Asymptotic behavoir of solutions of a biharmonic Dirichlet problem with large exponents*

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Abstract

We analyse the blow up phenomena of bounded integrable solutions of a semilinear fourth order elliptic problem with a large exponent under Dirichlet boundary conditions. We extend the results obtained by Ren-Wei in [16] and [17] for the biharmonic case.

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1 Introduction

The study concerning the asymptotic behavior of elliptic equations of fourth order equations is of considerable interest. A particular feature of biharmonic operator in $\mathbb{R}^4$ is conformally invariant. Let (M, g) be a smooth four-dimensional manifold. More precisely, if we consider a Panietz type of operator,

$$P_g^4\psi = \Delta_g^2\psi + div_g(\frac{2}{3}S_g - 2Ric_g)d\psi$$

where div_g denotes the divergence, d the differential and S_g, Ric_g denote the scalar and Ricci curvature of the metric g respectively. The above equation

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under a conformal transformation reduces to

$$P_g w + 2Q_g = 2\tilde{Q}_{g_w} e^{4w} \tag{1.1}$$

where Q_g is the $Q-$ curvature of the metric g, and Q_{g_w} is the $Q-$ curvature of the new metric g_w and Integration (1.1) over M, one obtains

$$k_g = \int_M Q_g = \int_M \tilde{Q}_g e^{4w}$$

where k_g is conformally-invariant. Hence we can write (1.1)

$$P_g w + 2Q_g = k_g \frac{\tilde{Q}_{g_w} e^{4w}}{\int_M \tilde{Q}_{g_w} e^{4w}} \tag{1.2}$$

When the manifold is the Euclidean space then (1.2) transforms to

$$\Delta^2 w = \rho \frac{h(x)e^{4w}}{\int_\Omega h(x)e^{4w}} \tag{1.3}$$

This type of problem has been extensively studied by Adimurthi, Robert and Struwe [3], Barakat-Dammak-Ouni-Pakard [5], Lin-Wei [14], [15] , Hebey-Robert [10] and many other people. Let $\Omega \subset \mathbb{R}^4$ be a smooth bounded domain. In this paper, we study the asymptotic behavior of a sequence of solutions

$$(S_p) \begin{cases} \Delta^2 u = (u^+)^p & \text{in } \Omega \\ u \not\equiv 0 & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

as the parameter $p \to +\infty$ under the assumption that u satisfies,

$$\int_\Omega (u^+)^{p+1} \leq \frac{C}{p} \tag{1.5}$$

for some $C > 0$ independent of p where $u^\pm = \max\{\pm u, 0\}$. Note that the least energy solution to (1.4) satisfy the condition (1.5). The equation has a very close relationship to (1.3). In course of this paper, we will introduce a blow-up solution for u, which along a subsequence converges to a entire solution of (1.3). In two dimensions an analogous problem was studied in Ren–Wei [16] and [17] in a star-shaped domain for the least energy solutions. For the biharmonic case the problem was studied by Takahasi [20], [21], with the convexity of the domain Ω and for positive solutions in the Navier boundary case. Ben Ayed-El Mehdi-Grossi [6] extended to non-convex domains and proved the single point condensation for least energy solutions, again for positive solutions in the Navier boundary case. Note that neither we have assumed the convexity of the domain Ω nor the positivity of the solution. Define

$$v_p := p u_p$$

Then we call $\mathcal{S}$ a blow-up set of a sequence v_{p_n} if

$\mathcal{S} = \{x \in \overline{\Omega} : \exists$ a subsequence of v_{p_n} and $x_n \in \Omega$ such that $x_n \to x$ and $v_{p_n}(x_n) \to +\infty\}$

Consider the functional $I_p : H_0^2(\Omega) \to \mathbb{R}$

$$I_p(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{1}{p+1}\int_\Omega (u^+)^{p+1} dx.$$

Any solution to (1.4) is in fact a critical point of the above functional and by regularity all solutions u of (1.4) are $C^\infty(\Omega) \cap C^4(\overline{\Omega})$.

Theorem 1.1. *If u_p is a solution of (1.4) satisfying (1.5), then as $p \to +\infty$ then,*

$$\lim_{p \to +\infty} \|u_p\|_\infty = \sqrt{e}.$$

This type of result is proved in [4] and [6] but only for u_p being a least energy solution and we prove the theorem in a general setting.

Theorem 1.2. *Then there exists a subsequence $v_{p,n}$ such that*
(f_1) $p\int_\Omega (u_p^+)^p \to 64\pi^2 N\sqrt{e}$ for some positive integer N.
(f_2) v_p has exactly $N-$ blow up points, and $\mathcal{S} = \{x_1, \cdots x_N\}$ such that $v_p \to v$ for $x \in \overline{\Omega} \setminus S$ and

$$v(x) = 64\pi^2\sqrt{e}\sum_{i=1}^{N} G(x, x_j)$$

where G is a Green's function of Δ^2 under Dirichlet boundary conditions that is

$$\begin{cases} \Delta^2 G(x-y) = \delta(x-y) & \text{in } \Omega \\ G(x,y) = \dfrac{\partial G}{\partial \nu}(x,y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

(f_3) Furthermore, the blow-up points $x_j \in \Omega$, $i \leq j \leq N$ satisfy the following relation

$$\nabla_x H(x_j, x_j) + \sum_{l \neq j} \nabla_x G(x_j, x_l) = 0 \tag{1.7}$$

where

$$H(x,y) = G(x,y) + \frac{log|x-y|}{8\pi^2} \tag{1.8}$$

is the regular part of the Green's function G.

The result is proved [6] for least energy solutions of (1.4). This result is more general in this context, as we precisely study the asymptotics of the blow-up solution in order to derive the result.

Corollary 1.3. *Let u_p be a least energy solution to (S_p). Then up to a subsequence*

(f_1) $p\int_\Omega (u_p^+)^p \to 64\pi^2\sqrt{e}$ *for some positive integer* N.
(f_2) $v_p \to v$ *for* $x \in \overline{\Omega} \setminus \{x_0\}$ *and*

$$v(x) = 64\pi^2\sqrt{e}G(x, x_0)$$

where G *is a Green's function of* Δ^2 *under Dirichlet boundary conditions that is where* x_0 *is a critical point of* $R(x) = H(x, x)$.

Note that by Boggio's principle the Greens function in a unit ball with Dirichlet boundary conditions is positive and explicitly given by this formula,

$$G(x, y) = \frac{1}{8\pi^2} \int_1^{\frac{[x,y]}{|x-y|}} \frac{v^2 - 1}{v^3} dv$$

where $[x, y] = \sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}$, see [8]. In the case of a ball positive solutions of (1.4) are radially symmetric which was proved in [7]. The main difficulty in working with fourth order equations in a general domain, is the absence of maximum principle in the Dirichlet case. More precisely, Dirichlet Green's function may become negative in some domains. It is important to note that for the Laplacian case, the method of moving planes has been used to show that the blow-up points are away from the boundary as in [16], and the process was extended by Lin and Wei for biharmonic problems with Navier boundary conditions [14]. In the Dirichlet case, we cannot apply the method of moving planes in order to exclude boundary blow-up as in Ren- Wei [16]. To overcome this difficulty, we use Pohozaev identity and strong pointwise estimates for blowing up solutions of (1.4) as in Robert- Wei [18]. In course of paper we will only prove Theorems 1.1 and 1.2. Our method can be used to study the above problem with polyharmonic operators.

2 Preliminary Lemmas

Lemma 2.1. *There exists* $C > 0$ *such that for all* $x, y \in \Omega$, $x \neq y$, *we have*

$$|G(x, y)| \leq C \ln\left(1 + \frac{1}{|x - y|}\right) \tag{2.1}$$

$$|\nabla^i G(x, y)| \leq C \frac{1}{|x - y|^i} \tag{2.2}$$

for $i \geq 1$. *Moreover there exists a constant* $C > 0$ *depending on* Ω *such that*

$$G(x, y) \geq -C. \tag{2.3}$$

Proof. The first two estimates is due to Krasovskiĭ [12]. Third result (2.3) can be found in [9] and in fact it tells us that negative part of the Green's function is bounded. □

Lemma 2.2. *Let u_p be a solution of (1.4) then $\|u_p\|_{L^\infty(\Omega)} \geq 1$. Moreover if $x_p \in \Omega$ be a blow-up point, then $u_p(x_p) > 0$ and in particular*

$$\lim_{p\to\infty} p\|u_p\|^{p-1}_{L^\infty(\Omega)} = +\infty.$$

Proof. Consider the problem

$$\begin{cases} \Delta^2\phi = \lambda_1\phi & \text{in } \Omega \\ \phi = \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Then it is easy to show that the first eigenvalue λ_1 is positive. Note that the first eigenfunction ϕ_1 may not be positive, or simple. Also for any solution u_p of (1.4) we have $\langle I_p'(u_p), u_p\rangle = 0$ and hence

$$\int_\Omega |\Delta u_p|^2 = \int_\Omega (u_p^+)^{p+1}.$$

Also from (2.4) we have $\lambda_1\int_\Omega |u_p|^2 \leq \int_\Omega |\Delta u_p|^2$ which implies that $\lambda_1\int_\Omega (u_p^+)^2 \leq \int_\Omega (u_p^+)^{p+1}$. Hence we have $\int_\Omega \{(u_p^+)^{p-1} - \lambda_1\}(u_p^+)^2 \geq 0$. Hence there exists a set of positive measure $\Omega' \subset \Omega$ such that $(u_p^+)^{p-1} \geq \lambda_1$ on Ω' which implies that there exist $x_p \in \Omega'$ such that $u_p(x_p)^{p-1} \geq \lambda_1$ and hence $\|u_p\|_{L^\infty} \geq \lambda_1^{\frac{1}{p-1}} \to 1$ as $p \to \infty$. Also note that for Hölder's inequality and for $p >> 1$,

$$\int_\Omega (u_p^+)^p \leq \left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p}{p+1}} \left(\int_\Omega |1|^{p+1}\right)^{\frac{1}{p+1}} = \left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p}{p+1}} |\Omega|^{\frac{1}{p+1}} \leq \frac{C}{p}$$

Hence by Green representation and (2.3)

$$u_p(x) = \int_\Omega G(x,y)(u_p^+(y))^p dy \geq -C\int_\Omega (u_p^+(y))^p dy \geq -\frac{C}{p}.$$

This implies that $\|u_p^-\|_{L^\infty(\Omega)} \leq \frac{C}{p}$. This implies that the points of blow of v_p are precisely the point of positive maxima of u_p. As a result $\lim_{p\to\infty} p\|u_p\|^{p-1}_{L^\infty(\Omega)} = +\infty$. ☐

Let x_p be a point in Ω such that $0 < u_p(x_p) = \|u_p\|_{L^\infty(\Omega)}$. Let $\varepsilon_p^4 p u_p^{p-1}(x_p) = 1$. Then by Lemma 2.2, $\varepsilon_p \to 0$ as $p \to \infty$. Let $\Omega_p := \frac{\Omega\backslash x_p}{\varepsilon_p}$. In order to prove Theorem 1.1, we have to study the blow-up sequence. Using an idea of Adimurth-Struwe [2] and Adimurthi-Grossi [4] we set

$$W_p(x) = p\frac{u_p(x_p + \varepsilon_p x) - p_p(x_p)}{u_p(x_p)}$$

for $x \in \Omega_p$. Then W_p satisfies the problem

$$\begin{cases} \Delta^2 W_p = \left(1 + \dfrac{W_p}{p}\right)_+^p & \text{in } \Omega_p \\ W_p = -p, \dfrac{\partial W_p}{\partial\nu} = 0 & \text{on } \partial\Omega_p \end{cases} \tag{2.5}$$

We will show that W_p converges W in $C^4_{loc}(\mathbb{R}^4)$ such that W satisfies

$$\begin{cases} \Delta^2 W = e^W & \text{in } \mathbb{R}^4 \\ \displaystyle\int_{\mathbb{R}^4} e^W = 64\pi^2. \end{cases} \tag{2.6}$$

Note that in this case the solution may not be unique as was studied by Lin [13] and Wei-Xu [22]. We need to show that the problem (2.6) is not a half space problem. Hence we prove the next lemma.

Lemma 2.3. *Then* $\lim_{p\to\infty} \frac{d(x_p, \partial\Omega)}{\varepsilon_p} = +\infty$.

Proof. If possible let $d(x_p, \partial\Omega) = O(\varepsilon_p)$. Then up to a rotation, we may assume that $\Omega_p \to (-\beta, +\infty) \times \mathbb{R}^3$. Let $R > 0$ and $x \in B_R(0) \cap \Omega_p$, then we have from the Green's function representation

$$\begin{aligned} |\nabla^i W_p(x)| &= \frac{p}{u_p(x_p)} |\varepsilon_p^i \nabla^i u(x_p + \varepsilon_p x)| \\ &= \frac{p\varepsilon_p^i}{u_p(x_p)} \left| \int_\Omega \nabla_x^i G(x_p + \varepsilon_p x, y)(u_p^+)^p dy \right| \\ &= \frac{p\varepsilon_p^i}{u_p(x_p)} \int_{B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} (u_p^+)^p dy \\ &+ \frac{p\varepsilon_p^i}{u_p(x_p)} \int_{\Omega\setminus B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} (u_p^+)^p dy \end{aligned}$$

Now in $\Omega \setminus B_{2\varepsilon_p R}(x_p)$ we have $|x_p + \varepsilon_p x - y| \geq |y - x_p| - \varepsilon_p |x| \geq R\varepsilon_p$, and $(u_p(y)^+)^p \leq u(x_p)^p$. Hence we have by definition of ε_p,

$$\begin{aligned} |\nabla^i W_p(x)| &\leq p u_p^{p-1}(x_p) \varepsilon_p^i \int_{B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} dy + p \int_\Omega (u_p^+)^p \\ &= \varepsilon_p^{i-4} \int_{B_{2\varepsilon_p R}(x_p)} \frac{1}{|x_p + \varepsilon_p x - y|^i} dy + p \int_\Omega (u_p^+)^p = O(R) \end{aligned}$$

Hence for $x \in B_R(0)$ such that

$$|W_p(x) - W_p(0)| \leq C|x|$$

But $W_p(0) = 0$ and hence W_p is uniformly bounded in a neighborhood of $\partial\Omega$. Choose $x \in \partial\Omega_p$ then we have $|W_p(x)| \leq C$ which implies $p \leq C$, a contradiction as $p \to \infty$. □

Lemma 2.4. *Then $W_p \to W$ as $p \to \infty$ in $C^4_{loc}(\mathbb{R}^4)$ where W satisfies (2.6). Moreover* $W(x) = -4\log(1 + \frac{|x|^2}{8\sqrt{6}})$.

Proof. As $\Omega_p \to \mathbb{R}^4$ as $p \to \infty$ and by previous Lemma we have $|\nabla^i W_p| \leq C$ for $x \in B_R(0)$. By standard elliptic estimate we can conclude that $W_p \to W$ as $p \to \infty$ in $C^4_{loc}(\mathbb{R}^4)$ where W satisfies

$$\Delta^2 W = e^W, \quad W(0) = 0$$

Also note that as $\left(1+\frac{W_p}{p}\right)_+^p \to e^W$ as $p\to\infty$ and hence by Fatou's lemma, we have

$$\int_{\mathbb{R}^4} e^W \le \liminf_{p\to\infty}\int_{\Omega_p}\left(1+\frac{W_p}{p}\right)_+^p = \liminf_{p\to\infty}\frac{1}{\varepsilon_p^4}\int_\Omega \frac{(u_p^+)^p}{\|u_p\|_\infty^p} \le C < +\infty$$

Now we show that in fact $W(x) = -4\log(1+\frac{|x|^2}{8\sqrt{6}})$. It is enough to show that

$$\int_{B_R(0)} |\Delta W|^2 \le CR^2$$

for any $R>0$, then by [13] and [22] will imply the result. We have

$$\Delta W_p(x) = \int_\Omega \Delta_x G(x_p+\varepsilon_p x, y)\left(1+\frac{W_p(y)}{p}\right)_+^p dy.$$

Hence we have

$$\int_{B_R(0)} |\Delta W_p(x)|dx \le C\int_\Omega \frac{p(u_p^+(y))^p}{\|u_p\|_\infty^p}\left(\int_{B_R(0)}\frac{dx}{|x_p+\varepsilon_p x - y|^2}\right)dy \le CR^2.$$

Hence for any $R>0$, we have $\int_{B_R(0)} |\Delta W(x)|^2dx \le CR^2$. Moreover, using the fact that $\lim_{R\to\infty}\int_{B_R}\left(1+\frac{|x|^2}{8\sqrt{6}}\right)^{-4} dx = 32|\mathbb{S}^3| = 64\pi^2$,

$$\lim_{R\to\infty}\lim_{p\to\infty}\int_{B_{R(0)}}\left(1+\frac{W_p}{p}\right)_+^p = 64\pi^2.$$

□

Lemma 2.5. *Let u_p be a solution of (1.4), then $p\int_\Omega (u_p^+)^{p+1} \ge C$ where $C>0$ is a constant independent of p.*

Proof. Using Gamma function $\frac{x^s}{\Gamma(s+1)} \le e^x$ for all $x\ge 0, s\ge 0$ and Moser Trudinger inequality we obtain for any solution of (1.4) we have

$$\|u_p\|_{L^{p+1}(\Omega)} \le D_{p+1}(p+1)^{\frac{1}{2}}\|\Delta u_p\|_{L^2(\Omega)}$$

and by Stirling's formula $\lim_{p\to\infty} D_{p+1} = (\frac{1}{64\pi^2 e})^{\frac{1}{2}}$ which implies that

$$\liminf_{p\to\infty} p\frac{\int_\Omega |\Delta u_p|^2}{(\int_\Omega (u_p^+)^{p+1})^{\frac{2}{p+1}}} \ge 64\pi^2 e$$

hence

$$\liminf_{p\to\infty} p\left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p-1}{p+1}} \ge 64\pi^2 e$$

which in fact implies that

$$\liminf_{p\to\infty} p\left(\int_\Omega (u_p^+)^{p+1}\right) \ge 64\pi^2 e \tag{2.7}$$

□

Lemma 2.6. *Let u_p be a solution of (1.4), then $\|u_p\|_{L^\infty(\Omega)} \leq C$ where C is a constant independent of p and hence there exist $c > 0, C > 0$ such that $c \leq p\int_\Omega (u_p^+)^p \leq C$ for $p >> 1$.*

Proof. We have by definition of ε_p and previous lemma

$$\begin{aligned}
\liminf_{p\to\infty} p\int_\Omega (u_p^+)^{p+1} &= \liminf_{p\to\infty} p\|u_p\|_\infty^{p+1}\varepsilon_p^4 \int_{\Omega_p} \left(1+\frac{W_p}{p}\right)_+^{p+1} dx \\
&= \liminf_{p\to\infty} \|u_p\|_\infty^2 \int_{\Omega_p} \left(1+\frac{W_p}{p}\right)_+^{p+1} \\
&\geq \liminf_{p\to\infty} \|u_p\|_\infty^2 \left(\int_{\mathbb{R}^4} e^W + o(1)\right) dx \\
&= 64\pi^2 \liminf_{p\to\infty} \|u_p\|_\infty^2. \qquad (2.8)
\end{aligned}$$

Hence we have $\|u_p\|_\infty \leq C$. Again we have by Hölder inequality we have

$$\begin{aligned}
\int_\Omega (u_p^+)^p &\leq \left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p}{p+1}} \left(\int_\Omega |1|^{p+1}\right)^{\frac{1}{p+1}} \\
&= \left(\int_\Omega (u_p^+)^{p+1}\right)^{\frac{p}{p+1}} |\Omega|^{\frac{1}{p+1}} \leq \frac{C}{p}.
\end{aligned}$$

For the left hand inequality,

$$\|u_p\|_\infty \int_\Omega (u_p^+)^p \geq \int_\Omega (u_p^+)^{p+1} \geq \frac{c}{p}.$$

This proves Lemma. Hence Theorem 1.1 follows. ☐

Then from (1.5) and Lemma 2.6, we have

$$\int_\Omega (u_p^+)^p dx = O\left(\frac{1}{p}\right) \text{as } p \to \infty.$$

Next we prove that the blow-up points of v_p are in the interior of Ω. Note that

$$\begin{cases} \Delta^2 v_p = p^{-(p-1)}(v_p^+)^p & \text{in } \Omega \\ v_p = 0 = \dfrac{\partial v_p}{\partial \nu} & \text{on } \partial\Omega \end{cases} \qquad (2.9)$$

Let $\varepsilon_0 > 0$ be fixed small number, define

$$\overline{\Omega} \setminus \Lambda = \left\{x \in \overline{\Omega} : \exists\, r_0 > 0 \ \text{ such that } p\int_{B(x,r_0)\cap\Omega} (u_p^+)^p < \varepsilon_0, \forall p\right\}$$

as in Lin and Wei [14]. Now we are going to prove that that in fact $\mathcal{S} = \Lambda$ and $\mathcal{S} \subset \Omega$ which means that there is no boundary blow-up. In order to do so we claim that that the blow-up points are isolated, finite and has quite a distance from the boundary.

Let $f_p = p(u_p^+)^p$. Note that as v_p may be a sign-changing solution of (2.9) we have, by Lemma 2.1,

$$v_p(x) = \int_\Omega G(x,y) f_p(y) dy \geq -pC \int_\Omega (u_p^+(y))^p dy \geq -C$$

and if y_p be a point of negative minimum of v_p then $v_p(y_p) \not\to -\infty$ as $p \to +\infty$. Hence there is only one-sided blow-up.
Let $x_{p,i}$ be a blow-up point of u_p. We say that $\mathcal{H}_k$ holds if there exists a $(x_{p,1}, \cdots x_{p,k}) \in \Omega^k$ such that
(i) $\lim_{p\to+\infty} \frac{|x_{p,i}-x_{p,j}|}{\varepsilon_{p,i}} = +\infty, \quad i \neq j$
(ii) $\lim_{p\to+\infty} \frac{d(x_{p,i},\partial\Omega)}{\varepsilon_{p,i}} = +\infty$
(iii) $\lim_{p\to+\infty} W_{p,i} = -4\log(1+\frac{|x|^2}{8\sqrt{6}})$ in $C^4_{loc}(\mathbb{R}^4) \;\; \forall i \in \{1,2,\cdots k\}$
Then it follows that $\mathcal{H}_1$ holds from the previous lemmas.

Lemma 2.7. *(a) Assume that $\mathcal{H}_k$ holds. Then either $\mathcal{H}_{k+1}$ holds or there exists a $C > 0$ such that*

$$\inf_{i=1,2\cdots k}\{|x-x_{p,i}|^4\} f_p \leq C \;\forall x \in \Omega \tag{2.10}$$

(b) Then there exist N such that $\mathcal{H}_N$ holds and there exists a $C > 0$ such that

$$\inf_{i=1,2\cdots k}\{|x-x_{p,i}|^4\} f_p \leq C \;\forall x \in \Omega \tag{2.11}$$

(c) For $j = 1,2,3$, there exists a $C > 0$ such that

$$\inf_{i=1,2\cdots k}\{|x-x_{p,i}|^j\} |\nabla^j v_p| \leq C \;\forall x \in \Omega \tag{2.12}$$

and hence for any compact set of $K \subset \overline{\Omega} \setminus \Lambda$ we have

$$\|\nabla^j v_p\|_{L^\infty(K)} \leq C \quad \forall j = 0,1,2,3.$$

(d) In particular, $\mathcal{S} = \Lambda$.

Proof. Let $w_p(x) = \inf_{i=1,2\cdots k}\{|x-x_{p,i}|^4\} f_p(x)$. Assume that $y_p \in \Omega$ such that $0 < w_p(y_p) = \|w_p\|_\infty \to \infty$ when $p \to \infty$. Define

$$A_p(x) = \frac{p u_p(y_p + \alpha_p x) - p u_p(y_p)}{u_p(y_p)}$$

Then $\Delta^2 A_p = (1 + \frac{A_p}{p})^p_+$ if $p\alpha_p^4 u_p^{p-1}(y_p) = 1$. Then

$$\begin{aligned} w_p(y_p) &= \inf_{i=1,\cdots k} p|y_p - x_{p,i}|^4 u_p^p(y_p) \\ &= \inf_{i=1,\cdots k} \frac{|y_p - x_{p,i}|^4}{\alpha_p^4} \end{aligned}$$

which implies that $\frac{|y_p - x_{p,i}|}{\alpha_p} \to +\infty$ for all $i = 1, 2 \cdots k$ as $p \to \infty$. Assume that there exist a k_0 such that $y_p - x_{p,k_0} = O(\varepsilon_p)$. Then $y_p - x_{p,k_0} = \theta_{p,k_0}\varepsilon_p$ for some $|\theta_{p,k_0}| \leq C$ and

$$\begin{aligned} |y_p - x_{p,k_0}|^4 f_p(y_p) &= |\theta_{p,k_0}|^4 \varepsilon_p^4 p u_p(x_{p,k_0} + \theta_{p,k_0}\varepsilon_p)^p \\ &\leq |\theta_{p,k_0}|^4 p \varepsilon_p^4 \|u_p\|^p \leq C \end{aligned}$$

which implies that $w_p(y_p)$ is bounded, a contradiction. Hence $\frac{|y_p - x_{p,i}|}{\varepsilon_p} \to +\infty$. Now we know that $w_p(y_p + \alpha_p x) \leq w_p(y_p)$ and hence we have,

$$\frac{f_p(y_p + \alpha_p x)}{f_p(y_p)} \leq \frac{\inf_{1,2\cdots k} |y_p - x_{p,i}|^4}{\inf_{1,2\cdots k} |y_p - x_{p,i} + \alpha_p x|^4}.$$

Let $x \in B_R(0)$. Let $\eta \in (0,1)$. Let $p \geq p(R)$ such that $\frac{|y_p - x_{p,i}|}{\alpha_p} \geq \frac{R}{\eta}$ for all $i = 1, \cdots k$, we have $|y_p - x_{p,i} + \alpha_p x| \geq (1-\eta)|y_p - x_{p,i}|$ and hence $\inf_{i=1,2\cdots k} |y_p - x_{p,i} + \alpha_p x|^4 \geq (1-\eta)^4 \inf_{i=1,2\cdots k} |y_p - x_{p,i}|^4$. This again implies that

$$\frac{(u_p^+(y_p + \alpha_p x))^p}{u_p^+(y_p)^p} \leq \frac{1}{(1-\eta)^4}$$

that is

$$\left(1 + \frac{A_p}{p}\right)_+^p \leq \frac{1}{(1-\eta)^4}$$

Hence using Lemma 2.3 and 2.4 we can prove that

$$\frac{d(y_p, \partial\Omega)}{\alpha_p} \to \infty \quad W_p \to -4\ln\left(1 + \frac{|x|^2}{8\sqrt{6}}\right)$$

in $C^4_{loc}(\mathbb{R}^4)$ as $p \to +\infty$. Letting $x_{p,k+1} := y_p$, then $\mathcal{H}_{k+1}$ holds.
(b) Suppose not then $\mathcal{H}_k$ holds for all k. Then we may choose $R > 0$ such that $B_{\varepsilon_p R}(x_i) \cap B_{\varepsilon_p R}(x_j) = \emptyset$ for all $i \neq j$. Then

$$\begin{aligned} C_1 &\geq p\int_\Omega (u_p^+)^p \geq \frac{\|u_p\|_\infty}{\varepsilon_p^4} \int_{\cup_{i=1}^k B_{\varepsilon_p R}(x_{p,i})} \frac{(u_p^+)^p}{\|u_p\|^p} \\ &\geq \frac{\|u_p\|_\infty}{\varepsilon_p^4} \sum_{i=1}^k \int_{B_{\varepsilon_p R}(x_{p,i})} \frac{(u_p^+)^p}{\|u_p\|^p} = \|u_p\|_\infty \sum_{i=1}^k \int_{B_R(0)} \left(1 + \frac{W_{p,i}}{p}\right)^p \\ &= 64\pi^2 \|u_p\|_\infty k + o(1) \end{aligned}$$

for all k, a contradiction. As a result, the blow-up points are isolated and finite.
(c) We have from Greens function representation, that

$$\begin{aligned} |\nabla^i v_p(x)| &= p\left|\int_\Omega \nabla_x^i G(x,y)(u_p^+(y))^p dy\right| \\ &\leq p\int_\Omega |\nabla_x^i G(x,y)|(u_p^+(y))^p dy \\ &\leq p\int_\Omega |x-y|^{-i}(u_p^+(y))^p dy \end{aligned} \tag{2.13}$$

Let $R_p(x) := \inf_{i=1,\cdots N} |x - x_{p,i}|$ and $\Omega_{p,i} = \{x \in \Omega : |x - x_{p,i}| = R_p(x)\}$. Then we have

$$p\int_{\Omega_{p,i}} |x-y|^{-j}(u_p^+)^p dy = p\int_{\Omega_{p,i}\cap B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p dy + p\int_{\Omega_{p,i}\setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p dy$$

Note that for $y \in \Omega_{p,i} \setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})$, we have

$$p|x-y|^{-j}(u_p^+)^p \le \frac{C}{|x-y|^j|x-x_{p,i}|^4}$$

and hence

$$p\int_{\Omega_{p,i}\setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p \le \int_{\Omega_{p,i}\setminus B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} \frac{1}{|x-y|^j|x-x_{p,i}|^4} \le \frac{C}{|x-x_{p,i}|^j}$$

When $\Omega_{p,i} \cap B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})$, we have $|x-y| \ge |x-x_{p,i}| - |y-x_{p,i}| \ge \frac{1}{2}|x-x_{p,i}|$ and

$$p\int_{\Omega_{p,i}\cap B_{\frac{|x-x_{p,i}|}{2}}(x_{p,i})} |x-y|^{-j}(u_p^+)^p dy \le \frac{C}{|x-x_{p,i}|^j}$$

Hence for any compact set $K \subset \overline{\Omega} \setminus \Lambda$ we have $\|v_p\|_{L^\infty(K)} \le C$.

(d) Now we prove that $\mathcal{S} = \Lambda$. Suppose $x_0 \notin \Lambda$, then from (c) we have v_p is uniformly bounded in $L^\infty(K)$ for some compact set K containing x_0 and hence $x_0 \notin \mathcal{S}$. Hence $\mathcal{S} \subset \Lambda$.

Let $x_0 \in \Lambda$, then definitely every compact set K containing x_0, $\|v_p\|_{L^\infty(K)} \to +\infty$ as $p \to \infty$, otherwise there exists $r > 0$ such that $\|v_p\|_{L^\infty(B_r(x_0))} \le C$ but $f_p = p^{1-p}(v_p^+)^p$, hence $f_p \to 0$ as $p \to \infty$ uniformly in $B_r(x_0)$ and this implies a contradiction as

$$p\int_{B_r(x_0)\cap\Omega} (u_p^+)^p \to 0 \text{ as } r \to 0$$

implying that $x_0 \notin \Lambda$. ☐

Lemma 2.8. *Suppose $u \in C^4(\overline{\Omega})$ be a solution of $\Delta^2 u = f(u)$. Let $F(u) = \int_0^u f(t)dt$ then*

$$4\int_\Omega F(u) = \int_{\partial\Omega} \langle x-y,\nu\rangle F(u)ds + \frac{1}{2}v^2\langle x-y,\nu\rangle ds + 2\frac{\partial u}{\partial \nu}vds + \int_{\partial\Omega}\left(\frac{\partial v}{\partial \nu}\langle x-y, Du\rangle + \frac{\partial u}{\partial \nu}\langle x-y, Dv\rangle - \langle Du, Dv\rangle\langle x-y,\nu\rangle\right)ds \quad (2.14)$$

where $-\Delta u = v$ and ν denotes the outward normal derivative of x on $\partial\Omega$. In particular, we have

$$4\int_\Omega \nu F(u)ds + \frac{1}{2}\int_{\partial\Omega} v^2\nu ds + \int_{\partial\Omega}\left\{\frac{\partial u}{\partial \nu}Du + \frac{\partial u}{\partial \nu}Dv - \langle Du, Dv\rangle\nu\right\}ds = 0 \quad (2.15)$$

Proof. This identity follows from [11]. ☐

3 Asymptotics of the blow-up solution

We know that locally $W_p \to W$ in $B_2(0) \setminus 0$ and hence

$$\Delta^2 W = e^W \text{in } B_2(0) \setminus \{0\}. \tag{3.1}$$

Choose $r > 0$ sufficiently small such that $W_p(r) > 0$ in $B_r(0)$, then integrating (2.6), we have

$$\int_{\partial B_r} \frac{\partial \Delta W_p}{\partial \nu} = \int_{B_r} \Delta^2 W_p = \int_{B_r(0)} \left(1 + \frac{W_p}{p}\right)^p dx := \alpha_p(r)$$

which implies that

$$\lim_{r\to 0} \int_{B_r(0)} \frac{\partial \Delta W}{\partial \nu} = \lim_{r\to 0} \lim_{p\to\infty} \alpha_p(r) := \alpha. \tag{3.2}$$

Now we want to compute the value of α. Now note that from (3.2) we have W satisfies

$$\Delta^2 W = e^W + \alpha\delta_0$$

where δ_0 is a Dirac mass concentrated at 0 and $\alpha > 0$. Hence

$$W(x) = \frac{\alpha}{8\pi^2} \ln \frac{1}{|x|} + g(x)$$

where g is smooth. Now let us apply Pohozaev identity to (2.6), noting the fact that $\nu = \frac{x}{|x|}$, we have

$$\begin{aligned} 4\int_{B_r(0)} \left(1 + \frac{W_p}{p}\right)^p dx - \int_{\partial B_r} |x| \left(1 + \frac{W_p}{p}\right)^p ds = \frac{1}{2} \int_{\partial B_r(0)} |\Delta W_p|^2 r ds \\ - \int_{\partial B_r(0)} \frac{\partial W_p}{\partial r} \frac{\partial \Delta W_p}{\partial r} r ds + \int_{\partial B_r(0)} \frac{\partial}{\partial r}\left(r \frac{\partial W_p}{\partial r}\right) \partial W_p ds \end{aligned} \tag{3.3}$$

Letting $p \to +\infty$ we obtain,

$$4 \lim_{p\to\infty} \alpha_p = 4\pi^2 \frac{\alpha^2}{64\pi^4}(1 + o_r(1))$$

where $o_r(1) \to 0$ as $r \to 0$. Now letting $r \to 0$, we obtain that $\alpha(64\pi^2 - \alpha) = 0$, which implies that $\alpha = 64\pi^2$ as $\alpha > 0$. Similarly applying Pohozaev identity on a ball of radius $r = \varepsilon_p \ln \frac{1}{\varepsilon_p}$, we obtain

$$|\alpha_p - 64\pi^2| \leq c\left(\ln \frac{1}{\varepsilon_p}\right)^{-1}$$

We now derive an important decay estimate for the blow-up solution.

Lemma 3.1. *For any $0 < \delta < 64\pi^2$, there exist $R_\delta > 0$ and $p_0 = p(\delta) \in \mathbb{N}$ such that if $|x| \geq 2R$ and $p \geq p_0$, then*

$$|W_p(x)| \leq -(8 - \delta) \ln |x|. \tag{3.4}$$

Proof. Note that $\int_{\mathbb{R}^4} e^W = 64\pi^2$ and $\lim_{p\to\infty}\int_{\Omega_p}(1+\frac{W_p}{p})_+^p \to 64\pi^2$. Hence given any $\delta > 0$, we can find $\varepsilon_\delta > 0$ and $r_\delta > 0$ such that

$$\int_{|y|\le r_\delta}\left(1+\frac{W_p}{p}\right)_+^p > 64\pi^2 - \frac{\delta}{2}$$

for some $p \ge p_0$. Note that the Green's function representation we can write u_p as

$$u_p(x) = \frac{1}{8\pi^2}\int_\Omega \ln\frac{1}{|x-y|}u_p^p(y)dy + f_p(x) \tag{3.5}$$

where f_p is uniformly bounded in $C^4(B_{\frac{3}{2}})$

$$u_p(x_p) = \frac{1}{8\pi^2}\int_\Omega \ln\frac{1}{|x_p-y|}u_p^p(y)dy + f_p(x_p). \tag{3.6}$$

By definition of W_p, we obtain from (3.5), (3.6)

$$\frac{pu_p(x)-pu_p(x_p)}{u_p(x_p)} = \frac{p}{8\pi^2}\int_\Omega \ln\frac{|x_p-y|}{|x-y|}\frac{u_p^p(y)}{u_p(x_p)}dy + f_p(x) - f_p(x_p).$$

Hence,

$$W_p(x) = \frac{1}{8\pi^2}\int_{\Omega_p}\ln\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p + f_p(x) - f_p(x_p)$$

then we can write,

$$W_p(x) = \frac{1}{8\pi^2}\int_{\Omega_p}\ln\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p + h_p(x)$$

where $h_p(x) = f_p(x_p+\varepsilon_p x) - f_p(x_p)$ and hence for all $i = 1, 2, 3, 4$ we have

$$\|\nabla^i h_p\|_{B_{\frac{1}{\varepsilon_p}}} \to 0 \tag{3.7}$$

and hence we have ΔW_p is uniformly bounded. As a result, we have $r > 0$ small, we have

$$
\begin{aligned}
W_p(x) &= \frac{1}{8\pi^2}\int_{\Omega_p} \ln\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p dy + O(1) \\
&= \frac{1}{8\pi^2}\int_{|y|\le r} \ln\frac{|y|}{|x||1-\frac{|y|}{|x|}|}\left(1+\frac{W_p}{p}\right)_+^p dy \\
&+ \frac{1}{8\pi^2}\int_{\Omega_p\cap\{r\le|y|\le 2|x-y|\}} \ln\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p dy \\
&+ \frac{1}{8\pi^2}\int_{\Omega_p\cap\{|y|\ge r,|y|\ge 2|x-y|\}} \ln\frac{|y|}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p dy + O(1) \\
&\le \frac{1}{8\pi^2}\ln\frac{2r}{|x|}\int_{|y|\le r}\left(1+\frac{W_p}{p}\right)_+^p dy + \frac{1}{8\pi^2}(\ln 2)\int_{\Omega_p\cap\{|y|\ge r\}}\left(1+\frac{W_p}{p}\right)_+^p dy \\
&+ \frac{1}{8\pi^2}\int_{\Omega_p\cap\{|y|\ge r,|y|\ge 2|x-y|\}} (\ln|y|)\left(1+\frac{W_p}{p}\right)_+^p dy \\
&+ \frac{1}{8\pi^2}\int_{\Omega_p\cap\{|y|\ge r,|y|\ge 2|x-y|\}} \ln\frac{1}{|x-y|}\left(1+\frac{W_p}{p}\right)_+^p + O(1) \\
&= \frac{1}{8\pi^2}\ln\frac{2r}{|x|}\int_{|y|\le r}\left(1+\frac{W_p}{p}\right)_+^p + \frac{1}{16\pi^2\delta}\ln 2|x| + O(1).
\end{aligned}
$$

Hence when $|x|\ge 2r$ we have $\ln\frac{2r}{|x|}\le 0$,

$$
\begin{aligned}
W_p(x) &\le \frac{1}{8\pi^2}\left(64\pi^2-\frac{\delta}{2}\right)\ln\frac{2r}{|x|} + \frac{1}{16\pi^2\delta}\ln 2|x| + C_\delta \\
&= (8-\delta)\left(\ln\frac{1}{|x|}\right) + C_\delta.
\end{aligned}
\tag{3.8}
$$

□

Lemma 3.2. *We have for* $\varepsilon_p\ln\frac{1}{\varepsilon_p}\le\varepsilon_p|x|\le 1$

$$
\left|\Delta W_p + \frac{\alpha_p}{4\pi^2}\frac{1}{|x|^2}\right| \le O\left(\left(\ln\frac{1}{\varepsilon_p}\right)^{-1}|x|^{-2}\right).
$$

Proof. For $\ln\frac{1}{\varepsilon_p}\le|x|\le\frac{1}{\varepsilon_p}$, let us define a radial function

$$
\tilde{\alpha}_p(|x|) = \int_{|y|\le r_0|x|}\left(1+\frac{W_p(y)}{p}\right)_+^p dy
$$

where $r_0\le\frac{1}{2}$. Note that

$$
\begin{aligned}
|\tilde{\alpha}_p(|x|)-\alpha_p| &\le \int_{|y|\ge r_0|x|}\left(1+\frac{W_p(y)}{p}\right)_+^p dy \le \int_{|y|\ge r_0|x|} e^{|W_p|} \\
&\le \int_{|y|\ge r_0|x|}\frac{1}{|y|^7}dy = O\left(\frac{1}{|x|^3}\right)
\end{aligned}
\tag{3.9}
$$

for $|x| \geq \ln \frac{1}{\varepsilon_p}$. Now we compute $R \leq |x| \leq \varepsilon_p^{-1}$ and using the fact that $\tilde{\alpha}_p(|x|) - \alpha_p = O(\frac{1}{|x|^3})$ we have

$$\begin{aligned}
\Delta W_p(x) &= -\frac{1}{4\pi^2}\int_{|y|\leq r_0|x|}\frac{1}{|x-y|^2}\Big(1+\frac{W_p(y)}{p}\Big)_+^p + O\Big(|x|^{-5}\Big) \\
&= -\frac{1}{4\pi^2}\int_{|y|\leq r_0|x|}\frac{1}{|x|^2}\Big(1+\frac{W_p(y)}{p}\Big)_+^p dy \\
&- \frac{1}{2\pi^2}\int_{|y|\leq r_0|x|}\frac{\langle x,y\rangle}{|x-y|^2}\Big(1+\frac{W_p(y)}{p}\Big)_+^p dy + O\Big(|x|^{-4}\Big) \\
&= -\frac{\tilde{\alpha}_p}{4\pi^2}\frac{1}{|x|^2} + O\Big(|x|^{-3}\Big) \\
&= -\frac{\alpha_p}{4\pi^2}\frac{1}{|x|^2} + O(|x|^{-3}) \\
&= -\frac{\tilde{\alpha}_p}{4\pi^2}\frac{1}{|x|^2} + O\Big(|x|^{-3}\Big) \\
&= -\frac{\alpha_p}{4\pi^2}\frac{1}{|x|^2} + O\Big(|x|^{-2}\Big(\ln\frac{1}{\varepsilon_p}\Big)^{-1}\Big). \qquad (3.10)
\end{aligned}$$

□

Lemma 3.3. *We have as $p \to +\infty$,*

$$\int_{B_{\frac{1}{\varepsilon_p}}(0)} |\Delta u_p|^2 dx = 128\pi^2\frac{\|u_p\|_\infty^2}{p^2}\ln\|u_p\|_\infty + o(1)$$

and

$$p\int_{B_{\frac{1}{\varepsilon_p}}(0)} (u_p^+)^{p+1}dx = 64\pi^2\|u_p\|_\infty^2 + o(1).$$

Proof. Note that from definition of ε_p we have

$$\ln p + 4\ln\varepsilon_p + (p-1)\ln\|u_p\|_\infty = 0 \qquad (3.11)$$

For integration on $B_{\frac{1}{\varepsilon_p}}(0)$, we break the domain of integration in $B_1(0) \cup (B_{\frac{1}{\varepsilon_p}}(0) \setminus B_1(0))$. Note $\int_{B_1}|\Delta u_p|^2 = O(\frac{1}{p})$. Hence we only need to compute this integral in the annulus $B_{\frac{1}{\varepsilon_p}}(0)\setminus B_1(0)$. Now $\Delta W_p\|u_p\|_\infty = p\varepsilon_p^2\Delta u_p(x_p+\varepsilon_p x)$. Using Lemma 3.2 and integrating both sides we obtain

$$\begin{aligned}
\int_{B_{\frac{1}{\varepsilon_p}}(0)\setminus B_1(0)} |\Delta u_p|^2 dx &= \frac{\alpha_p^2}{(4\pi^2)^2}\frac{\|u_p\|_\infty^2}{p^2}\omega_4\ln\frac{1}{\varepsilon_p} + o(1) \\
&= 512\pi^2\frac{\|u_p\|_\infty^2}{p^2}\ln\frac{1}{\varepsilon_p} + o(1) \\
&= 128\pi^2\frac{\|u_p\|_\infty^2}{p}\ln\|u_p\|_\infty + o(1) \qquad (3.12)
\end{aligned}$$

where $\omega_4 = 2\pi^2$ is the volume of unit sphere in $\mathbb{R}^4$. The other asymptotic follows trivially from the decay estimate in Lemma 3.1 and dominated convergence theorem which follows . □

Now we proceed to prove

Proof of Theorem 1.1. We know

$$p \int_{B_{\frac{1}{\varepsilon_p}}} |\Delta u_p|^2 dx + o(1) = p \int_{B_{\frac{1}{\varepsilon_p}}} (u_p^+)^{p+1} + o(1)$$

and hence from Lemma 3.3 we have

$$\ln \|u_p\|_\infty \to \frac{1}{2}$$

which implies that $\|u_p\|_\infty \to \sqrt{e}$ as $p \to +\infty$. □

Lemma 3.4. *Let $x_i = \lim_{p\to\infty} x_{p,i} \in \overline{\Omega}$ and $\mathcal{S} = \{x_1, \cdots x_N\}$. Then v_p is uniformly bounded for all $x \in \overline{\Omega} \setminus \mathcal{S}$.*

Proof. Since x_i are isolated, there exist a $R > 0$ such that $\Omega' = \Omega \setminus \cup_{i=1}^N \overline{B_R}(x_i)$ is connected. Then $|\nabla v_p| \le C$ for all $x \in \Omega'$ by Lemma 2.7. Let $x' \in \partial\Omega \cap \partial\Omega'$, then $|v_p(x) - v_p(x')| \le C$ for all $x \in \Omega'$. But this implies v_p is uniformly bounded in Ω'. As a result $u_p^+(x) \le \frac{C}{p}$ for $x \in \overline{\Omega} \setminus \mathcal{S}$. □

Lemma 3.5. *Then there exist $\gamma_j > 0$ $j = 1, 2, \cdots N$ such that*

$$\lim_{p\to\infty} v_p(x) = \sum_{j=1}^{N} \gamma_j G(., x_j) \ \ in \ C^4_{loc}(\mathbb{R}^4).$$

Proof. Since v_p is uniformly bounded in $C(\overline{\Omega} \setminus \mathcal{S})$, and by standard regularity we have $v_p \to v$ as $p \to \infty$ in $C^4(\overline{\Omega} \setminus \mathcal{S})$. Then for $r > 0$ small such that $|u_p(y)|^{p-1} u_p(y) > 0$ in $B_r(x_j)$ and hence we have

$$v_p(x) = p \int_\Omega G(x,y)(u_p^+(y))^p dy = \sum_{j=1}^{N} p \int_{B_r(x_j)} G(x,y) u_p^p(y) dy + o(1)$$

and noting that the $G(x, .)$ is continuous in $\overline{\Omega} \setminus \{x\}$, we obtain

$$v_p(x) = \sum_{j=1}^{N} p \int_{B_r(x_j)} G(x,y) u_p^p(y) dy + o(1)$$

where

$$\gamma_j = \lim_{r\to 0} \lim_{p\to+\infty} p \int_{B_r(x_j)} u_p^p(y) dy.$$

□

Lemma 3.6. *We claim that $\gamma_j = 64\pi^2\sqrt{e}$.*

Proof. We have

$$\lim_{p\to\infty} v_p(x) = \sum_{j=1}^{N} \gamma_j G(x, x_j)$$

Without loss of generality, we assume $N = 1$ and $x = x_1 = 0$, then we have $x \in B_R(0) \subset \overline{\Omega}$, the Green's function is

$$G(x,0) = \frac{1}{8\pi^2}\left(\ln\frac{R}{|x|} + \frac{|x|^2}{2R^2} - \frac{1}{2}\right)$$

Choose a r small such that $0 < r < R$ such that $v_p > 0$ in $B_r(0)$. Since $v_p \to \gamma_1 G(x, x_1)$ in C^4_{loc}, we have

$$v_p = \gamma_1(G(x,0) + h_p),\ z_p = -\Delta v_p = \gamma_1(-\Delta G(x,0) - \Delta h_p) \text{ on } \partial B_r$$

where

$$-\Delta G(x,0) = -\frac{1}{8\pi^2}\left(\frac{2}{|x|^2} + \frac{1}{R^2} - \frac{3|x|}{R^2}\right)$$

and $|\nabla^i h| \le C$ for $i = 1, 2, 3$. We will use a local Pohozaev identity from (2.8) in $B_r(0)$. We have

$$\begin{aligned} \frac{4p^2}{(p+1)}\int_{B_r(0)} (u_p^+)^{p+1}dx &= \frac{p^2}{(p+1)}\int_{\partial B_r} \langle x,\nu\rangle (u_p^+)^{p+1}ds \\ &+ \frac{1}{2}\int_{\partial B_r}\langle x,\nu\rangle z_p^2 ds + 2\int_{\partial B_r}\frac{\partial v_p}{\partial \nu} z_p ds \\ &+ \int_{\partial B_r}\left\{\frac{\partial z_p}{\partial\nu}\langle x, Dv_p\rangle ds + \frac{\partial v_p}{\partial \nu}\langle x, Dz_p\rangle - \langle Dv_p, Dz_p\rangle\langle x,\nu\rangle\right\}ds \end{aligned}$$

where $z_p = -\Delta u_p$. Then we have $\frac{p^2}{(p+1)}\int_{\partial B_r(x_0)}\langle x,\nu\rangle (u_p^+)^{p+1}ds \to 0$ and
$\frac{1}{2}\int_{\partial B_r(0)}\langle x,\nu\rangle z_p^2 ds \to \frac{\gamma_1^2}{16\pi^2} + O(r)$
$2\int_{\partial B_r(0)} \frac{\partial v_p}{\partial \nu} z_p \to -\frac{\gamma_1^2}{8\pi^2} + O(r)$
$\int_{\partial B_r} \frac{\partial z_p}{\partial \nu}\langle x, Dv_p\rangle ds \to \frac{\gamma_1^2}{8\pi^2} + O(r)$
$\int_{\partial B_r} \frac{\partial v_p}{\partial \nu}\langle x, Dz_p\rangle \to \frac{\gamma_1^2}{8\pi^2} + O(r)$
$\int_{\partial B_r}\langle Dv_p, Dz_p\rangle\langle x,\nu\rangle \to \frac{\gamma_1^2}{8\pi^2} + O(r)$. This implies that

$$\frac{p^2}{(p+1)}\int_{B_r(x_0)} u_p^{p+1}dx = \frac{\gamma_1^2}{64\pi^2} + O(r)$$

This again implies that

$$\frac{1}{p+1}\int_{B_r(0)} u_p^{p+1} = \frac{1}{64\pi^2}\left(\int_{B_r} u_p^p\right)^2 + O(r) \tag{3.13}$$

Using dominated convergence theorem, we have

$$\begin{aligned} p\int_{B_r} u_p^p + o(1) &= p\|u_p\|_\infty^p \varepsilon_p^4 \int_{B_{\frac{r}{\varepsilon_p}}(0)} \left(1+\frac{W_p}{p}\right)^p dx \\ &= \|u_p\|_\infty \int_{B_{\frac{r}{\varepsilon_p}}(0)} \left(1+\frac{W_p}{p}\right)^p \\ &\geq \|u_p\|_\infty \int_{\mathbb{R}^4} e^W dx = 64\pi^2 \|u_p\|_\infty \end{aligned} \tag{3.14}$$

$$\frac{p}{p+1}\|u_p\|_\infty \int_{B_r(0)} (u_p^+)^p \geq \frac{p}{p+1}\int_{B_r(0)} (u_p^+)^{p+1}. \tag{3.15}$$

Hence we have from (3.13), (3.14) and (3.15) such that

$$64\pi^2 \lim_{p\to\infty} \|u_p\|_\infty \leq \liminf_{p\to\infty} \lim_{r\to 0} p\int_{B_r} u_p^p dx \leq \limsup_{p\to\infty} \lim_{r\to 0} p\int_{B_r} u_p^p dx \leq 64\pi^2 \lim_{p\to\infty} \|u_p\|_\infty$$

and as a result we have

$$\lim_{p\to\infty}\lim_{r\to 0} p\int_{B_r} u_p^p dx = 64\pi^2\sqrt{e}.$$

□

Now we show that there is no boundary blow-up, by the use of Pohozaev identity.

Lemma 3.7. *In particular,* $\mathcal{S}\cap\partial\Omega=\emptyset$.

Proof. It is enough to prove that

$$\lim_{r\to 0}\lim_{p\to+\infty} p\int_{\Omega\cap B_r(x_0)} (u_p^+)^p dx = 0$$

if $x_0\in\partial\Omega\cap\Lambda$. If possible let $x_0\in\partial\Omega\cap\mathcal{S}$, then we have

$$\lim_{r\to 0}\lim_{p\to+\infty} p\int_{\Omega\cap B_r(x_0)} (u_p^+)^p \geq \varepsilon_0$$

which implies that

$$p\int_{\Omega\cap B_r(x_0)} (u_p^+)^p \geq \frac{\varepsilon_0}{2} \tag{3.16}$$

for all $p >> 1$ and $r > 0$. Without loss of generality, we may assume that $\Lambda\cap B_\delta(x_0)=\{x_0\}$. Let $y_p = x_0+\rho_{k,r}\nu(x_0)$ where

$$\rho_{p,r} = \frac{\int_{\partial\Omega}\langle x-x_0,\nu\rangle(\Delta v_p)^2}{\int_{\partial\Omega}\langle \nu(x_0),\nu\rangle(\Delta v_p)^2}$$

where $r << r_1$ such that $\frac{1}{2}\leq\langle\nu(x_0),\nu\rangle\leq 1$ for $x\in\overline{B}_r(x_0)\cap\Omega$. Here $\nu(x)$ is an outer normal vector to $T_{x_0}\partial\Omega$ at x. Then it follows that $|\rho_{p,r}|\leq 2r$ and

$$\int_{\partial\Omega\cap B_r(x_0)} \langle x-y_p,\nu\rangle(\Delta v_p)^2 dx = 0 \tag{3.17}$$

and hence

$$\int_{\partial\Omega\cap B_r(x_0)}\langle x-x_0+\rho_{p,r}\nu(x_0),\nu\rangle(\Delta v_p)^2dx=0 \tag{3.18}$$

Now applying the Pohozaev identity on $\Omega\cap B_r(x_0)$ with $y=y_p$, we obtain

$$\begin{aligned}\frac{4p^2}{(p+1)}\int_{\Omega\cap B_r(x_0)}(u_p^+)^{p+1}dx &= \frac{p^2}{(p+1)}\int_{\Omega\cap\partial B_r(x_0)}\langle x-y_p,\nu\rangle(u_p^+)^{p+1}ds\\ &- 2\int_{\Omega\cap\partial B_r(x_0)}\frac{\partial v_p}{\partial\nu}\Delta v_p ds+\int_{\Omega\cap\partial B_r(x_0)}\langle x-y_p,\nu\rangle(\Delta v_p)^2ds\\ &- \int_{\Omega\cap\partial B_r(x_0)}\langle x-y_p,\nabla v_p\rangle\frac{\partial\Delta v_p}{\partial\nu}ds-\int_{\Omega\cap\partial B_r(x_0)}\langle x-y_p,\nabla\Delta v_p\rangle\frac{\partial v_p}{\partial\nu}ds\\ &+ \int_{\Omega\cap\partial B_r(x_0)}\langle x-y_p,\nu\rangle\langle\nabla v_p,\nabla\Delta v_p\rangle ds\end{aligned} \tag{3.19}$$

As $v_p(x)\to\sum_{i=1}^N\gamma_iG(x,x_i)$ in $C^3(\overline{\Omega}\setminus\mathcal{S})$ follows from the previous lemma. Again by the boundary values $G(x,x_0)=0$. Also note that last five terms in right hand side $O(r^3)$ and hence we have

$$\begin{aligned}\frac{4p^2}{(p+1)}\int_{\Omega\cap B_r(x_0)}(u_p^+)^{p+1}dx &= \frac{p^2}{(p+1)}\int_{\Omega\cap\partial B_r(x_0)}\langle x-y_p,\nu\rangle(u_p^+)^{p+1}ds\\ &+ O(r^3)=O(r^3)\end{aligned}$$

as $\frac{p^2}{p+1}|\int_{\Omega\cap\partial B_r(x_0)}\langle x-y_p,\nu\rangle(u_p^+)^{p+1}ds|\le o_p(1)r^4$. Hence

$$\lim_{p\to\infty}\lim_{r\to\infty}\frac{p^2}{(p+1)}\int_{\Omega\cap B_r(x_0)}(u_p^+)^{p+1}dx=0$$

Now by Hölder's inequality we have

$$p\int_{\Omega\cap B_r(x_0)}(u_p^+)^pdx=\left(\frac{p^2}{(p+1)}\int_{\Omega\cap B_r(x_0)}(u_p^+)^{p+1}\right)^{\frac{p}{p+1}}\left(p^{-\frac{p-1}{p+1}}(p+1)^{\frac{p}{p+1}}\right)$$

and hence

$$\lim_{p\to\infty}\lim_{r\to\infty}p\int_{\Omega\cap B_r(x_0)}(u_p^+)^pdx=0$$

a contradiction to (3.16). □

Proof of Theorem 1.2. Now we prove the identity in (f_3). Let $r>0$ be a small number such that $B_r(x_i)\subset\overline{\Omega}$ and $B_r(x_i)\cap B_r(x_j)=\emptyset$. It is enough to prove the identity for $i=1$. Now applying Pohozaev identity to v_p on the domain $\Omega\setminus B_r(x_1)$ we have

$$\int_{\partial(\Omega\setminus B_r(x_1))}\nu F(v_p)+\frac{1}{2}\int_{\partial(\Omega\setminus B_r(x_1))}z_p^2\nu+\int_{\partial(\Omega\setminus B_r(x_1))}\left\{\frac{\partial z_p}{\partial\nu}Dv_p+\frac{\partial v_p}{\partial\nu}Dz_p-\langle Dz_p,Dv_p\rangle\nu\right\}ds=0$$

where $z_p = -\Delta v_p$, where $F(v_p) = \frac{p^2}{(p+1)}(u_p^+)^{p+1}$. Using the boundary values we have

$$\int\limits_{\partial B_r(x_1)} \nu F(v_p) + \frac{1}{2} \int\limits_{\partial(\Omega \setminus B_r(x_1))} z_p^2 \nu + \int\limits_{\partial B_r(x_1)} \left\{ \frac{\partial z_p}{\partial \nu} D v_p + \frac{\partial v_p}{\partial \nu} D z_p - \langle D z_p, D v_p \rangle \nu \right\} ds = 0.$$

Letting $p \to \infty$, we obtain

$$\frac{1}{2} \int_{\partial(\Omega \setminus B_r(x_1))} z^2 \nu + \int_{\partial B_r(x_1)} \left\{ \frac{\partial z}{\partial \nu} D v + \frac{\partial v}{\partial \nu} D z - \langle D z, D v \rangle \nu \right\} ds = 0$$

where $z = -\Delta v$. Now we have previous lemmas, we have $v(x) = 64\pi^2\sqrt{e}\sum_{j=1}^N G(x, x_j)$ and $-\Delta v(x) = 64\pi^2\sqrt{e}\sum_{j=1}^N (-\Delta G(x, x_j))$. But from (1.8), we have

$$\frac{v}{64\pi^2\sqrt{e}} = H(x, x_1) + \sum_{j=2}^{N} G(x, x_j) - \frac{1}{8\pi^2} \ln|x - x_1| \tag{3.20}$$

$$\frac{z}{64\pi^2\sqrt{e}} = (-\Delta)H(x, x_1) + \sum_{j=2}^{N}(-\Delta)G(x, x_j) + \frac{1}{8\pi^2}\frac{1}{|x - x_1|^2} \tag{3.21}$$

Hence we have

$$\frac{1}{(64\pi^2\sqrt{e})^2} \int_{\partial(\Omega \setminus B_r(x_1))} z^2 \nu = O(r^2)$$

By using mean value theorem [19],

$$\begin{aligned}
\frac{1}{(64\pi^2\sqrt{e})^2} \int_{\partial B_r(x_1)} \frac{\partial z}{\partial \nu} D v &= \int_{\partial B_r(x_1)} (-\frac{1}{2\pi^2 r^3} + O(1))\left(\nabla_x(H(x, x_1) + \sum_{j=2}^{N} \nabla_x G(x, x_j) + \frac{x}{8\pi^2 r^2}\right) \\
&= -\frac{1}{2\pi^2 r^3} \int_{\partial B_r(x_1)} \left(\nabla_x H(x, x_1) + \sum_{j=2}^{k} \nabla_x G(x, x_j)\right) + O(r) \\
&= -\left(\nabla_x H(x_1^\star, x_1) + \sum_{j=2}^{k} \nabla_x G(x_1^\star, x_j)\right) + O(r)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(64\pi^2\sqrt{e})^2} \int_{\partial B_r(x_1)} \frac{\partial v}{\partial \nu} D z &= \int_{\partial B_r(x_1)} (-\frac{x}{2\pi^2 r^4} + O(1))\left(\nabla_x(H(x, x_1) + \sum_{j=2}^{N} \nabla_x G(x, x_j) + \frac{x}{8\pi^2 r^2}\right) \\
&= -\frac{1}{2\pi^2 r^3} \int_{\partial B_r(x_1)} \left(\nabla_x H(x, x_1) + \sum_{j=2}^{k} \nabla_x G(x, x_j)\right) + O(r) \\
&= -\left(\nabla_x H(x_2^\star, x_2) + \sum_{j=2}^{k} \nabla_x G(x_2^\star, x_j)\right) + O(r)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(64\pi^2\sqrt{e})^2} \int_{\partial B_r(x_1)} \langle D z, D v \rangle \nu &= \left\langle (-\frac{x}{2\pi^2 r^4} + O(1)), \left(\nabla_x(H(x, x_1) + \sum_{j=2}^{N} \nabla_x G(x, x_j) + \frac{x}{8\pi^2 r^2}\right)\right\rangle \\
&= -\left(\nabla_x H(x_3^\star, x_1) + \sum_{j=2}^{k} \nabla_x G(x_3^\star, x_j)\right) + O(r)
\end{aligned}$$

where $x_i^\star \in B_r(x_1)$ for $i = 1, 2, 3$. Letting $r \to 0$, we have $x_i^\star \to x_1$ and hence we have

$$\nabla_x H(x_1, x_1) + \sum_{j=2}^{k} \nabla_x G(x_1, x_j) = 0.$$

We can obtain the other identities in a similar way. □

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