GLOBAL SOLUTION BRANCH AND MORSE INDEX ESTIMATES OF A SEMILINEAR ELLIPTIC EQUATION WITH SUPER-CRITICAL EXPONENT

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Abstract. We consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = u^p + \lambda u \ \text{ in } B, \\ u > 0 \ \text{ in } B, \quad u = 0 \ \text{ on } \partial B \end{cases} \tag{0.1}$$

where B denotes the unit ball in $\mathbb{R}^N$, $N \geq 3$, $\lambda > 0$ and $p > (N+2)/(N-2)$. According to classical bifurcation theory, the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathscr{C}$ of solutions (λ, u) of (0.1), where μ_1 is the principle eigenvalue of $-\Delta$ in B with Dirichlet boundary data. It is known that there is a unique value $\lambda = \lambda_* \in (0, \mu_1)$ such that (0.1) has a radial singular solution $u_*(|x|)$. Let $p_c > \frac{N+2}{N-2}$ be the Joseph-Lundgren exponent. We show that the structure of the branch $\mathscr{C}$ changes for $p \geq p_c$ and $(N+2)/(N-2) < p < p_c$. For $(N+2)/(N-2) < p < p_c$, $\mathscr{C}$ turns infinitely many times around λ_* which implies that the all the singular solutions have infinite Morse index. For $p \geq p_c$, we show that all solutions (regular or singular) have finite Morse index. For $N \geq 12$ and $p > p_c$ large, we show that all solutions (regular or singular) have exactly Morse index one. As a consequence, we prove that any regular solution intersects with the singular solution exactly once and regular solution exists (and is unique) only when $\lambda \in (\lambda_*, \mu_1)$.

1. Introduction

Let B be the unit ball in $\mathbb{R}^N$ $(N \geq 3)$. In this paper, we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = u^p + \lambda u \ \text{ in } B, \\ u > 0 \ \text{ in } B, \quad u = 0 \ \text{ on } \partial B \end{cases} \tag{1.1}$$

where

$$p > p_N := \frac{N+2}{N-2}, \quad \lambda \in \mathbb{R}.$$

By [8], any solution of Problem (1.1) is radially symmetric. It is easy to see that there exist no solutions for (1.1) if $\lambda \geq \mu_1$ or $\lambda \leq 0$, where μ_1 is the principle eigenvalue of $-\Delta$ in B with Dirichlet data. According to classical bifurcation theory [15], the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded

1991 *Mathematics Subject Classification.* Primary 35B45; Secondary 35J40.

Key words and phrases. Branch of positive radial solutions, infinitely many turning points, Morse index, semilinear elliptic problems with super-critical exponent, Bessel function.

branch $\mathscr{C}$ of solutions (λ, u). In this paper we are interested in the structure of the branch $\mathscr{C}$.

For $1 < p < \frac{N+2}{N-2}$, there exists at most one solution of (1.1) when $\lambda < \mu_1$ ([17]). When $p = \frac{N+2}{N-2}$, Brezis and Nirenberg established that (1.1) is solvable for $\bar{\lambda} < \lambda < \mu_1$ where $\bar{\lambda} = 0$ when $N \geq 4$ and $\bar{\lambda} = \frac{1}{4}\mu_1$ when $N = 3$. The situation drastically changes as soon as $p > \frac{N+2}{N-2}$. Del Pino-Dolbeault-Musso [5] constructed bubble-tower solutions of (1.1) for a slightly supercritical exponent in dimension $N \geq 4$. Budd and Norbury [2] used formal asymptotic and numerical computations to derive some qualitative properties of the solution branch when $N = 3$, $p > 5$: they found that before reaching $\lambda = 0$, the solution curve turns right and oscillates infinitely many times in the form of an exponentially damped sinusoidal along a line $\lambda = \lambda_*$. Merle and Peletier [11] proved that there is a unique value $\lambda = \lambda_* > 0$ such that a singular solution u_* exists for (1.1). Moreover,

$$u_*(r) = A(p,N)r^{-2/(p-1)}\{1 - B(p,N)r^2 + o(r^2)\} \text{ as } r \to 0, \tag{1.2}$$

where

$$A(p,N) = \Big[\frac{2}{p-1}\Big(N-2-\frac{2}{p-1}\Big)\Big]^{1/(p-1)}, \quad B(p,N) = 4\lambda_*\Big(N-1-\frac{3}{p-1}\Big)^{-1}. \tag{1.3}$$

Merle-Peletier-Serrin [12] studied the asymptotic behavior of the positive solutions $(\lambda_p, u_p) \in \mathscr{C}$ as $p \to \infty$. Recently, using geometric theory of dynamical system, Dolbeault and Flores [6] rigorously proved the numerical computations in [2] in the case of $p < p_c$—the Joseph-Lungren exponent (see (1.5)).

An analogue problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p \text{ in } B, \\ u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B \end{cases} \tag{1.4}$$

has been completely understood. It is known [10] that there exists a unique $\lambda_* > 0$ such that the solution to (1.4) exists only when $\lambda \leq \lambda_*$. Let

$$p_c := \begin{cases} \infty & \text{if } 2 \leq N \leq 10 \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11, \end{cases} \tag{1.5}$$

be the so-called Joseph-Lundgren exponent introduced in [10]. When $p < p_c$, there exists another number $\lambda^* < \lambda_*$ such that regular solution is smooth up to $\lambda = \lambda_*$, singular solution exists at $\lambda = \lambda^*$, and solutions branch turns infinitely many times at $\lambda = \lambda^*$. For $p \geq p_c$, there are no secondary branch and the singular solution exists precisely at $\lambda = \lambda_*$ and it is stable.

The exponent p_c has long been known to play an important role in semilinear heat equations with power-like nonlinearities. See Gui-Ni-Wang [9], Polacik-Yanagida [13], Fila-Winkler-Yanagida [7] and Wang [16] and the references therein.

The study of (1.1) turns out to be more difficult and delicate than (1.4). The main difficulty is that any solution (regular or singular) to (1.1) is unstable. An important question is then to estimate the Morse index of the solutions (regular or singular). In this paper we shall see that p_c also plays an important role in the structure of the branch $\mathscr{C}$, i.e., $\mathscr{C}$ turns infinitely many times around $\lambda = \lambda_*$ provided $(N+2)/(N-2) < p < p_c$, but this does not occur when $p \geq p_c$. The first conclusion was obtained recently in [6] by using tools of the geometric theory of dynamical systems. Nothing is known for the case $p \geq p_c$. The results obtained in this paper for the case $p \geq p_c$ are new. In some cases, we obtain the optimal results. We will use different methods to deal with this problem. We show that the Morse index of u_* is ∞ provided $(N+2)/(N-2) < p < p_c$, but it is finite when $p \geq p_c$. To show the first conclusion, we use some arguments similar to those in [4]. We also present some sufficient conditions to see that the Morse index of u_* is 1 when $p \geq p_c$. Note that this holds only for $N \geq 11$ and this is optimal.

The main results of this paper are summarized in the following three theorems:

Theorem 1.1. *Assume $N \geq 3$, $(N+2)/(N-2) < p < p_c$. There is a unique number $\lambda_* > 0$ such that, given any integer $k \geq 1$, there exist at least k bounded radial solutions of (1.1) for any λ sufficiently close to λ_*. In particular, there are infinitely many classical solutions of (1.1) for $\lambda = \lambda_*$.*

Theorem 1.2. *For $N \geq 11$ and $p \geq p_c$, the Morse index of any solution (singular or regular) u_λ of (1.1) is finite. As a consequence, the graph of any regular solution intersects with that of the singular solution at most finitely many times.*

Theorem 1.3. *For $N \geq 12$ and $p \geq p_c^2(N)$, where $p_c^2(N) \geq p_c$ can be computed explicitly, the Morse index of any solution (singular or regular) u_λ of (1.1) is exactly 1. As a consequence, the regular solution intersects with the singular solution only once and the regular solution exists and is unique when $\lambda \in (\lambda_*, \mu_1)$.*

Theorem 1.1 has been proved by Dolbeault and Flores [6], using geometric dynamical system method. Here we shall give a PDE proof. Theorems 1.2 and 1.3 are new. Theorem 1.3 gives a complete description of the solution branch. The existence of another supercritical exponent $p_c^2 \geq p_c$ is interesting.

A simple estimate on the number $p_c^2(N)$ can be given as follows. Let $J_\nu(r)$ be the Bessel function satisfying

$$J_\nu'' + \frac{1}{r} J_\nu' + \left(1 - \frac{\nu^2}{r^2}\right) J_\nu = 0, \quad r \in (0, \infty). \tag{1.6}$$

Let $j_{1,\nu}$ and $j_{2,\nu}$ be the first two zeroes of $J_\nu(r)$. Then Theorem 1.3 holds under the following condition

$$j_{1,\frac{N-2}{2}} \le j_{2,\nu} \tag{1.7}$$

where

$$\nu = \frac{1}{2}\sqrt{(N-2-\frac{4}{p-1})^2 - 8(N-2-\frac{2}{p-1})} \ge 0 \text{ when } p \ge p_c. \tag{1.8}$$

Notice that $\mu_1 = j^2_{1,\frac{N-2}{2}}$. We also note that as $p \to +\infty$, condition (1.7) becomes

$$j_{1,\frac{N-2}{2}} \le j_{2,\frac{\sqrt{(N-2)(N-10)}}{2}} \tag{1.9}$$

which can be shown to hold when $N \ge 12$. This is also the reason behind the condition $N \ge 12$ in Theorem 1.3.

Finally, let us also observe the following fact

$$\begin{array}{ll} \frac{2p}{p-1}\left(N-2-\frac{2}{p-1}\right) > \frac{(N-2)^2}{4} & \text{for } \frac{N+2}{N-2} < p < p_c \\ \frac{2p}{p-1}\left(N-2-\frac{2}{p-1}\right) \le \frac{(N-2)^2}{4} & \text{for } p \ge p_c. \end{array}$$

This also implies

$$\begin{array}{ll} \left(N-2-\frac{4}{p-1}\right)^2 - 8\left(N-2-\frac{2}{p-1}\right) < 0 & \text{for } \frac{N+2}{N-2} < p < p_c \\ \left(N-2-\frac{4}{p-1}\right)^2 - 8\left(N-2-\frac{2}{p-1}\right) > 0 & \text{for } p > p_c \\ \left(N-2-\frac{4}{p-1}\right)^2 - 8\left(N-2-\frac{2}{p-1}\right) = 0 & \text{for } p = p_c. \end{array} \tag{1.10}$$

The organization of the paper is as follows: in Section 2, we prove Theorem 1.1. In Section 3, we show that the Morse index of any solution is finite and then we prove Theorem 1.2. In Section 4, we prove that under some conditions the Morse index of any solution is one and thus prove Theorem 1.3. We leave the proof of one key theorem to Section 5. In Section 6, we present some estimates on the exponent p_c^2.

2. The case for $(N+2)/(N-2) < p < p_c$: Proof of Theorem 1.1

In this section we present a PDE proof of Theorem 1.1. We only need to consider the equation

$$\begin{cases} -u'' - \frac{N-1}{r}u' = u^p + \lambda u \ \text{ in } (0,1) \\ u > 0 \ \text{ in } (0,1), \ \ u(1) = 0 \end{cases} \tag{2.1}$$

Let $w = \lambda^{-1/(p-1)}u$. We see that w satisfies the problem

$$\begin{cases} -w'' - \frac{N-1}{r}w' = \lambda[w^p + w] \ \text{ in } (0,1) \\ w > 0 \ \text{ in } (0,1), \ \ w(1) = 0. \end{cases} \tag{2.2}$$

We also know that there is a unique $\lambda = \lambda_*$, such that (2.2) has a singular solution w_*. Introducing the changes:

$$t = \lambda_*^{1/2} r, \quad W(t) = w_*(r)$$

we see that W satisfies the problem

$$\text{(2.3)} \qquad \begin{cases} -W'' - \frac{N-1}{t} W' = W^p + W \ \text{ in } (0, \lambda_*^{1/2}) \\ W > 0 \ \text{ in } (0, \lambda_*^{1/2}), \ \ W(\lambda_*^{1/2}) = 0. \end{cases}$$

Moreover, we can choose subsequences $\{(\lambda_n, u_n)\} \equiv (\lambda_n, u_{\lambda_n})\} \subset \mathscr{C}$ with $\lambda_n \to \lambda_*$, $\max_B u_n \to \infty$ as $n \to \infty$. Then by the changes

$$t = \lambda_n^{1/2} r, \quad W_n(t) = \lambda_n^{-1/(p-1)} u_n(r)$$

we see that W_n satisfies the problem

$$\text{(2.4)} \qquad \begin{cases} -W_n'' - \frac{N-1}{t} W_n' = W_n^p + W_n \ \text{ in } (0, \lambda_n^{1/2}) \\ W_n > 0 \ \text{ in } (0, \lambda_n^{1/2}), \ \ W_n(\lambda_n^{1/2}) = 0. \end{cases}$$

It is known from Theorem 1.2 of [11] that $W_n \to W$ in $C^0_{loc}(0, \lambda_*^{1/2})$ as $n \to \infty$.

We have the following proposition.

Proposition 2.1. *For any sequence $\{(\lambda_n, W_n)\} \equiv \{(\lambda_n, W_{\lambda_n})\}$ with $\lambda_n \to \lambda_*$, $\max_B W_n \to \infty$ and any $M >> 1$, there is $N^* = N^*(M) > 1$ such that for $n > N^*$, the graph of W_n intersects with that of W at least M times in $(0, \min\{\lambda_n^{1/2}, \lambda_*^{1/2}\})$.*

Proposition 2.1 implies that the Morse index of W is ∞ and hence the Morse index of u_* is ∞. Indeed, we see that $W - W_n$ has at least M zeroes in $(0, \min\{\lambda_n^{1/2}, \lambda_*^{1/2}\})$ and thus there are at least $[\frac{M}{2}] - 1$ intervals I_i $(i = 1, 2, \dots, [\frac{M}{2}] - 1)$ on which $W - W_n > 0$. We also see that $h_n^i := W - W_n$ satisfies

$$\text{(2.5)} \qquad \begin{cases} -\Delta h_n^i < pW^{p-1} h_n^i + h_n^i \ \text{ in } I_i, \\ h_n^i > 0 \ \text{ in } I_i, \ \ h_n^i = 0 \ \text{ on } \partial I_i \end{cases}$$

Multiplying h_n^i on both the sides of the equation in (2.5) and integrating it on the annular domain $\Omega_i := \{y : |y| \in I_i\}$, we see that

$$\int_{\Omega_i} [|\nabla h_n^i|^2 - (pW^{p-1} + 1)(h_n^i)^2] < 0.$$

Since each $h_n^i \in H_0^1(B_*)$, where $B_* := \{y : \ |y| < \lambda_*^{1/2}\}$,

$$\int_{B_*} h_n^i h_n^j dy = 0, \quad i \neq j,$$

the arbitrariness of M implies that the Morse index of W is ∞.

Proof of Proposition 2.1

The existence of $\{(\lambda_n, W_n)\}$ is known from the existence of the sequence $\{(\lambda_n, u_n)\}$. On the contrary, this is a sequence $\{(\lambda_n, W_n)\}$ satisfying the conditions in this proposition, we have that there is $t_0 \in (0, \min\{\lambda_n^{1/2}, \lambda_*^{1/2}\})$ independent of n such that $W > W_n$ in $(0, t_0)$ (note that $W(0) = \infty$). Since $W_n(t) = \lambda_n^{-1/(p-1)} u_n$, it follows from Theorem 1.2 of [11] that

$$W_n \to W \ \text{ in } C^0_{loc}(0, \lambda_*^{1/2}) \text{ as } n \to \infty.$$

Let $z_n(\rho) = W_n/W$, $\rho = \ln t$. Then z_n satisfies

$$z_n''(\rho) + \Big[N - 2 + \frac{2tW_t(t)}{W}\Big] z_n'(\rho) + t^2 W^{p-1}[z_n^p - z_n] = 0, \quad \rho < \ln t_0. \tag{2.6}$$

Since $w_*(r) = \lambda_*^{-1/(p-1)} u_*(r)$ and $W(t) = w_*(r)$ with $t = \lambda_*^{1/2} r$, we see from (1.2) that

$$W(t) = A(p,N) t^{-2/(p-1)} (1 - B(p,N)\lambda_*^{-1} t^2 + o(t^2)) \ \text{ as } t \to 0. \tag{2.7}$$

Thus,

$$\frac{2tW_t(t)}{W(t)} \le -\frac{4}{p-1} \ \text{ for } t \text{ sufficiently small} \tag{2.8}$$

$$\frac{2tW_t(t)}{W(t)} \to -\frac{4}{p-1} \ \text{ as } t \to 0 \tag{2.9}$$

and

$$t^2 W^{p-1}(t) \le \frac{2}{p-1}\Big(N - 2 - \frac{2}{p-1}\Big) \ \text{ for } t \text{ sufficiently small} \tag{2.10}$$

$$t^2 W^{p-1}(t) \to \frac{2}{p-1}\Big(N - 2 - \frac{2}{p-1}\Big) \ \text{ as } t \to 0. \tag{2.11}$$

Since $z_n(\rho) < 1$ for $\rho \in (-\infty, \ln t_0)$ and $W(t) \to \infty$ as $t \to 0$, we see that there exists $-\infty < T < \ln t_0$ such that

$$z_n''(\rho) + g_1(\rho) z_n'(\rho) \ge 0 \ \text{ for } \rho \in (-\infty, T) \tag{2.12}$$

where $g_1(\rho) = N - 2 + \frac{2tW_t(t)}{W(t)}$. Thus,

$$\exp\Big(\int_{-\infty}^{\rho} g_1(s)ds\Big) z_n'(\rho) \ge \exp\Big(\int_{-\infty}^{\tau} g_1(s)ds\Big) z_n'(\tau) \ \text{ if } t \ge \tau > -\infty. \tag{2.13}$$

We know that $g_1(\rho) \to N - 2 - \frac{4}{p-1}$ as $\rho \to -\infty$ and $z_n(\rho) \to 0$ as $\rho \to -\infty$. Then it follows from (2.13) that

$$z_n'(\rho) \ge 0 \ \text{ for } -\infty < \rho < T. \tag{2.14}$$

We see that $z_n'(\rho) \not\equiv 0$ since $W_n \not\equiv W$.

Let $\omega_n(\rho) = 1 - z_n(\rho)$. Then by (2.6), we see

$$\omega_n''(\rho) + g_1(\rho)\omega_n' + g_2^n(\rho)\omega_n = 0, \quad \omega_n > 0 \text{ on } (-\infty, T) \tag{2.15}$$

where

$$g_2^n(\rho) = t^2 W^{p-1}(t)\Big[\frac{z_n^p - z_n}{1 - z_n}\Big].$$

Since $z_n \to 1$ in $C^0_{loc}(-\infty, T)$ as $n \to \infty$, we see that

$$\frac{z_n^p - z_n}{1 - z_n} \to -(p-1) \text{ in } C^0_{loc}(-\infty, T) \text{ as } n \to \infty.$$

This, the fact that

$$t^2 W^{p-1}(t) \to \frac{2}{p-1}\Big(N - 2 - \frac{2}{p-1}\Big) \text{ as } t \to 0$$

and

$$\Big(N - 2 - \frac{4}{p-1}\Big)^2 - 8\Big(N - 2 - \frac{2}{p-1}\Big) < 0 \text{ for } (N+2)/(N-2) < p < p_c$$

imply that there exists $T^* \in (-\infty, T)$ such that for any interval $[T_2, T_1] \subset (-\infty, T^*)$ and n sufficiently large

$$[g_1]^2 - 4g_2^n < 0 \text{ in } [T_2, T_1]. \tag{2.16}$$

Thus there exist b_1 and c_1 such that $b_1^2 - 4c_1 < 0$, $g_1(\rho) < b_1$ and $g_2^n(\rho) > c_1$ if $\rho \in [T_2, T_1]$. Observe that any solution of

$$Z''(\rho) + b_1 Z'(\rho) + c_1 Z(\rho) = 0 \tag{2.17}$$

is oscillatory; in particular, there exist $T_2 < a_2 < b_2 < T_1$ such that $Z(a_2) = Z(a_3) = 0$, $Z > 0$ in (a_2, b_2) (and hence $Z'(a_2) > 0 > Z'(b_2)$). Multiplying (2.15) by Z and (2.17) by ω_n, we have

$$\omega_n'' Z + g_1(\rho)\omega_n' Z + g_2^n(\rho)\omega_n Z = 0 \text{ on } [a_2, b_2] \tag{2.18}$$

$$Z''\omega_n + b_1 Z'\omega_n + c_1 Z\omega_n = 0 \text{ on } [a_2, b_2]. \tag{2.19}$$

Subtracting (2.19) from (2.18) yields

$$(Z\omega_n' - Z'\omega_n)' + (g_1(\rho)\omega_n' Z - b_1 Z'\omega_n) + (g_2^n - c_1)\omega_n Z = 0 \text{ on } [a_2, b_2].$$

Thus, by the fact that $g_1(\rho) < b_1$, $g_2^n(\rho) > c_1$ and $\omega_n' \le 0$, we have

$$(Z\omega_n' - Z'\omega_n)' + b_1(\omega_n' Z - Z'\omega_n) < 0 \text{ on } (a_2, b_2)$$

and hence

$$e^{b_1 b_2} Z'(b_2)\omega_n(b_2) > e^{b_1 a_2} Z'(a_2)\omega_n(a_2).$$

This is impossible (note that $Z'(a_2) > 0 > Z'(b_2)$). This completes the proof. □

To prove Theorem 1.1, we only need to obtain similar results for the problem (2.2). According to classical theory [15], the point $(\mu_1, 0)$ is a bifurcation point from which emanates an unbounded branch $\mathscr{A}$ of solutions (λ, w) of (2.2).

We show the following theorem.

Theorem 2.2. *Let $N \geq 3$ and $\frac{N+2}{N-2} < p < p_c$. Then the radial solution branch $\mathscr{A}$ of (2.2) possesses infinitely many turning points around $\lambda = \lambda_*$.*

It is clear that Theorem 2.2 implies that the conclusions of Theorem 1.1 hold. To prove Theorem 2.2, we first prove the following lemma.

Lemma 2.3. *For any $\kappa \in (0, \infty)$, there is at most one $\tilde{\lambda} := \tilde{\lambda}(\kappa) \in (0, \mu_1)$ with $(\tilde{\lambda}, w_{\tilde{\lambda}}) \in \mathscr{A}$ and $w_{\tilde{\lambda}}(0) = \kappa$.*

Proof. Suppose that there are λ_1, $\lambda_2 \in (0, \mu_1)$ with $\lambda_1 \neq \lambda_2$ and $(\lambda_1, w_{\lambda_1})$, $(\lambda_2, w_{\lambda_2}) \in \mathscr{A}$ such that $w_{\lambda_1}(0) = w_{\lambda_2}(0) = \kappa$. If we set $w_j \equiv w_{\lambda_j}$ for $j = 1, 2$, then

$$-w_j'' - \frac{N-1}{r} w_j' = \lambda_j [w_j^p + w_j], \quad w_j(0) = \kappa, \quad w_j'(0) = 0, \quad w_j(1) = 0. \tag{2.20}$$

Let $t = \lambda_j^{1/2} r$ and $z_j(t) = w_j(r)$. We see that z_j satisfies the problem

$$-z_j'' - \frac{N-1}{t} z_j' = z_j^p + z_j, \quad z_j(0) = \kappa, \quad z_j'(0) = 0, \quad z_j(\lambda_j^{1/2}) = 0. \tag{2.21}$$

Since z_1 and z_2 satisfies the same initial values $z_j(0) = \kappa$, $z_j'(0) = 0$, the standard ODE theory implies $\lambda_1 = \lambda_2$, which is a contradiction. This completes the proof. $\square$

Lemma 2.3 implies that the radial solution branch $\mathscr{A}$ of (2.2) does not possess secondary bifurcation point.

Proof of Theorem 2.2

The proof of Theorem 2.2 can be obtained from Proposition 2.1 and arguments similar to those in [4].

Arguments similar to those in the proof of Proposition 2.1 imply that for any $M >> 1$, we can find $N^* = N^*(M)$ such that for $n > N^*$, the Morse index of W_n is at least $[\frac{M}{2}] - 1$. (Note that instead of choosing $h_n^i = W - W_n$, we can choose $h_n^i = W_n - W$.) We can argue as in Subsection 2.1 of [3], in the space $C^1([0,1]) \times \mathbb{R}$, to find an analytic solution curve of (2.2): $\lambda = \tilde{\lambda}(s)$, $w = \tilde{w}(s)$ for $s \geq 0$, such that

$$\begin{aligned}
&\|\tilde{w}\|_\infty \to \infty \ \text{ as } s \to \infty \\
&(\tilde{w}(s), \tilde{\lambda}(s)) \in \mathscr{A} \ \text{ for } s \geq 0 \\
&(\tilde{w}(0), \tilde{\lambda}(0)) = (0, \mu_1) \ \text{ and} \\
&I - \tilde{\lambda}(s) A'(\tilde{w}(s)) \ \text{ is invertible except at isolated points}
\end{aligned}$$

where $A'(\tilde{w}(s)) := G^{-1}[(p\tilde{w}^{p-1}(s)+1)I]$ and $G(h) = r^{1-N}(r^{N-1}h'(r))'$ with $h'(0) = 0$, $h(1) = 0$. We see from Lemma 2.3 that the curve has no intersection. Let us denote this curve by T and $\mu_{i,\tilde{\lambda}(s)}(\tilde{w}(s))$ be the ith eigenvalue, counting the multiplicity, of

$$-G - \tilde{\lambda}(s)[p\tilde{w}^{p-1}(s) + 1]I \tag{2.22}$$

on $(0,1)$ with the Dirichlet boundary condition. By our comments above, $\mu_{i,\tilde{\lambda}(s)}(\tilde{w}(s))$ are continuous and piecewise analytic, and have only isolated zeroes. We will show that $\mu_{i,\tilde{\lambda}(s)}(\tilde{w}(s)) < 0$ for s large. This means that for any $\zeta > 0$, equation (2.22) has at least ζ negative eigenvalues for s large. Hence we see that there is a sequence $\{s_i\}$ with $s_i \to \infty$ as $i \to \infty$ such that the number of negative eigenvalues of (2.22) (counting multiplicity) changes at s_i. (Recall that $\mu_{i,\tilde{\lambda}(0)}(\tilde{w}(0)) = \mu_i(-G) - \mu_1 \to +\infty$ as $i \to \infty$). Each $(\tilde{w}(s_i), \tilde{\lambda}(s_i))$ must be a bifurcation point. We also see that each $(\tilde{w}(s_i), \tilde{\lambda}(s_i))$ is either a turning point and a point of secondary bifurcation. Our Lemma 2.3 implies that it is not a secondary bifurcation point. Thus, it must be a turning point.

To prove our claim on $\mu_{i,\tilde{\lambda}(s)}$ for large s, we need to consider positive solutions (w_i, λ_i) of (2.2) such that $\lambda_i \to \lambda_*$ as $i \to \infty$ and $\|w_i\|_\infty \to \infty$ as $i \to \infty$. Thus, we see that there is a s_i with $s_i \to \infty$ such that $\tilde{\lambda}(s_i) = \lambda_i$ and $\tilde{w}(s_i) = w_i$. By the changes:

$$z_i(\rho) = w_i(r), \quad \rho = \lambda_i^{1/2} r$$

we see that z_i satisfies the problem

$$-z_i'' - \frac{N-1}{\rho} z_i' = z_i^p + z_i \text{ in } (0, \lambda_i^{1/2}), \quad z_i(\lambda_i^{1/2}) = 0. \tag{2.23}$$

It is known from Proposition 2.1 that for any $M >> 1$, the Morse index of z_i in $H_0^1(\tilde{B}_i)$, where $\tilde{B}_i = \{y : \ |y| < \lambda_i^{1/2}\}$, is bigger than M provided i sufficiently large. Hence, returning to the original scaling we see that there is at least $(M-1)$-dimensional subspace E_i of $H_0^1(B)$ such that

$$\int_B \left[|\nabla h|^2 - \tilde{\lambda}(s)(p\tilde{w}^{p-1}(s) + 1)h^2\right] < 0$$

for h is in the unit sphere of E_i and s large. By the variational characterization of eigenvalues, this implies that $\mu_{i,\tilde{\lambda}(s)}(\tilde{w}(s)) < 0$ for $1 \le i \le M-1$ if s is large. Since M is arbitrary, this proves our claim and completes the proof of Theorem 2.2. □

3. The case for $p \geq p_c$: Proof of Theorem 1.2

In this section we will study the structure of radial solution branch $\mathscr{C}$ for $p \geq p_c$. We first show that for any $(\lambda, u_\lambda) \in \mathscr{C}$, the Morse index of u_λ is bounded. Then we prove Theorem 1.2.

We state our first result.

Theorem 3.1. *There exists an integer $C \geq 1$ independent of λ such that*

$$1 \leq m(u_\lambda) \leq C. \tag{3.1}$$

To prove Theorem 3.1, we need the following key estimate

Lemma 3.2. *Let u_λ be a regular or singular solution of (1.1). Then it holds*

$$u_\lambda(r) \leq A(p, N) r^{-\frac{2}{p-1}} \text{ for } r \in [0, 1], \tag{3.2}$$

where $A(p, N)$ is given in (1.3).

Proof: We first see that (3.2) holds for a regular solution u_λ. To prove (3.2), we introduce the Emden-Fowler transformation for u_λ:

$$v_\lambda(t) = r^{\frac{2}{p-1}} u_\lambda(r), \quad t = \ln r.$$

Then v_λ satisfies the following problem (without loss of generality, we omit the subscript λ on v_λ in the following)

$$v'' + \alpha v' + v^p - \beta v + \lambda e^{2t} v = 0, \quad t \in (-\infty, 0), \quad v(0) = 0, \quad v(-\infty) = 0 \tag{3.3}$$

where

$$\alpha = N - 2 - \frac{4}{p-1}, \quad \beta = A(p, N)^{p-1}.$$

Now we show that

$$v(t) \leq \beta^{1/(p-1)} (= A(p, N)) \text{ for } t \in (-\infty, 0) \tag{3.4}$$

which implies (3.2).

Suppose on the contrary. Since $v(-\infty) = 0$, we see that there is a $-\infty < \tilde{t} < 0$ such that $v(\tilde{t}) = A(p, N)$ but $v(t) < A(p, N)$ for $t \in (-\infty, \tilde{t})$.

Let $v(t) = A(p, N) + \varphi$. We see that $\varphi(t) \leq 0$ for $t \in (-\infty, \tilde{t}]$ and

$$v^p(t) \geq A(p, N)^p + pA(p, N)^{p-1}\varphi \text{ for } t \in (-\infty, \tilde{t}). \tag{3.5}$$

Hence

$$\begin{aligned} 0 &= \varphi'' + \alpha\varphi' + (A(p, N) + \varphi)^p - \beta(A(p, N) + \varphi) + \lambda e^{2t}(A(p, N) + \varphi) \\ &\geq \varphi'' + \alpha\varphi' + (p-1)\beta\varphi. \end{aligned}$$

Let $\varphi(t) = e^{\tau_1 t}\tilde{\varphi}(t)$, where

$$\tau_1 = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - 4(p-1)\beta}$$

we see that $\tilde{\varphi}$ satisfies

$$\begin{cases} \tilde{\varphi}'' + (\alpha + 2\tau_1)\tilde{\varphi}' \le 0, & t \in (-\infty, \tilde{t}) \\ \tilde{\varphi}(\tilde{t}) = 0, \quad \tilde{\varphi}(-\infty) = 0 \end{cases} \tag{3.6}$$

where we use the fact that (see (1.10))

$$\alpha^2 - 4(p-1)\beta \ge 0 \ \text{ for } p \ge p_c$$

and

$$\tau_1 = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - 4(p-1)\beta} < 0.$$

This contradicts the maximum principle sine $\tilde{\varphi} \le 0$ in $(-\infty, \tilde{t})$. This contradiction implies that (3.2) holds.

To see that the estimate (3.2) holds for u_*, we see that under the changes:

$$v_*(t) = r^{2/(p-1)}u_*(r), \quad t = \ln r$$

v_* satisfies the problem

$$v_*'' + \alpha v_*' - \beta v_* + v_*^p + \lambda_* e^{2t}v_* = 0 \ \ t \in (-\infty, 0), \ \ v_*(-\infty) = A(p,N), \ \ v_*(0) = 0. \tag{3.7}$$

But (1.2) implies that

$$v_*(t) = A(p,N) - A(p,N)B(p,N)e^{2t} + o(e^{2t}) \ \text{ for } t \text{ near } -\infty. \tag{3.8}$$

Therefore, if we set $v_* = A(p,N) + \varphi$, we see that $\varphi(t) = -A(p,N)B(p,N)e^{2t} + o(e^{2t}) < 0$ for t near $-\infty$ and

$$e^{-\tau_1 t}\varphi(t) \to 0 \ \text{ as } t \to -\infty.$$

The fact that

$$v_*(t) \le A(p,N) \ \text{ for } t \in (-\infty, 0)$$

can be obtained by arguments similar to those in the proof of (3.4).

This proves Lemma 3.2.

□

Next we prove that the Morse index of u_λ is finite.

Proof of Theorem 3.1. By (3.2), (1.10) and the Hardy's inequality, we see that

$$\int_B [|\nabla\phi|^2 - pu_\lambda^{p-1}\phi^2] \ge \int_B [|\nabla\phi|^2 - \frac{(N-2)^2}{4}\frac{\phi^2}{r^2}] \ge 0 \tag{3.9}$$

for any $\phi \in H_0^1(B)$. We can easily see that $m(u_\lambda) \geq 1$, since $u_\lambda \in H_0^1(B)$, and

$$\int_B [|\nabla u_\lambda|^2 - (pu_\lambda^{p-1} + \lambda)u_\lambda^2] = (1-p)\int_B u_\lambda^{p+1} < 0.$$

To show that the Morse index of u_λ is finite, we use a contradiction argument. On the contrary, there is a sequence $\{(\lambda_n, u_n)\} \equiv \{(\lambda_n, u_{\lambda_n})\} \subset \mathscr{C}$ such that

$$m(u_n) \to \infty \text{ as } n \to \infty.$$

We can choose a subsequence (still denoted by $\{(\lambda_n, u_n)\}$) such that $\lambda_n \to \hat{\lambda}$ as $n \to \infty$. Thus, for n sufficiently large, the number of negative eigenvalues (counting multiplicity) of the problem

$$-\Delta\varphi = [pu_n^{p-1} + \lambda_n]\varphi + \sigma\varphi \text{ in } B, \quad \varphi = 0 \text{ on } \partial B \tag{3.10}$$

is large. Therefore, the first eigenvalue σ_1^n of the problem (3.10) satisfies

$$\sigma_1^n + \hat{\lambda} < 0.$$

This is a contradiction since

$$0 \leq \int_B [|\nabla\varphi_1^n|^2 - pu_n^{p-1}(\varphi_1^n)^2] = (\sigma_1^n + \lambda_n)\int_B (\varphi_1^n)^2 < 0$$

where $\varphi_1 \in H_0^1(B)$ is the first eigenfunction of (3.10) corresponding to σ_1^n.

This proves Theorem 3.1.

□

Proofs of Theorems 1.2

The first conclusion is known from Theorem 3.1. We only need to show that the graph of any regular solution u_λ of (1.1) intersects with that of u_* at most finitely many times.

On the contrary, there exists (λ, u_λ) such that the graph of u_λ intersects with that of u_* infinitely many times. There are three cases here: $\lambda < \lambda_*$, $\lambda > \lambda_*$ and $\lambda = \lambda_*$. For $\lambda \geq \lambda_*$, we can show that $m(u_*) = \infty$. This contradicts Theorem 3.1. Indeed, since the graph of u_λ intersects with that of u_* infinitely many times, there are infinitely many intervals $J_i \subset (0,1)$ $(i = 1, 2, \ldots)$ such that $u_* > u_\lambda$ in J_i. Let

$$h_i = \begin{cases} u_* - u_\lambda, & \text{in } J_i \\ 0, & \text{in } (0,1)\backslash J_i. \end{cases}$$

We see that

$$\int_{B_i} [|\nabla h_i|^2 - (pu_*^{p-1} + \lambda_*)h_i^2]dx < 0$$

provided $\lambda \geq \lambda_*$, where $B_i = \{x, \ |x| \in J_i\}$. Note that

$$-\Delta h_i < pu_*^{p-1}h_i + \lambda_* h_i \quad \text{in } J_i.$$

This implies that $m(u_*) = \infty$.

For $\lambda < \lambda_*$, we can show that $m(u_\lambda) = \infty$. This contradicts Theorem 3.1 again. Similarly, there are infinitely many intervals $J_k \subset (0,1)$ $(k = 1,2,\ldots)$ such that $u_\lambda > u_*$ in J_k. Let

$$h_k = \begin{cases} u_\lambda - u_*, & \text{in } J_k \\ 0, & \text{in } (0,1)\backslash J_k. \end{cases}$$

We see that

$$\int_{B_k} [|\nabla h_k|^2 - (pu_\lambda^{p-1} + \lambda)h_k^2]dx < 0$$

provided $\lambda < \lambda_*$, where $B_k = \{x,\ |x| \in J_k\}$. Note that

$$-\Delta h_k < pu_\lambda^{p-1}h_k + \lambda h_k \quad \text{in } J_k.$$

This implies that $m(u_\lambda) = \infty$.

The proof above also implies that the graphs of any two different regular solutions can only intersect finitely many times. This completes the proof.

□

4. Morse index one solutions and Proof of Theorem 1.3

In the this section, we show that under some conditions, the Morse index of any solution is exactly one and thus prove Theorem 1.3.

To this end, it is vital to study the following linearized operator at the singular solution

$$-\Delta h = (pu_*^{p-1} + \lambda_*)h \ \text{ in } B. \tag{4.1}$$

First, we note that under the Emden-Fowler transformation:

$$\psi(t) = r^{2/(p-1)}h(r), \quad t = \ln r$$

we see that $\psi(t)$ satisfies the equation

$$\psi'' + \alpha\psi' - \beta\psi + pv_*^{p-1}\psi + \lambda_* e^{2t}\psi = 0 \ \ t \in (-\infty, 0). \tag{4.2}$$

Since $v_*(t) \to A(p,N)$ as $t \to -\infty$ and the characteristic equation of

$$\psi'' + \alpha\psi' + (p-1)\beta\psi = 0$$

is

$$\xi^2 + \alpha\xi + (p-1)\beta = 0,$$

we see that (4.2) has two fundamental solutions $\psi_1(t)$ and $\psi_2(t)$ with

$$\psi_1(t) \sim e^{\xi_1 t}, \quad \psi_2(t) \sim e^{\xi_2 t} \ \text{ as } t \to -\infty \tag{4.3}$$

where

$$\xi_1 = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - 4(p-1)\beta}, \quad \xi_2 = -\frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - 4(p-1)\beta}. \tag{4.4}$$

Note that

$$\xi_1 < 0, \quad \xi_2 < 0, \quad |\xi_2| > |\xi_1| > 2 \tag{4.5}$$

provided that $p \geq p_c$.

Our main result on (4.1) is the following.

Theorem 4.1. *There exists $P := p_c^2(N) > 0$ such that for $p \geq \max\{p_c, P, 2\}$, then $\psi_1(0) < 0$. As a consequence, Problem (4.1) has a weak radial solution $h \in H^1(B)$ with $h(1) \neq 0$.*

We discuss several applications of Theorem 4.1. The proof of it is delayed to the next section.

As a first corollary of Theorem 4.1, we have

Corollary 4.2. *Problem (4.1) does not have any weak radial solution in $H_0^1(B)$.*

Proof. Suppose on the contrary, there is a weak radial function $k \in H_0^1(B)$ satisfying

$$-\Delta k = (pu_*^{p-1} + \lambda_*)k.$$

By the regularity of $-\Delta$, we see that $k \in C^1((0,1])$. Multiplying h on both the sides of (4.1) and integrating it on B, we see that $h(1)k'(1) = 0$. The maximum principle implies that $k'(1) < 0$. Then $h(1) = 0$, a contradiction. □

Next we obtain the following theorem.

Theorem 4.3. *Assume that the conditions in Theorem 4.1 hold. Then the Morse index of u_* is 1.*

Proof. To see this, we show that for any sequence $\{(\lambda_n, u_n)\} \subset \mathscr{C}$ satisfying $\lambda_n \to \lambda_*$, $\max_B u_n \to \infty$,

$$m(u_n) = 1. \tag{4.6}$$

Considering the eigenvalue problem

$$-\Delta k = (pu_n^{p-1} + \lambda_n)k + \kappa k \text{ in } B, \quad k = 0 \text{ on } \partial B, \tag{4.7}$$

suppose $m(u_n) \geq 2$, then the second eigenfunction $k_n \in H_0^1(B) \cap C^2(0,1)$ corresponding to the second eigenvalue $\kappa_n < 0$ of (4.7) satisfies that k_n changes sign in $(0,1)$ and $k_n(0) = \max_B k_n$, $\|k_n\|_{L^2(B)} = 1$. We also see

$$\lambda_n + \kappa_n > 0 \text{ and } |\kappa_n| \leq C$$

since $\int_B[|\nabla k_n|^2 - pu_n^{p-1}k_n^2] > 0$ and $m(u_n) \le C$ (see Theorem 3.1). The fact $u_n \to u_*$ in $C^0(B\backslash\{0\})$ as $n \to \infty$ and the regularity of $-\Delta$ imply that $\kappa_n \to \tilde\kappa \le 0$, $k_n \to \tilde k$ in $C^2(B\backslash\{0\})$ as $n \to \infty$ (we can choose subsequences if necessary) and $\tilde k \in H_0^1(B) \cap C^2(B\backslash\{0\})$, which changes sign in $(0,1)$, satisfies

$$-\Delta\tilde k = (pu_*^{p-1} + \lambda_*)\tilde k + \tilde\kappa\tilde k \ \text{ in } B, \quad \tilde k = 0 \ \text{ on } \partial B. \tag{4.8}$$

(Note that $\int_B |\nabla k_n|^2 \le p\beta + \lambda_n$.) We easily see that $\tilde\kappa < 0$, otherwise, $\tilde\kappa = 0$, contradicts to Corollary 4.2.

The Emden-Fowler transformation as above implies that $\hat k(t) = r^{2/(p-1)}\tilde k(r)$ satisfies the problem

$$\hat k'' + \alpha\hat k' + pv_*^{p-1}\hat k - \beta\hat k + (\lambda_* + \tilde\kappa)e^{2t}\hat k = 0 \ \ t \in (-\infty, 0), \ \ \hat k(0) = 0. \tag{4.9}$$

Moreover, $\hat k$ changes sign in $(-\infty, 0)$. By a little variant of the proof of Theorem 4.1 (note that $\tilde\kappa y(t) > 0$ for $t \le t_*$ in (5.8)), we see that the fundamental solution $\hat k_*$ satisfying $\hat k_*(t) \sim e^{\xi_1 t}$ at $t = -\infty$ has the property $\hat k_*(0) < 0$. Thus, Corollary 4.2 implies that $\hat k$ can not exist. This completes the proof. □

Corollary 4.4. *Under the conditions of Theorem 4.3, (1.1) has no regular solution for* $\lambda \le \lambda_*$.

Proof. We first consider the case $\lambda < \lambda_*$. On the contrary, we see that there are two possibilities: (i) $u_* > u_\lambda$ in $[0,1)$, (ii) the graph of u_λ intersects with that of u_* more than two times in $(0,1)$.

Indeed, if the graph of u_λ intersects with that of u_* just once in $(0,1)$, we can easily see that $\lambda > \lambda_*$. Making the changes:

$$t = \lambda_*^{1/2} r, \ \ w_*(t) = \lambda_*^{-1/(p-1)} u_*(r)$$

and

$$t = \lambda^{1/2} r, \ \ w_\lambda(t) = \lambda^{-1/(p-1)} u_\lambda(r)$$

we see that $w_*(t)$ and $w_\lambda(t)$ satisfy the problems

$$-\Delta w_* = w_*^p + w_*, \ \ w_*(\lambda_*^{1/2}) = 0$$

$$-\Delta w_\lambda = w_\lambda^p + w_\lambda, \ \ w_\lambda(\lambda^{1/2}) = 0$$

respectively. Since there is $r_0 \in (0,1)$ such that $u_*(r_0) = u_\lambda(r_0)$, then $u_\lambda(r) > u_*(r)$ for $r \in (r_0, 1)$. This implies $w_\lambda(t) > w_*(t)$ for $t \in (\lambda^{1/2} r_0, \lambda^{1/2})$. It is clear that $\lambda_*^{1/2} > \lambda_*^{1/2} r_0 > \lambda^{1/2} r_0$. Since $w_*(\lambda_*^{1/2}) = 0$ and $w_\lambda(\lambda^{1/2}) = 0$, we see that $\lambda_*^{1/2} < \lambda^{1/2}$. This contradicts the fact $\lambda < \lambda_*$.

Now we show that (i) and (ii) are also impossible.

For Case (i), we see that

$$-\Delta u_* = u_*^p + \lambda_* u_* \ \text{ in } B, \quad u_* = 0 \ \text{ on } \partial B$$

and

$$-\Delta u_\lambda = u_\lambda^p + \lambda u_\lambda \ \text{ in } B, \quad u_\lambda = 0 \ \text{ on } \partial B.$$

Multiplying u_λ on both the sides of the equation of u_* and integrating it on B, we obtain

$$\int_B (u_*^{p-1} - u_\lambda^{p-1}) u_* u_\lambda = (\lambda - \lambda_*) \int_B u_* u_\lambda < 0$$

a contradiction.

For Case (ii), we can obtain that $m(u_*) \geq 2$. Indeed, there are $r_0 < r_1 < r_2 \leq 1$ such that $u_* > u_\lambda$ in $(0, r_0)$, $u_* < u_\lambda$ in (r_0, r_1), $u_* > u_\lambda$ in (r_1, r_2). Making the changes:

$$t = \lambda_*^{1/2} r, \quad w_*(t) = \lambda_*^{-1/(p-1)} u_*(r)$$

and

$$t = \lambda^{1/2} r, \quad w_\lambda(t) = \lambda^{-1/(p-1)} u_\lambda(r)$$

we see that w_* and w_λ satisfy the problem

$$\begin{cases} -w_*'' - \frac{N-1}{t} w_*' = w_*^p + w_* & \text{in } (0, \lambda_*^{1/2}) \\ w_* > 0 \ \text{ in } (0, \lambda_*^{1/2}), \quad w_*(\lambda_*^{1/2}) = 0 \end{cases}$$

and

$$\begin{cases} -w_\lambda'' - \frac{N-1}{t} w_\lambda' = w_\lambda^p + w_\lambda & \text{in } (0, \lambda^{1/2}) \\ w_\lambda > 0 \ \text{ in } (0, \lambda^{1/2}), \quad w_\lambda(\lambda^{1/2}) = 0 \end{cases}$$

respectively. Now we claim that the graph of w_* intersects with that of w_λ at least two times in $(0, \lambda^{1/2})$. Since $u_* < u_\lambda$ in (r_0, r_1) and $\lambda^{-1/(p-1)} \geq \lambda_*^{-1/(p-1)}$, we see that there is an interval $J \subset (0, \lambda^{1/2})$ such that $w_\lambda > w_*$ in J. On the other hand, since $w_*(0) = \infty$, we see that there is an interval $(0, \tilde{t})$ such that $w_* > w_\lambda$ in this interval. Moreover, since $w_\lambda(\lambda^{1/2}) = 0$, $w_*(\lambda_*^{1/2}) = 0$ and $\lambda_* > \lambda$, we see that there is another interval $(\hat{t}, \lambda^{1/2})$ such that $w_* > w_\lambda$ in this interval. This implies that our claim is true. Let $w_\lambda \equiv 0$ in $[\lambda^{1/2}, \lambda_*^{1/2}]$. Arguments similar to those in the proof of Proposition 2.1 imply that $m(w_*) \geq 2$. This implies that $m(u_*) \geq 2$ and contradicts the fact $m(u_*) = 1$.

It remains to consider the case $\lambda = \lambda_*$. Suppose that there is a regular solution u^* for (1.1) with $\lambda = \lambda_*$. We see from arguments as the above that its graph intersects with the graph of u_* exactly once. Now we show that $m(u_*) \geq 2$. Arguments similar to those in the proof of Theorem 4.3 imply that there exist $\zeta < 0$ with $\lambda_* + \zeta > 0$ and $m \in H_0^1(B)$ with $m(r) > 0$ for $r \in (0,1)$ such that

$$\int_B [|\nabla m|^2 - (pu_*^{p-1} + \lambda_*)m^2] = \zeta \int_B m^2.$$

Thus, ζ is the first eigenvalue and m is the first eigenfunction of the problem

$$-\Delta h = (pu_*^{p-1} + \lambda_*)h + \kappa h \quad \text{in } B \quad h = 0 \text{ on } \partial B.$$

On the other hand, we see that there is $r_0 \in (0,1)$ such that $u_*(r) > u^*(r)$ for $r \in (0, r_0)$ and $u_*(r_0) = u^*(r_0)$. Let

$$m_*(r) = \begin{cases} u_*(r) - u^*(r) & \text{for } r \in (0, r_0) \\ 0 & \text{otherwise} \end{cases}$$

We see that

$$\int_{B_*} [|\nabla m_*|^2 - (pu_*^{p-1} + \lambda_*)m_*^2] < 0.$$

Thus, $m(u_*) \geq 2$. This contradicts $m(u_*) = 1$. This completes the proof. □

Corollary 4.5. *Under the conditions of Theorem 4.3, the graph of a regular solution u_λ intersects with that of u_* only once in $(0,1)$.*

Proof. Using the changes:

$$t = \lambda_*^{1/2} r, \quad w_*(t) = \lambda_*^{-1/(p-1)} u_*(r)$$

and

$$t = \lambda^{1/2} r, \quad w_\lambda(t) = \lambda^{-1/(p-1)} u_\lambda(r)$$

we see that $w_*(t)$ and $w_\lambda(t)$ satisfy the problems

$$-\Delta w_* = w_*^p + w_*, \quad w_*(\lambda_*^{1/2}) = 0$$

$$-\Delta w_\lambda = w_\lambda^p + w_\lambda, \quad w_\lambda(\lambda^{1/2}) = 0$$

respectively. Since $w_\lambda(0) < w_*(0)$ and $\lambda > \lambda_*$, we easily see that the graph of w_λ intersects with that of w_*. This implies that the graph of u_λ must intersect with that of u_*. If the graph of u_λ intersects with that of u_* more than two times, we can easily obtain that $m(u_*) \geq 2$. This is a contradiction. □

Proof of Theorem 1.3.

Theorem 1.3 follows from Theorem 4.3, Corollaries 4.4 and 4.5.

□

5. Proof of Theorem 4.1

This section is devoted to the proof of Theorem 4.1. We show that ψ_1 changes sign only once in $(-\infty,0)$ and

$$\psi_1(0)<0. \tag{5.1}$$

We first show that ψ_1 must change sign in $(-\infty,0)$. In fact, since $\psi_1>0$ as $t\to-\infty$, we assume $\psi_1>0$ in $(-\infty,0)$. We see that v_* satisfies

$$Lv_*=(p-1)v_*^p$$

where

$$L\psi=\psi''+\alpha\psi'-\beta\psi+pv_*^{p-1}\psi+\lambda_*e^{2t}\psi.$$

Thus

$$\begin{aligned}(p-1)\int_{-\infty}^0 v_*^p\psi_1e^{\alpha t} &= (p-1)\int_{-\infty}^0 e^{\alpha t}(Lv_*)\psi_1\\ &= e^{\alpha t}(v_*'\psi_1-v_*\psi_1')|_{-\infty}^0\\ &= v_*'(0)\psi_1(0).\end{aligned}$$

This is impossible since $v_*'(0)<0$.

Let $\tilde\psi(t)=e^{\frac{\alpha}{2}t}\psi_1(t)$. We see that $\tilde\psi(t)\sim e^{\frac{t}{2}\sqrt{\alpha^2-4(p-1)\beta}}$. Thus

$$\tilde\psi(-\infty)=0 \ \text{ for } p>p_c. \tag{5.2}$$

(Note that $\alpha^2-4(p-1)\beta>0$ provided $p>p_c$.) Moreover, $\tilde\psi$ satisfies the problem

$$\tilde\psi''-(\beta+\frac{\alpha^2}{4})\tilde\psi+pv_*^{p-1}\tilde\psi+\lambda_*e^{2t}\tilde\psi=0,\quad \tilde\psi(-\infty)=0. \tag{5.3}$$

Let $J_\nu(r)$ denote the Bessel function satisfying (1.6). We denote the first two zeroes of $J_\nu(r)$ by $j_{1,\nu}$ and $j_{2,\nu}$. Under the Emden-Fowler transformations:

$$\psi_\nu(t)=J_\nu(r),\quad t=\ln r$$

we see that

$$\psi_\nu''(t)-\nu^2\psi_\nu+e^{2t}\psi_\nu=0,\ \ t\in(-\infty,\infty) \tag{5.4}$$

and the first and second zeros of $\psi_\nu(t)$ are $\ln j_{1,\nu}$ and $\ln j_{2,\nu}$.

Let $\rho=t-\ln j_{i,\nu}(i=1,2)$ and $\varphi_\nu(\rho)=\psi_\nu(t)$. We see from [14] that for $\nu>0$, the first and the second eigenvalues of the problem

$$\varphi_\nu''-\nu^2\varphi_\nu+\eta e^{2\rho}\varphi_\nu=0,\quad \varphi_\nu(0)=0 \tag{5.5}$$

have the following asymptotic expansions

$$\eta_{1,\nu}=(j_{1,\nu})^2=\nu^2-2\tilde a_1\nu^{4/3}+O(\nu^{2/3}) \tag{5.6}$$

(5.7) $$\eta_{2,\nu} = (j_{1,\nu})^2 = \nu^2 - 2\tilde{a}_2\nu^{4/3} + O(\nu^{2/3})$$

where $\tilde{a}_1 > \tilde{a}_2$. The corresponding eigenfunctions are

$$\varphi_1(\rho) = \psi_\nu(\rho + \ln j_{1,\nu}), \quad \rho \in (-\infty, 0)$$

$$\varphi_2(\rho) = \psi_\nu(\rho + \ln j_{2,\nu}), \quad \rho \in (-\infty, 0).$$

Let $w_0(t) = A(p,N)(1-\theta e^{2t})$ with some $0 < \theta < B(p,N)$, where $B(p,N)$ is given in (1.2) (we will choose θ below). Let $v_*(t) = w_0(t) + y(t)$. Then,

$$v_*^p = (w_0(t) + y(t))^p \geq A(p,N)^p(1 - p\theta e^{2t}) + pw_0^{p-1}y(t).$$

Moreover, y satisfies the equation

(5.8)

$$y'' + \alpha y' - \beta y + pw_0^{p-1}y + \lambda_* e^{2t}y + \lambda_* e^{2t}A(p,N)(1-\theta e^{2t}) - \theta[4 + 2\alpha + (p-1)\beta]A(p,N)e^{2t} \leq 0.$$

We now choose t_* such that

(5.9) $$\lambda_*(1 - \theta e^{2t_*}) - \theta(\lambda_* + 4 + 2\alpha + (p-1)\beta) = 0.$$

We see that for $t \leq t_*$,

$$\lambda_*(1 - \theta e^{2t}) - \theta(\lambda_* + 4 + 2\alpha + \beta(p-1)) \geq 0.$$

Hence y satisfies the equation

(5.10) $$y'' + \alpha y' - \beta y + pw_0^{p-1}y + \lambda_* e^{2t}y \leq 0, \quad t \leq t_*.$$

On the other hand, for any p, we have

$$pw_0^{p-1} \leq p\beta[1 - \min\{(p-1, 1\}\theta e^{2t}].$$

Let $\tilde{y}(t) = e^{\frac{\alpha}{2}t}y(t)$. Then $\tilde{y}(t)$ satisfies

(5.11) $$\tilde{y}''(t) - (\frac{\alpha^2}{4} - \beta(p-1))\tilde{y} + (\lambda_* + pw_0^{p-1} - p\beta) \leq 0.$$

Note that

$$\tilde{y}(t) < 0, \;\; \tilde{y}'(t) < 0 \;\text{ for } t \text{ near } -\infty$$

and

(5.12) $$\lambda_* e^{2t} + pw_0^{p-1} - p\beta \leq (\lambda_* - p\beta\min\{p-1, 1\}\theta)e^{2t}.$$

Since the first zero of $\psi_\nu(t)$ is $\ln j_{1,\nu}$, if we set

$$t = \ln j_{1,\nu} + s, \;\; \hat{\psi}_\nu(s) = \psi_\nu(t)$$

we see that $\hat{\psi}_\nu(s)$ satisfies the problem

(5.13) $$\hat{\psi}_\nu'' - \nu^2\hat{\psi}_\nu + j_{1,\nu}^2 e^{2s}\hat{\psi}_\nu = 0, \;\; s < 0, \quad \hat{\psi}_\nu(0) = 0.$$

Therefore, to keep $\tilde{y}(t) < 0$ for $t \in (-\infty, t_*)$, we only need to have

$$(\lambda_* - p\min\{p-1, 1\}\beta\theta)e^{2t_*} \le j_{1,\nu}^2, \tag{5.14}$$

with $\nu^2 = \frac{\alpha^2}{4} - (p-1)\beta$. Indeed, setting

$$\hat{y}(s) = \tilde{y}(t), \quad t = t_* + s, \tag{5.15}$$

we see from (5.11) that

$$\hat{y}'' - (\frac{\alpha^2}{4} - \beta(p-1))\hat{y} + \Theta(s)\hat{y} \le 0, \quad s < 0 \tag{5.16}$$

where

$$\Theta(s) = (\lambda_* - p\beta\min\{p-1, 1\}\theta)e^{2t_*}e^{2s}.$$

It follows from (5.13) and (5.16) that if

$$(\lambda_* - p\beta\min\{p-1, 1\}\theta)e^{2t_*} \le j_{1,\nu}^2$$

then

$$\hat{y}(s) < 0 \quad \text{for } s < 0.$$

This implies

$$\tilde{y}(t) < 0 \ \text{ for } t \in (-\infty, t_*).$$

Thus, we obtain another estimate for t_*,

$$e^{2t_*} \le \frac{j_{1,\nu}^2}{\lambda_* - p\min\{p-1, 1\}\beta\theta}. \tag{5.17}$$

Combining (5.9) and (5.17), we need

$$e^{2t_*} \le \min\Big(\frac{1}{\theta} - \frac{4 + 2\alpha + (p-1)\beta}{\lambda_*}, \ \frac{j_{1,\nu}^2}{\lambda_* - p\min\{p-1, 1\}\beta\theta}\Big). \tag{5.18}$$

Now we choose

$$\frac{1}{\theta} - \frac{4 + 2\alpha + (p-1)\beta}{\lambda_*} = \frac{j_{1,\nu}^2}{\lambda_* - p\min\{p-1, 1\}\beta\theta}.$$

Then, for $p \ge 2$,

$$\theta = \frac{\lambda_*}{p\beta}\tilde{\theta} \tag{5.19}$$

where

$$\tilde{\theta} = \frac{Q + p\beta + j_{1,\nu}^2 - \sqrt{(Q + p\beta + j_{1,\nu}^2)^2 - 4p\beta Q}}{2Q} \tag{5.20}$$

and

$$Q = 4 + 2\alpha + (p-1)\beta = 4\Big(N - 1 - \frac{3}{p-1}\Big) = \frac{16\lambda_*}{B(p, N)}.$$

Since $(Q+p\beta+j_{1,\nu}^2)^2-4p\beta Q>(Q-p\beta+j_{1,\nu}^2)^2$, we see that

$$\tilde{\theta}<\frac{p\beta}{Q}=\frac{p\beta B(p,N)}{16\lambda_*}.$$

Therefore,

$$\theta<\frac{B(p,N)}{16}. \tag{5.21}$$

Thus,

$$e^{2t_*}=\frac{p\beta-Q\tilde{\theta}}{\lambda_*\tilde{\theta}}.$$

By arguments above, we see that the solution $\tilde{\psi}(t)$ of (5.3) has no zero in $(-\infty,t_*)$. Indeed, since $v_*^{p-1}<w_0^{p-1}$ in $(-\infty,t_*)$, under the change: $t=t_*+s$, we see that

$$\begin{aligned} pv_*^{p-1}(t_*+s)+\lambda_*e^{2t_*}e^{2s} &< pw_0^{p-1}(t_*+s)+\lambda_*e^{2t_*}e^{2s}\\ &\le p\beta+[\lambda_*-\min\{p-1,1\}p\beta\theta]e^{2t_*}e^{2s}\\ &\le p\beta+j_{1,\nu}^2e^{2s}. \end{aligned}$$

The Sturm comparison principle implies that $\tilde{\psi}$ has no zero in $(-\infty,t_*)$, otherwise, $\hat{\psi}_\nu$ (with $\nu^2=\frac{\alpha^2}{4}-(p-1)\beta$) has a zero in $(-\infty,0)$, this is impossible.

On the other hand, we have that

$$v_*(t_*)\le w_0(t_*)=\frac{Q\tilde{\theta}}{p\beta}\beta^{1/(p-1)}. \tag{5.22}$$

Since $u_*'(r)<0$ for $r\in(0,1)$, we see that $e^{-\frac{2}{p-1}t}v_*(t)$ is decreasing in $(-\infty,0)$. Then

$$e^{-\frac{2}{p-1}t}v_*(t)\le e^{-\frac{2}{p-1}t_*}v_*(t_*)\quad \text{for } t\in[t_*,0).$$

The fact

$$e^{-\frac{2}{p-1}t_*}=\Big[\frac{\lambda_*\tilde{\theta}}{p\beta-Q\tilde{\theta}}\Big]^{1/(p-1)}$$

and (5.22) imply that for $t\in[t_*,0)$,

$$pv_*^{p-1}(t)e^{-2t}\le\Big(\frac{Q\tilde{\theta}}{p\beta}\Big)^{p-1}\frac{\lambda_*\tilde{\theta}}{p\beta-Q\tilde{\theta}}p\beta. \tag{5.23}$$

Since

$$(Q+p\beta+j_{1,\nu}^2)^2-4p\beta Q=(Q-p\beta+j_{1,\nu}^2)^2+4p\beta j_{1,\nu}^2$$

and

$$\frac{4p\beta j_{1,\nu}^2}{(Q-p\beta+j_{1,\nu}^2)^2}\ \text{ is bounded for any } p$$

(note that $1 \le p\beta \le (N-2)^2/4$ for $p \ge p_c$), we see that

$$\begin{aligned} p\beta - Q\tilde{\theta} &= \frac{1}{2}\Big[-(Q - p\beta + j_{1,\nu}^2) + |Q - p\beta + j_{1,\nu}^2| \sqrt{1 + \frac{4p\beta j_{1,\nu}^2}{(Q - p\beta + j_{1,\nu}^2)^2}} \Big] \\ &> \tau \frac{p\beta j_{1,\nu}^2}{Q - p\beta + j_{1,\nu}^2} \end{aligned}$$

with $0 < \tau < 1$ independent of p provided

$$Q - p\beta + j_{1,\nu}^2 > 0.$$

(Note that $(1+x)^{1/2} = 1 + \frac{1}{2}(1+\xi)^{-1/2}x \ge 1 + \frac{1}{2}(1+x)^{-1/2}x$.) The case $Q - p\beta + j_{1,\nu}^2 \le 0$ can be treated similarly. Indeed, if $Q - p\beta + j_{1,\nu}^2 = 0$, then

$$p\beta = Q\tilde{\theta} + \sqrt{p\beta j_{1,\nu}^2}.$$

Thus,

$$\frac{Q\tilde{\theta}}{p\beta} = 1 - \frac{\sqrt{p\beta j_{1,\nu}^2}}{p\beta} := \tilde{\tau} < 1.$$

If $Q - p\beta + j_{1,\nu}^2 < 0$, we see that

$$p\beta - Q\tilde{\theta} > \tau \frac{p\beta j_{1,\nu}^2}{|Q - p\beta + j_{1,\nu}^2|}.$$

We also can choose $\tilde{\tau}$ similarly. Therefore, without loss of generality, we only consider the case $Q - p\beta + j_{1,\nu}^2 > 0$. Thus

$$\frac{Q\tilde{\theta}}{p\beta} < \frac{Q - p\beta + (1-\tau) j_{1,\nu}^2}{Q - p\beta + j_{1,\nu}^2} := \tilde{\tau} < 1$$

where $\tilde{\tau}$ is independent of p. Therefore, it follows from (5.23) that

(5.24)

$$p v_*^{p-1}(t) e^{-2t} \le \frac{\mu_1 p\beta (Q - p\beta + (1-\tau) j_{1,\nu}^2)}{Q\tau j_{1,\nu}^2} \tilde{\tau}^{p-1} \le \frac{\mu_1 (N-2)^2 (Q - 1 + (1-\tau) j_{1,\nu}^2)}{4Q\tau j_{1,\nu}^2} \tilde{\tau}^{p-1}.$$

Now, we conclude that when

$$p \ge 1 + \frac{\ln \frac{4(\mu_1 - \lambda_*) Q\tau j_{1,\nu}^2}{\mu_1 (N-2)^2 (Q - 1 + (1-\tau) j_{1,\nu}^2)}}{\ln \tilde{\tau}}, \tag{5.25}$$

$$p v_*^{p-1}(t) e^{-2t} \le \mu_1 - \lambda_* \ \text{ for } t \in [t_*, 0) \tag{5.26}$$

where μ_1 is the first eigenvalue of the problem

$$-\Delta v = \mu v \ \text{ in } B, \quad v = 0 \ \text{ on } \partial B.$$

Let

$$P := P(N) = 1 + \frac{\ln \frac{4(\mu_1 - \lambda_*)Q\tau j_{1,\nu}^2}{\mu_1 (N-2)^2 (Q - 1 + (1-\tau) j_{1,\nu}^2)}}{\ln \tilde{\tau}}.$$

Then,

$$pv_*^{p-1}(t)e^{-2t} + \lambda_* \leq \mu_1 \ \text{ for } t \in [t_*, 0) \tag{5.27}$$

provided $p \geq \max\{p_c, P, 2\}$.

On the other hand, we see that the first eigenfunction $q_1(t)$ corresponding to μ_1 satisfies the problem

$$q_1'' - \frac{(N-2)^2}{4} q_1 + \mu_1 e^{2t} q_1 = 0 \ \text{ in } (-\infty, 0), \quad q_1(-\infty) = q_1(0) = 0. \tag{5.28}$$

Moreover, q_1 does not change sign in $(-\infty, 0)$. It follows from (5.3) that $\tilde{\psi}(t)$ satisfies the problem

$$\tilde{\psi}''(t) - \frac{(N-2)^2}{4}\tilde{\psi} + (pv_*^{p-1}e^{-2t} + \lambda_*)e^{2t}\tilde{\psi} = 0, \quad \tilde{\psi}(-\infty) = 0. \tag{5.29}$$

Thus, (5.27) and Sturm comparison principle imply that $\tilde{\psi}$ can not have two zeroes in $[t_*, 0]$. Otherwise, q_1 will have a zero in $(t_*, 0)$, a contradiction. Thus, $\tilde{\psi}$ has only one zero in $(-\infty, 0)$ and $\tilde{\psi}(0) < 0$. This implies $\psi_1(0) < 0$. This completes the proof. □

Remark 5.1. Following the same arguments as above, we can also show that for any $\lambda \in (\lambda_*, \mu_1)$, there exists P_λ sufficiently large such that for $p \geq P_\lambda$, $m(u_\lambda) = 1$. Note that P_λ depends on $(\mu_1 - \lambda)$, then $P_\lambda \to \infty$ as $\lambda \to \mu_1$.

6. More Estimates on $p_c^2(N)$

In this section we provide more conditions to guarantee that $m(u_*) = 1$. As seen in last section, we just need to show $\psi_1(0) < 0$.

Proposition 6.1. *There exists $P := P_1(N) > 0$ such that for $p \geq \max\{p_c, P, 2\}$ and*

$$\left(j_{2,\frac{N-2}{2}}^2 j_{1,\nu}^2 - j_{1,\frac{N-2}{2}}^4\right)\left(j_{1,\frac{N-2}{2}}^4 + Q j_{2,\frac{N-2}{2}}^2\right) + p\beta j_{1,\frac{N-2}{2}}^4 j_{2,\frac{N-2}{2}}^2 \geq 0, \tag{6.1}$$

where $j_{k,\xi}$ denote the k-th positive zero of the Bessel function $J_\xi(x)$, $\nu = \frac{\alpha^2}{4} - (p-1)\beta$, $Q = 4\left(N - 1 - \frac{3}{p-1}\right)$, then $\psi_1(0) < 0$. Thus, $m(u_*) = 1$. Moreover, $m(u_\lambda) = 1$ for $\lambda_* < \lambda < \mu_1$.

Proof. Arguments similar to those in the proof of Theorem 4.1 imply that, by choosing

$$P_1(N) := 1 + \frac{\ln \frac{4(\mu_2 - \mu_1)Q\tau j_{1,\nu}^2}{\mu_1 (N-2)^2 (Q - 1 + (1-\tau) j_{1,\nu}^2)}}{\ln \tilde{\tau}}, \tag{6.2}$$

where μ_2 is the second eigenvalue of the problem

$$-\Delta v = \mu v \text{ in } B, \quad v = 0 \text{ on } \partial B,$$

we see that for $p \geq \max\{p_c, 2, P_1(N)\}$,

$$pv_*^{p-1}(t)e^{-2t} \leq \mu_2 - \mu_1 < \mu_2 - \lambda_* \text{ for } t \in [t_*, 0). \tag{6.3}$$

Therefore,

$$pv_*^{p-1}(t)e^{-2t} + \lambda_* \leq \mu_2 \text{ for } t \in [t_*, 0) \tag{6.4}$$

provided $p \geq \max\{p_c, P_1(N), 2\}$.

On the other hand, we see that the second eigenfunction $q_2(t)$ corresponding to μ_2 satisfies the problem

$$q_2'' - \frac{(N-2)^2}{4} q_2 + \mu_2 e^{2t} q_2 = 0 \text{ in } (-\infty, 0), \quad q_2(-\infty) = q_2(0) = 0. \tag{6.5}$$

Moreover, q_2 changes sign only once in $(-\infty, 0)$, and $q_2(\ln(j_{1,(N-2)/2}/j_{2,(N-2)/2}) = 0$. Now we require

$$t_* > \ln(j_{1,(N-2)/2}/j_{2,(N-2)/2}). \tag{6.6}$$

Thus,

$$e^{2t_*} > e^{2\ln\left(\frac{j_{1,(N-2)/2}}{j_{2,(N-2)/2}}\right)} = \left(\frac{j_{1,(N-2)/2}}{j_{2,(N-2)/2}}\right)^2.$$

$$p\beta - Q\tilde{\theta} \geq \vartheta^2 \mu_1 \tilde{\theta}, \tag{6.7}$$

where

$$\vartheta = \frac{j_{1,(N-2)/2}}{j_{2,(N-2)/2}}.$$

Then

$$2Qp\beta \geq (\vartheta^2\mu_1 + Q)\left[Q + p\beta + j_{1,\nu}^2 - \sqrt{(Q + p\beta + j_{1,\nu}^2)^2 - 4p\beta Q}\right].$$

Therefore,

$$\vartheta^4 \mu_1^2 + \vartheta^2 Q\mu_1 \leq \vartheta^2 j_{1,\nu}^2 \mu_1 + \vartheta^2 p\beta\mu_1 + Qj_{1,\nu}^2. \tag{6.8}$$

That is,

$$\left(j_{2,\frac{N-2}{2}}^2 j_{1,\nu}^2 - j_{1,\frac{N-2}{2}}^4\right)\left(j_{1,(N-2)/2}^4 + Qj_{2,(N-2)/2}^2\right) + p\beta j_{1,\frac{N-2}{2}}^4 j_{2,\frac{N-2}{2}}^2 \geq 0. \tag{6.9}$$

It follows from (5.3) that $\tilde{\psi}(t)$ satisfies the problem

$$\tilde{\psi}''(t) - \frac{(N-2)^2}{4}\tilde{\psi} + (pv_*^{p-1}e^{-2t} + \lambda_*)e^{2t}\tilde{\psi} = 0, \quad \tilde{\psi}(-\infty) = 0. \tag{6.10}$$

Thus, (6.4), (6.6) and the Sturm comparison principle imply that $\tilde{\psi}$ can not have two zeroes in $[t_*, 0]$. Otherwise, q_2 will have two zeros in $(-\infty, 0)$, a contradiction. Thus, $\tilde{\psi}$ has only one zero in $(-\infty, 0)$ and $\tilde{\psi}(0) < 0$. This implies $\psi_1(0) < 0$. Therefore, we have $m(u_*) = 1$. Similar arguments imply $m(u_\lambda) = 1$ for $\lambda_* < \lambda < \mu_1$.

We see from [14] that for $k = 1, 2$

$$\mu + \frac{a_k}{2^{1/3}}\mu^{1/3} < j_{k,\mu} < \mu + \frac{a_k}{2^{1/3}}\mu^{1/3} + \frac{3}{20}a_k^2\frac{2^{1/3}}{\mu^{1/3}} \tag{6.11}$$

where $a_1 = 2.33811$, $a_2 = 4.08795$. Using the Matlab, we see that when $N = 12$ and $p \geq 12.154$; $N = 13$ and $p \geq 4.858$; $N = 14$ and $p \geq 3.313$; $N = 15$ and $p \geq 2.644$; $N = 16$ and $p \geq 2.27$; $N = 17$ and $p \geq 2.032$; $N = 18$ and $p \geq 1.868$; $N = 19$ and $p \geq 1.747$; $N = 20$ and $p \geq 1.655$; $N = 30$ and $p \geq 1.2870$, (6.9) holds. When $N = 11$, (6.9) holds for very large p. Since $\nu^2 = \frac{\alpha^2}{4} - (p-1)\beta = 0$ for $p = p_c$, our arguments can not be used to deal with this case. Thus, $m(u_*) = 1$ for $N \geq 12$ provided that p is suitable large . This completes the proof. □

Another estimate for p_c^2 is given by the following.

Proposition 6.2. *Assume*

$$\mu_1 \leq j_{2,\nu}^2 \tag{6.12}$$

with $\nu = \frac{\alpha^2}{4} - (p-1)\beta$. *Then* $\psi_1(0) < 0$. *Therefore,* $m(u_*) = 1$. *Moreover,* $m(u_\lambda) = 1$ *for all* $\lambda_* < \lambda < \mu_1$.

Proof. Since

$$pv_*^{p-1} + \lambda_* e^{2t} < p\beta + \mu_1 e^{2t}$$

if $\mu_1 \leq j_{2,\nu}^2$ and $\tilde{\psi}$ has two zeros in $(-\infty, 0]$, we have that the second eigenfunction $\tilde{q}_2(t)$ corresponding to the problem

$$\tilde{q}_2''(t) - \nu^2\tilde{q}_2(t) + j_{2,\nu}^2 e^{2t}\tilde{q}_2(t) = 0, \quad \tilde{q}_2(-\infty) = \tilde{q}_2(0) = 0$$

has two zeros in $(-\infty, 0)$. This is a contradiction. Therefore, $\psi_1(0) < 0$. This implies $m(u_*) = 1$. Since $pu_\lambda^{p-1} \leq p\beta$ for $\lambda_* < \lambda < \mu_1$, we can easily see that $m(u_\lambda) = 1$ for any regular solution u_λ provided that (6.12) holds.

Using the Matlab, we see that when $N = 12$ and $p \geq 13.451$; $N = 13$ and $p \geq 5.1532$; $N = 14$ and $p \geq 3.4708$; $N = 15$ and $p \geq 2.7487$; $N = 30$ and $p \geq 1.1343$, (6.12) hold. This implies that for p large, $m(u_*) = 1$. Since $\nu^2 = \frac{\alpha^2}{4} - (p-1)\beta = 0$ for $p = p_c$, we see that (6.12) can not hold for $p = p_c$. Thus, our argument can not be used to deal with the case $p = p_c$. Moreover, when $N = 11$, (6.12) only holds for very large p. □

We also have the following condition to guarantee $m(u_*) = 1$.

Proposition 6.3. *Let t_* be as in Theorem 4.1. Then $\psi_1(0) < 0$ provided*

$$|t_*| \leq \frac{\pi}{\sqrt{\lambda_* - (\frac{\alpha^2}{4} - \beta(p-1))}}. \tag{6.13}$$

Proof. We see that

$$pv_*^{p-1} + \lambda_* e^{2t} \leq p\beta + \lambda_* \text{ in } (-\infty, 0).$$

Suppose that $\tilde{\psi}$ has two zeroes at $\hat{t}_1$, $\hat{t}_2$ in $(t_*, 0]$, then we have

$$|t_*| > \frac{\pi}{\sqrt{\lambda_* - (\frac{\alpha^2}{4} - \beta(p-1))}}. \tag{6.14}$$

On the contrary, we see that there is a solution $\ell(t)$ of the problem

$$\ell'' - [(\frac{\alpha^2}{4} - \beta(p-1)) - \lambda_*]\ell = 0 \tag{6.15}$$

such that $\ell > 0$ in $[\hat{t}_1, \hat{t}_2]$. Let $e(t) = \frac{\tilde{\psi}(t)}{\ell(t)}$. We see that $e(t)$ satisfies the equation

$$e''(t) + \frac{2\ell'}{\ell}e'(t) + [(pv_*^{p-1} + \lambda_* e^{2t}) - (p\beta + \lambda_*)]e(t) = 0$$

and $e(t)$ has two zeroes $\hat{t}_1$, $\hat{t}_2$. Moreover, $e < 0$ in $(\hat{t}_1, \hat{t}_2)$. Thus, e has a minimum at some point in $(\hat{t}_1, \hat{t}_2)$, this is impossible. This also implies that if

$$|t_*| \leq \frac{\pi}{\sqrt{\lambda_* - (\frac{\alpha^2}{4} - \beta(p-1))}} \tag{6.16}$$

then $\tilde{\psi}$ changes sign only once in $(-\infty, 0)$ and $\tilde{\psi}(0) < 0$. Thus, $\psi_1(0) < 0$. The condition (6.16) is equivalent to

$$\frac{1}{2}\ln\frac{(1-\tilde{\theta})\lambda_*}{j_{1,\nu}^2} \leq \frac{\pi}{\sqrt{\lambda_* - (\frac{\alpha^2}{4} - \beta(p-1))}} \tag{6.17}$$

or

$$\frac{1}{2}\ln\frac{\lambda_*\tilde{\theta}}{p\beta - Q\tilde{\theta}} \leq \frac{\pi}{\sqrt{\lambda_* - (\frac{\alpha^2}{4} - \beta(p-1))}}. \tag{6.18}$$

We only need to check the following conditions

$$\frac{1}{2}\ln\frac{(1-\tilde{\theta})\mu_1}{j_{1,\nu}^2} \leq \frac{\pi}{\sqrt{\mu_1 - (\frac{\alpha^2}{4} - \beta(p-1))}} \tag{6.19}$$

or

$$\frac{1}{2}\ln\frac{\mu_1\tilde{\theta}}{p\beta - Q\tilde{\theta}} \leq \frac{\pi}{\sqrt{\mu_1 - (\frac{\alpha^2}{4} - \beta(p-1))}}. \tag{6.20}$$

□

Remark 6.4. It is known from the numerical data that to guarantee $m(u_*) = 1$, p will be very large if $N = 11$. This implies that the bound obtained in Theorem 4.1 is reasonable. For $N \geq 12$, we only need a suitable large p. Our arguments can not be used to deal with the case $p = p_c$. This case is still unclear, it will be interesting to know the Morse index of u_* for this case. We also see that the Morse index of all the radial solutions of (1.1) in $\mathscr{C}$ are 1 provided that our conditions hold.

Acknowledgments: The research of the first author is supported partially by NSFC (10871060). The research of the second author is supported by an Earmarked Grant from RGC of Hong Kong and a Focused Research Scheme of CUHK.

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