varepsilon,P}|^2$ and $U^4_{\varepsilon,P}$ are of the same order and are the lowest order term in the energy expansion and hence we have from (9.1) and $R \gg 1$

$$
\begin{aligned}
\varepsilon^2\int_\Omega |\nabla U_{\varepsilon,P}|^2 &= \varepsilon^4\frac{H(P)}{2}\int_{B^2_{\frac{\delta}{\varepsilon}}(0)} U(y',0)\frac{\partial U(y',0)}{\partial r}|y'|dy' + o\left(\varepsilon^4\left(\log(\log\frac{1}{\varepsilon})\right)\right) \\
&= \varepsilon^4\frac{H(P)}{4}\int_R^{\delta/\varepsilon}\frac{1}{r(\log r)}dr + o\left(\varepsilon^4\left(\log\left(\log\frac{1}{\varepsilon}\right)\right)\right) \\
&= -\varepsilon^4\frac{H(P)}{4}\left(\log\left(\log\frac{1}{\varepsilon}\right)\right) + o\left(\varepsilon^4\left(\log\left(\log\frac{1}{\varepsilon}\right)\right)\right).
\end{aligned}
$$

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