

# Spherical solutions to a nonlocal free boundary problem from diblock copolymer morphology

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## Abstract

The  $\Gamma$ -limit of the Ohta-Kawasaki density functional theory of diblock copolymers is a non-local free boundary problem. For some values of block composition and the nonlocal interaction, an equilibrium pattern of many spheres exists in a three dimensional domain. A sub-range of the parameters is found where the multiple sphere pattern is stable. This stable pattern models the spherical phase in the diblock copolymer morphology. The spheres are approximately round. They satisfy an equation that involves their mean curvature and a quantity that depends nonlocally on the whole pattern. The locations of the spheres are determined via a Green's function of the domain.

**Key words.** Spherical phase, diblock copolymer morphology, sphere coarsening, interface oscillation.

**2000 AMS subject classification.** 35R35, 82B24, 82D60.

**Abbreviated title.** Spherical solutions.

## 1 Introduction

A diblock copolymer melt is a soft material, characterized by fluid-like disorder on the molecular scale and a high degree of order at a longer length scale. A molecule in a diblock copolymer is a linear sub-chain of A-monomers grafted covalently to another sub-chain of B-monomers. Because of the repulsion between the unlike monomers, the different type sub-chains tend to segregate, but as they are chemically bonded in chain molecules, segregation of sub-chains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs: micro-domains rich in A monomers and micro-domains rich in B monomers emerge as a result. These micro-domains form patterns that

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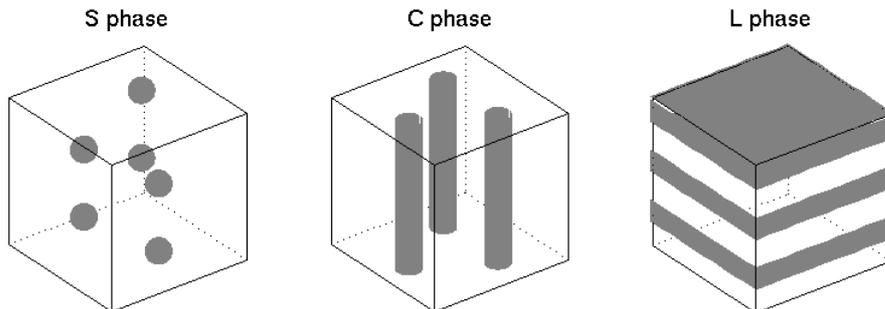


Figure 1: The spherical, cylindrical, and lamellar morphology phases commonly observed in diblock copolymer melts. The dark color indicates the concentration of type A monomers, and the white color indicates the concentration of type B monomers.

are known as morphology phases. Various phases, including lamellar, cylindrical, spherical, gyroid, have been observed in experiments. See Bates and Fredrickson [1] for more on block copolymers.

This paper deals with the spherical phase of the block copolymer morphology (Figure 1, Plot 1). Let  $a \in (0, 1)$  be the block composition fraction which is the number of the A-monomers divided by the number of all the A- and B-monomers in a chain molecule. The spherical phase occurs when  $a$  is relatively close to 0 (or close to 1), and the A-monomers (or B-monomers respectively) form small balls in space. We will mathematically construct a pattern with a number of balls.

The model we use here is a nonlocal free boundary problem derived from the Ohta-Kawasaki density functional theory of diblock copolymers [18]. Let  $D$  be a bounded and sufficiently smooth domain in  $R^3$  occupied by a diblock copolymer melt in the spherical phase. Let  $E$  be a subset of  $D$  where A-monomers concentrate. Then  $D \setminus E$  is the subset where B-monomers concentrate. Denote the part of the boundary of  $E$  that is in  $D$  by  $\partial_D E$  which is the set of the interfaces separating the A-rich micro-domains from the B-rich micro-domains. Denote the Lebesgue measure of  $E$  by  $|E|$ . Given a block composition fraction  $a \in (0, 1)$ , one has  $|E| = a|D|$ . Moreover there exists a number  $\lambda$  such that at every point on  $\partial_D E$

$$H(\partial_D E) + \gamma(-\Delta)^{-1}(\chi_E - a) = \lambda. \quad (1.1)$$

Here  $H(\partial_D E)$  is the mean curvature of  $\partial_D E$  viewed from  $E$ ,  $\gamma$  is a positive parameter, and  $\chi_E$  is the characteristic function of  $E$ , i.e.  $\chi_E(x) = 1$  if  $x \in E$ , and  $\chi_E(x) = 0$  if  $x \in D \setminus E$ . The expression  $(-\Delta)^{-1}(\chi_E - a)$  is the solution  $v$  of the problem

$$-\Delta v = \chi_E - a \text{ in } D, \quad \partial_\nu v = 0 \text{ on the boundary of } D, \quad \bar{v} = 0$$

where the bar over a function is the average of the function over its domain, i.e.

$$\bar{v} = \frac{1}{|D|} \int_D v(x) dx.$$

Because  $(-\Delta)^{-1}$  is a nonlocal operator, the free boundary problem (1.1) is nonlocal.

The equation (1.1) is the Euler-Lagrange equation of the free energy  $J$  of the system. The functional  $J$  is given by

$$J(E) = |D\chi_E|(D) + \frac{\gamma}{2} \int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2 dx, \quad E \in \Sigma. \quad (1.2)$$

The admissible set  $\Sigma$  of the functional  $J$  is the collection of all measurable subsets of  $D$  of measure  $a|D|$  and of finite perimeter, i.e.

$$\Sigma = \{E \subset D : E \text{ is Lebesgue measurable, } |E| = a|D|, \chi_E \in BV(D)\}. \quad (1.3)$$

Here  $BV(D)$  is the space of functions of bounded variation on  $D$ . In (1.2),  $|D\chi_E|(D)$  is the perimeter of  $E$ . When  $\partial E$  is smooth, this is merely the surface area of  $\partial E \cap D$ . For a more general  $E$ ,  $\chi_E$  is a BV-function and  $D\chi_E$  is a vector valued finite measure. We denote the magnitude of this measure by  $|D\chi_E|$  which is a positive, finite measure. The perimeter of  $E$  is defined to be the size of  $D$  under this measure. The operator  $(-\Delta)^{-1/2}$  is the positive square root of  $(-\Delta)^{-1}$ .

The main difficulty in (1.1) stems from the nonlocal term. Without it, i.e. if  $\gamma = 0$ , (1.1) would just be the equation of constant mean curvature. However with the nonlocal term the curvature of a solution in general is not constant. One exception occurs in the study of the lamellar phase (Figure 1, Plot 3) where interfaces are parallel planes (Ren and Wei [22, 25]). The solution we are looking for in this paper is a union of a number of disconnected sets each of which is close to a small round ball. The solution is hence termed a spherical solution.

Nishiura and Ohnishi [16] formulated the Ohta-Kawasaki theory on a bounded domain as a singularly perturbed variational problem with a nonlocal term and also identified the free boundary problem (1.1). Ren and Wei [22] showed that (1.2) is a  $\Gamma$ -limit of the singularly perturbed variational problem. See the last section for more discussion on the Ohta-Kawasaki theory and  $\Gamma$ -convergence.

Since then much work has been done mathematically to these problems. The lamellar phase (Figure 1, Plot 3) is studied by Ren and Wei [22, 24, 25, 29, 30], Fife and Hilhorst [9], Choksi and Ren [4], Chen and Oshita [2], and Choksi and Sternberg [6]. The result obtained by Müller [15] is related to the lamellar phase in the case  $a = 1/2$ , as observed in [16]. Radially symmetric bubble and ring patterns are studied by Ren and Wei [23, 28, 31]. The cylindrical phase (Figure 1, Plot 2) is studied by Ren and Wei [21, 20]. The gyroid phase is numerically studied by Teramoto and Nishiura [33]. Triblock copolymers are studied by Ren and Wei [26, 27]. A diblock copolymer - homopolymer blend is studied by Choksi and Ren [5]. Also see Ohnishi *et al* [17], and Choksi [3].

In this paper we use a variant of the Lyapunov-Schmidt reduction procedure to construct spherical solutions. This argument was first successfully used in this context in the study of the cylindrical phase of diblock copolymers by Ren and Wei [21, 20].

## 2 Main results

The Green's function of  $-\Delta$  is denoted by  $G$ . It is a sum of two parts:

$$G(x, y) = \frac{1}{4\pi|x - y|} + R(x, y). \quad (2.1)$$

The regular part of  $G(x, y)$  is  $R(x, y)$ . The Green's function satisfies the equation

$$-\Delta_x G(x, y) = \delta(x - y) - \frac{1}{|D|} \text{ in } D, \quad \partial_{\nu(x)} G(x, y) = 0 \text{ on } \partial D, \quad \overline{G(\cdot, y)} = 0 \text{ for every } y \in D. \quad (2.2)$$

Here  $\Delta_x$  is the Laplacian with respect to the  $x$ -variable of  $G$ , and  $\nu(x)$  is the outward normal direction at  $x \in \partial D$ . We set

$$F(\xi_1, \xi_2, \dots, \xi_K) = \sum_{k=1}^K R(\xi_k, \xi_k) + \sum_{k=1}^K \sum_{l=1, l \neq k}^K G(\xi_k, \xi_l), \quad (2.3)$$

for  $\xi_k \in D$  and  $\xi_k \neq \xi_l$  if  $k \neq l$ . Because  $G(x, y) \rightarrow \infty$  if  $|x - y| \rightarrow 0$  and  $R(x, x) \rightarrow \infty$  if  $x \rightarrow \partial D$ ,  $F$  admits at least one global minimum.

The average sphere radius is

$$\rho = \left( \frac{3a|D|}{4\pi K} \right)^{1/3}. \quad (2.4)$$

The main result of this paper is the following existence theorem.

**Theorem 2.1** *Let  $K \geq 2$  be an integer.*

1. *For every  $\epsilon > 0$  there exists  $\delta > 0$ , depending on  $\epsilon$ ,  $K$  and  $D$  only, such that if*

$$\gamma\rho^3 > 3 + \epsilon, \quad (2.5)$$

$$\left| \gamma\rho^3 - \frac{3(n+2)(2n+1)}{2} \right| > \epsilon n^2, \quad \text{for all } n = 2, 3, 4, \dots, \quad (2.6)$$

and

$$\rho < \delta, \quad (2.7)$$

then there exists a solution  $E$  of (1.1).

2. *The solution  $E$  is a union of  $K$  approximate balls. The radius of each ball is close to  $\rho$ .*
3. *Let the centers of these balls be  $\zeta_1, \zeta_2, \dots, \zeta_K$ . Then  $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_K)$ , is close to a global minimum of the function  $F$ .*

We have opted for a rather general existence theorem. The solution found in the theorem is not necessarily stable. The stability of the solution depends on how (2.6) is satisfied.

**Theorem 2.2** *If (2.6) is satisfied because*

$$\gamma\rho^3 - \frac{3(n+2)(2n+1)}{2} < -\epsilon n^2, \quad \text{for all } n \geq 2, \quad (2.8)$$

then the sphere solution is stable. Otherwise if (2.6) is satisfied but

$$\epsilon n^2 < \gamma\rho^3 - \frac{3(n+2)(2n+1)}{2}, \quad \text{and } \gamma\rho^3 - \frac{3(n+3)(2n+3)}{2} < -\epsilon(n+1)^2 \quad (2.9)$$

for some  $n \geq 2$ , then the sphere solution is unstable.

When we delete intervals around  $\frac{3(n+2)(2n+1)}{2}$ ,  $n = 2, 3, \dots$ , in (2.6), the width of the intervals,  $2\epsilon n^2$ , grows as  $n$  becomes large. At some point an interval will include nearby members in the sequence  $2n(n+1)$ . When this happens,  $\gamma\rho^3$  can not be placed above such  $\frac{3(n+2)(2n+1)}{2}$ . This implies that there exists  $C(\epsilon) > 0$  depending on  $\epsilon$  such that

$$\gamma < \frac{C(\epsilon)}{\rho^3}. \quad (2.10)$$

Combing this with (2.5) we see that  $\rho$  and  $\gamma$  are in a somewhat narrow parameter range

$$\rho < \delta, \quad \frac{3+\epsilon}{\rho^3} < \gamma < \frac{C(\epsilon)}{\rho^3}, \quad (2.11)$$

and  $\gamma\rho^3$  must stay away from the sequence  $\frac{3(n+2)(2n+1)}{2}$ ,  $n = 2, 3, \dots$ , in the sense of (2.6). From (2.11) one sees that  $\rho$  must be small and  $\gamma$  be appropriately large.

The condition (2.5) prevents coarsening. By coarsening we mean that some balls become larger and other balls shrink and disappear.

The gap condition (2.6) controls interface oscillation. Interface oscillation refers to a phenomenon that oscillations appear on the boundary of a ball. The gap condition also suggests bifurcations to oscillating solutions. Elsewhere gap conditions have appeared in constructing layered solutions for singularly perturbed problems. See Malchiodi and Montenegro [12], M. del Pino, M. Kowalczyk and Wei [8], Pacard and Ritoré [19], and the references therein.

The solution found in Theorem 2.1 may be unstable because of interface oscillation. The condition (2.8) in Theorem 2.2 eliminates this possibility. Under (2.8)  $\rho$  and  $\gamma$  must satisfy a more stringent requirement

$$\rho < \delta, \quad \frac{3+\epsilon}{\rho^3} < \gamma < \frac{30-4\epsilon}{\rho^3}. \quad (2.12)$$

This means that  $\gamma\rho^3$  must stay to the left of the sequence  $\frac{3(n+2)(2n+1)}{2}$ ,  $n = 2, 3, \dots$ . If (2.9) holds, we have an unstable mode that tends to bring oscillations to the spheres.

The spheres in the solution we construct are approximately round, with the same approximate radius. Theorem 2.1, Part 3, asserts that the sphere centers must minimize  $F$  approximately.

We can even determine the optimal number of balls in a spherical pattern. Because of (2.11), we write

$$\gamma = \frac{\mu}{a} = \frac{\mu}{\left(\frac{4\pi K}{3|D|}\right)\rho^3}. \quad (2.13)$$

Now  $a$  and  $\mu$  are the parameters of the problem. We hold  $\mu$  fixed and make  $a$  and hence  $\rho$  small.

With (2.13) and (2.4) the leading order of the free energy is calculated from the formula in Lemma 8.1

$$4\pi\rho^2 K + \frac{\gamma}{2} \left(\frac{8\pi\rho^5 K}{15}\right) = 4\pi K^{1/3} \left(\frac{3a|D|}{4\pi}\right)^{2/3} + \frac{\mu 4\pi}{15a} \left(\frac{3a|D|}{4\pi}\right)^{5/3} K^{-2/3}. \quad (2.14)$$

With respect to  $K$  the last quantity is minimized at

$$K = \frac{|D|\mu}{10\pi}. \quad (2.15)$$

Note that the choice (2.15) of  $K$  does not violate the condition (2.12), since with this  $K$ ,

$$\gamma = \frac{\mu}{a} = \mu \frac{3|D|}{K 4\pi\rho^3} = \mu \frac{3|D|}{4\pi\rho^3} \frac{10\pi}{\mu|D|} = \frac{30}{4\rho^3}. \quad (2.16)$$

The number (2.15) gives the optimal number of spheres in a spherical pattern.

### 3 Approximate solutions

From now on throughout the rest of the paper we are given  $\epsilon > 0$ , and  $\gamma$  and  $\rho$  satisfy (2.5) and (2.6).

Let  $U_1$  be a small neighborhood in  $D^K$  of the set  $\{\eta : F(\eta) = \min_{\xi \in D^K} F(\xi)\}$ , and  $U_2$  be the set

$$U_2 = \{(r_1, r_2, \dots, r_K) \in R^K : r_k \in ((1 - \delta_2)\rho, (1 + \delta_2)\rho), k = 1, 2, \dots, K, \sum_{k=1}^K \frac{4\pi r_k^3}{3} = a|D|\}. \quad (3.1)$$

The constant  $\delta_2$  is positive, small and depending on  $\epsilon$ . It will be fixed later in the proofs of Lemmas 5.3 and 8.2. Define

$$U = U_1 \times U_2. \quad (3.2)$$

Let  $\xi_1, \xi_2, \dots, \xi_K$  be  $K$  distinct points in  $D$  such that  $\xi = (\xi_1, \xi_2, \dots, \xi_K)$  is in  $U_1$ , and  $r = (r_1, r_2, \dots, r_K)$  be in  $U_2$ . Denote the ball centered at  $\xi_k$  of radius  $r_k$  by  $B_k$ . The union of the  $B_k$ 's is  $B$ :

$$B = \bigcup_{k=1}^K B_k = \bigcup_{k=1}^K \{x \in R^3 : |x - \xi_k| < r_k\}. \quad (3.3)$$

With  $U_1$  close to  $\{\eta : F(\eta) = \min_{\kappa \in D^K} F(\kappa)\}$  and  $\rho$  sufficiently small, the  $B_k$ 's are all inside  $D$  and disjoint.

**Lemma 3.1** *When  $E$  is  $B$ , the left side of (1.1) is*

$$\frac{1}{r_k} + \gamma \left[ \frac{r_k^2}{3} + \frac{4\pi r_k^3}{3} R(\xi_k, \xi_k) + \sum_{l \neq k} \frac{4\pi r_l^3}{3} G(\xi_k, \xi_l) \right] + O(\rho)$$

at each  $\xi_k + r_k \theta_k$ .

*Proof.* At a boundary point  $\xi_k + r_k \theta_k$  of  $B_k$ , the curvature is  $\frac{1}{r_k}$ .

We compute  $v_k = (-\Delta)^{-1}(\chi_{B_k} - \frac{4\pi r_k^3}{3|D|})$ . Define

$$P_k(x) = \begin{cases} -\frac{|x - \xi_k|^2}{6} + \frac{r_k^2}{2} & \text{if } |x| < r_k \\ \frac{r_k^3}{3|x - \xi_k|}, & \text{if } |x| \geq r_k \end{cases}.$$

Then  $-\Delta P_k = \chi_{B_k}$ . Write  $v_k(x) = P_k(x) + Q_k(x, \xi_k)$ . Clearly

$$-\Delta Q_k(x, \xi_k) = -\frac{4\pi r_k^3}{3|D|}, \quad \partial_{\nu(x)} Q_k(x, \xi_k) = -\partial_{\nu} \frac{4\pi r_k^3}{3} \frac{1}{4\pi|x - \xi_k|} \text{ on } \partial D, \quad \overline{Q_k(\cdot, \xi_k)} = -\overline{P_k}.$$

From (2.2) we see that  $Q_k(x, \xi_k)$  and  $\frac{4\pi r_k^3}{3} R(x, \xi_k)$  satisfy the same equation and the same boundary condition, where  $R$  is the regular part of the Green's function  $G$ . Therefore they can differ only by a

constant. This constant is  $\overline{Q_k(\cdot, \xi_k)} - \frac{4\pi r_k^3}{3} \overline{R(\cdot, \xi_k)}$ . But  $\overline{v_k} = \overline{G(\cdot, \xi_k)} = 0$  implies that this constant is

$$-\overline{P_k} + \frac{4\pi r_k^3}{3} \frac{1}{4\pi|x - \xi_k|} = \frac{4\pi r_k^5}{3} \frac{1}{10|D|}$$

by direct calculation. Hence

$$Q_k(x, \xi_k) = \frac{4\pi r_k^3}{3} R(x, \xi_k) + \frac{4\pi}{3} \frac{r_k^5}{10|D|},$$

and

$$v_k(x) = P_k(x) + \frac{4\pi r_k^3}{3} R(x, \xi_k) + \frac{4\pi}{3} \frac{r_k^5}{10|D|}. \quad (3.4)$$

Let  $v = (-\Delta)^{-1}(\chi_B - a) = \sum_l v_l$ . Then at  $\xi_k + r_k \theta_k$

$$\begin{aligned} v(\xi_k + r_k \theta_k) &= \frac{r_k^2}{3} + \frac{4\pi r_k^3}{3} R(\xi_k + r_k \theta_k, \xi_k) + \sum_{l \neq k} \frac{4\pi r_l^3}{3} G(\xi_k + r_k \theta_k, \xi_l) + \sum_{l=1}^K \frac{4\pi}{3} \frac{r_l^5}{10|D|} \\ &= \frac{r_k^2}{3} + \frac{4\pi r_k^3}{3} R(\xi_k, \xi_k) + \sum_{l \neq k} \frac{4\pi r_l^3}{3} G(\xi_k, \xi_l) + O(\rho^4). \end{aligned} \quad (3.5)$$

The lemma follows from (2.10).  $\square$

**Lemma 3.2** *The free energy of  $B$  is*

$$\begin{aligned} J(B) &= \sum_{k=1}^K 4\pi r_k^2 + \frac{\gamma}{2} \left\{ \sum_{k=1}^K \left[ \frac{8\pi r_k^5}{15} + \left( \frac{4\pi}{3} \right)^2 r_k^6 R(\xi_k, \xi_k) \right] \right. \\ &\quad \left. + \sum_{k=1}^K \sum_{l=1, l \neq k}^K \left( \frac{4\pi}{3} \right)^2 r_k^3 r_l^3 G(\xi_k, \xi_l) + \sum_{k=1}^K \sum_{l=1}^K \left( \frac{4\pi}{3} \right)^2 \left( \frac{r_k^3 r_l^5}{10|D|} + \frac{r_k^5 r_l^3}{10|D|} \right) \right\}. \end{aligned}$$

*Proof.* The local part of the free energy is just  $\sum_{k=1}^K 4\pi r_k^2$ . The nonlocal part of the free energy is

$$\begin{aligned} &\int_D |(-\Delta)^{-1/2}(\chi_B - a)|^2 dx \\ &= \int_D (\chi_B - a)v(x) dx = \sum_{l=1}^K \int_{B_l} v(x) dx \\ &= \sum_{l=1}^K \sum_{k=1}^K \int_{B_l} v_k(x) dx = \sum_{l=1}^K \sum_{k=1}^K \left[ \int_{B_l} P_k(x) dx + \int_{B_l} Q_k(x, \xi_k) dx \right] \end{aligned}$$

There are two possibilities. When  $l = k$ , from the definition of  $P_k$  we find

$$\int_{B_k} P_k(x) dx = \frac{8\pi r_k^5}{15}. \quad (3.6)$$

For the integral of  $Q_k$ , we have

$$\int_{B_k} Q_k(x, \xi_k) = \frac{4\pi r_k^3}{3} \int_{B_k} R(x, \xi_k) + \left(\frac{4\pi}{3}\right)^2 \frac{r_k^8}{10|D|}.$$

Since  $R(x, \xi_k) - \frac{1}{6|D|}|x - \xi_k|^2$  is harmonic in  $x$ , by the Mean Value Theorem for harmonic functions

$$\begin{aligned} \int_{B_k} R(x, \xi_k) dx &= \int_{B_k} \left( R(x, \xi_k) - \frac{1}{6|D|}|x - \xi_k|^2 \right) dx + \int_{B_k} \frac{1}{6|D|}|x - \xi_k|^2 dx \\ &= \frac{4\pi r_k^3}{3} R(\xi_k, \xi_k) + \frac{4\pi}{3} \frac{r_k^5}{10|D|} \end{aligned} \quad (3.7)$$

Hence

$$\int_{B_k} v_k = \frac{8\pi r_k^5}{15} + \left(\frac{4\pi}{3}\right)^2 r_k^6 R(\xi_k, \xi_k) + \left(\frac{4\pi}{3}\right)^2 \frac{r_k^8}{5|D|}.$$

When  $l \neq k$ , for  $x \in B_l$ , since  $P_k$  is harmonic,

$$\begin{aligned} \int_{B_l} v_k &= \int_{B_l} P_k dx + \frac{4\pi r_k^3}{3} \int_{B_l} R(x, \xi_k) dx + \left(\frac{4\pi}{3}\right)^2 \frac{r_k^5 r_l^3}{10} \\ &= \frac{4\pi}{3} r_l^3 \frac{r_k^3}{3|\xi_k - \xi_l|} + \frac{4\pi r_k^3}{3} \left[ \int_{B_l} \left( R(x, \xi_k) - \frac{1}{6|D|}|x - \xi_l|^2 \right) dx + \int_{B_l} \frac{1}{6|D|}|x - \xi_l|^2 dx \right] \\ &\quad + \left(\frac{4\pi}{3}\right)^2 \frac{r_k^5 r_l^3}{10|D|} \\ &= \left(\frac{4\pi}{3}\right)^2 \frac{r_k^3 r_l^3}{4\pi|\xi_k - \xi_l|} + \left(\frac{4\pi}{3}\right)^2 r_k^3 r_l^3 R(\xi_k, \xi_l) + \left(\frac{4\pi}{3}\right)^2 \left( \frac{r_k^3 r_l^5}{10|D|} + \frac{r_k^5 r_l^3}{10|D|} \right) \end{aligned}$$

Finally the nonlocal part of the free energy is

$$\begin{aligned} \int_D (\chi_B - a)v dx &= \sum_{k=1}^K \left[ \frac{8\pi r_k^5}{15} + \left(\frac{4\pi}{3}\right)^2 r_k^6 R(\xi_k, \xi_k) \right] \\ &\quad + \sum_{k=1}^K \sum_{l=1, l \neq k}^K \left[ \left(\frac{4\pi}{3}\right)^2 \frac{r_k^3 r_l^3}{4\pi|\xi_k - \xi_l|} + \left(\frac{4\pi}{3}\right)^2 r_k^3 r_l^3 R(\xi_k, \xi_l) \right] \\ &\quad + \sum_{k=1}^K \sum_{l=1}^K \left(\frac{4\pi}{3}\right)^2 \left( \frac{r_k^3 r_l^5}{10|D|} + \frac{r_k^5 r_l^3}{10|D|} \right). \end{aligned} \quad (3.8)$$

The lemma now follows.  $\square$

## 4 Perturbed spheres

We perturb each ball  $B_k$  considered in the last section. A perturbed ball denoted by  $E_{\phi_k}$  is described by a function  $\phi_k = \phi_k(\theta_k)$ ,  $\theta_k \in S^2$ :

$$E_{\phi_k} = \{ \xi_k + t\theta_k : \theta_k \in S^2, t \in [0, (r_k^3 + \phi_k(\theta_k))^{1/3}] \} \quad (4.1)$$

Each  $\phi_k$  is small compared to  $r_k^3$  so that  $r_k^3 + \phi_k(\theta_k)$  is positive. The  $\phi_k$ 's also satisfy

$$\sum_{k=1}^K \int_{S^2} \phi_k(\theta_k) d\theta_k = 0. \quad (4.2)$$

Here the integral is a surface integral over  $S^2$  and

$$d\theta_k = \sin \theta_{k,2} d\theta_{k,1} d\theta_{k,2} \quad (4.3)$$

is the surface element on  $S^2$ . Hence the total volume inside the perturbed spheres remains fixed:

$$\sum_{k=1}^K |E_{\phi_k}| = \sum_k \int_{S^2} \int_0^{(r_k^3 + \phi_k(\theta_k))^{1/3}} t^2 dt d\theta_k = \sum_k \int_{S^2} \left( \frac{r_k^3}{3} + \frac{\phi_k(\theta_k)}{3} \right) d\theta_k = \sum_k \frac{4\pi r_k^3}{3} = a|D|.$$

The union of the  $E_{\phi_k}$ 's is  $E_\phi$ :

$$E_\phi = \bigcup_{k=1}^K E_{\phi_k}. \quad (4.4)$$

With these notations  $B = E_0$ .

We let  $\theta = (\theta_1, \theta_2, \dots, \theta_K)$  and  $\phi(\theta) = (\phi_1(\theta_1), \phi_2(\theta_2), \dots, \phi_K(\theta_K))$ . Each  $\theta_k$  is identified by its latitude and longitude  $(\theta_{k,1}, \theta_{k,2})$ , namely

$$\theta_k = (\cos \theta_{k,1} \sin \theta_{k,2}, \sin \theta_{k,1} \sin \theta_{k,2}, \cos \theta_{k,2}). \quad (4.5)$$

To express surface area in terms of  $\phi_k$ , first define

$$L(s, p, q, \beta) = s^{-1/3} \sqrt{\frac{p^2}{9 \sin^2 \beta} + \frac{q^2}{9} + s^2}, \quad (4.6)$$

and then define

$$L_k(\phi_k, \frac{\partial \phi_k}{\partial \theta_{k,1}}, \frac{\partial \phi_k}{\partial \theta_{k,2}}, \theta_{k,2}) = r_k^2 L\left(1 + \frac{\phi_k}{r_k^3}, \frac{1}{r_k^3} \frac{\partial \phi_k}{\partial \theta_{k,1}}, \frac{1}{r_k^3} \frac{\partial \phi_k}{\partial \theta_{k,2}}, \theta_{k,2}\right) \quad (4.7)$$

The surface area of  $\partial_D E_\phi$  can be expressed as

$$\sum_{k=1}^K |D\chi_{E_{\phi_k}}|(D) = \sum_{k=1}^K \int_{S^2} L_k(\phi_k, \frac{\partial \phi_k}{\partial \theta_{k,1}}, \frac{\partial \phi_k}{\partial \theta_{k,2}}, \theta_{k,2}) d\theta_k. \quad (4.8)$$

The nonlocal part of  $J$  in (1.2) may be written in terms of  $\phi$  as

$$\frac{\gamma}{2} \int_D |(-\Delta)^{-1/2} (\chi_{E_\phi} - a)|^2 dx = \frac{\gamma}{2} \int_{E_\phi} \int_{E_\phi} G(x, y) dx dy. \quad (4.9)$$

The first variation of  $J$  can now be written as

$$J'(E_\phi)(w) = \sum_{k=1}^K \int_{S^2} \left[ \frac{\partial L_k}{\partial \phi_k} w_k + \frac{\partial L_k}{\partial \theta_{k,1}} w_{k,1} + \frac{\partial L_k}{\partial \theta_{k,2}} w_{k,2} \right] d\theta_k + \quad (4.10)$$

$$\sum_{k=1}^K \int_{S^2} w_k(\theta_k) \left[ \sum_{l=1}^K \frac{\gamma}{3} \int_{E_{\phi_l}} G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) dy \right] d\theta_k. \quad (4.11)$$

Here we have used short hand notations  $\phi_{k,1} = \frac{\partial \phi_k}{\partial \theta_{k,1}}$  and  $\phi_{k,2} = \frac{\partial \phi_k}{\partial \theta_{k,2}}$ , and so on. From (4.10) we define a second order, quasilinear, elliptic operator

$$\mathcal{H}_k(\phi_k)(\theta_k) = \frac{1}{\sin \theta_{k,2}} \left[ \frac{\partial L_k}{\partial \phi_k} \sin \theta_{k,2} - \frac{\partial}{\partial \theta_{k,1}} \left( \frac{\partial L_k}{\partial \phi_{k,1}} \sin \theta_{k,2} \right) - \frac{\partial}{\partial \theta_{k,2}} \left( \frac{\partial L_k}{\partial \phi_{k,2}} \sin \theta_{k,2} \right) \right]. \quad (4.12)$$

This is just the mean curvature of the perturbed sphere  $\partial E_{\phi_k}$  at  $\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k$ , multiplied by  $\frac{1}{3}$ . The second part (4.11) of the first variation of  $J$  gives rise to a nonlocal operator

$$\phi \rightarrow \sum_{l=1}^K \frac{\gamma}{3} \int_{E_{\phi_l}} G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) dy. \quad (4.13)$$

This is just

$$\frac{\gamma}{3} (-\Delta)^{-1} (\chi_{E_{\phi}} - a) (\xi_k + (r_k^3 + \theta_k)^{1/3} \theta_k),$$

the nonlocal part of (1.1) at  $\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k$ , multiplied by  $\frac{1}{3}$ .

There are two cases in the sum over  $l$  in (4.13), when  $l = k$  we write

$$\begin{aligned} & \frac{\gamma}{3} \int_{E_{\phi_k}} G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) dy \\ &= \frac{\gamma}{3} \int_{E_{\phi_k}} \frac{dy}{4\pi |\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k - y|} + \frac{\gamma}{3} \int_{E_{\phi_k}} R(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) dy. \end{aligned}$$

We denote the last two terms by

$$\mathcal{A}_k(\phi_k)(\theta_k) = \frac{\gamma}{3} \int_{E_{\phi_k}} \frac{dy}{4\pi |\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k - y|} \quad (4.14)$$

$$\mathcal{B}_k(\phi_k)(\theta_k) = \frac{\gamma}{3} \int_{E_{\phi_k}} R(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) dy. \quad (4.15)$$

When  $l \neq k$  in (4.13) we let

$$\mathcal{C}_{kl}(\phi_k, \phi_l)(\theta_k) = \frac{\gamma}{3} \int_{E_{\phi_l}} G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) dy. \quad (4.16)$$

The left side of (1.1) (multiplied by  $\frac{1}{3}$ ) now becomes

$$\mathcal{H}_k(\phi_k)(\theta_k) + \mathcal{A}_k(\phi_k)(\theta_k) + \mathcal{B}_k(\phi_k)(\theta_k) + \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l)(\theta_k)$$

at  $\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k$ . Let us define

$$\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_K) \quad (4.17)$$

where

$$\mathcal{S}_k(\phi)(\theta_k) = \mathcal{H}_k(\phi_k)(\theta_k) + \mathcal{A}_k(\phi_k)(\theta_k) + \mathcal{B}_k(\phi_k)(\theta_k) + \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l)(\theta_k) + \lambda(\phi). \quad (4.18)$$

Here  $\lambda(\phi)$  is a number, independent of  $k$ . It is given by

$$\lambda(\phi) = -\frac{1}{K} \sum_{k=1}^K \overline{[\mathcal{H}_k(\phi_k) + \mathcal{A}_k(\phi_k) + \mathcal{B}_k(\phi_k) + \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l)]}. \quad (4.19)$$

The bar over the quantity here stands for the average of the quantity over  $S^2$ . With this definition of  $\lambda$ ,

$$\sum_{k=1}^K \overline{\mathcal{S}_k(\phi_k)} = 0. \quad (4.20)$$

The operator  $\mathcal{S}$  maps from

$$\mathcal{X} = \{\phi = (\phi_1, \phi_2, \dots, \phi_K) : \phi_k \in W^{2,p}(S^2), k = 1, 2, \dots, K, \sum_{k=1}^K \overline{\phi_k} = 0\} \quad (4.21)$$

to

$$\mathcal{Y} = \{q = (q_1, q_2, \dots, q_K) : q_k \in L^p(S^2), k = 1, 2, \dots, K, \sum_{k=1}^K \overline{q_k} = 0\}. \quad (4.22)$$

For technical reasons  $p$  is assumed to be in the range

$$2 < p < \infty. \quad (4.23)$$

The equation (1.1) now becomes

$$\mathcal{S}(\phi) = 0. \quad (4.24)$$

By defining

$$\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_K), \text{ where } \mathcal{C}_k(\phi_1, \phi_2, \dots, \phi_K) = \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l), \quad (4.25)$$

we write

$$\mathcal{S} = \mathcal{H} + \mathcal{A} + \mathcal{B} + \mathcal{C} + \lambda. \quad (4.26)$$

In the map  $\mathcal{S}$  the inputs  $\phi_1, \phi_2, \dots, \phi_k$  only interact in  $\mathcal{C}$  and  $\lambda$ . The other operators can be written in the block matrix form

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_1 & 0 & \dots & 0 \\ 0 & \mathcal{H}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{H}_K \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 & \dots & 0 \\ 0 & \mathcal{A}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{A}_K \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & 0 & \dots & 0 \\ 0 & \mathcal{B}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{B}_K \end{bmatrix}, \quad (4.27)$$

where each entry in a matrix is an operator from  $W^{2,p}(S^2)$  to  $L^p(S^2)$ . The scalar operator  $\lambda$  gives the projection  $-(\lambda_\xi(\phi), \lambda_\xi(\phi), \dots, \lambda_\xi(\phi))$  of  $\mathcal{H}(\phi) + \mathcal{A}(\phi) + \mathcal{B}(\phi) + \mathcal{C}(\phi)$  to the one dimensional space spanned by  $(1, 1, \dots, 1)$ .

Let us write down the first Fréchet derivatives of these operators. For simplicity we write

$$\phi_{k,i} = \frac{\partial \phi_k}{\partial \theta_{k,i}}, \quad \phi_{k,ij} = \frac{\partial^2 \phi_k}{\partial \theta_{k,ij}}, \quad u_{k,i} = \frac{\partial u_k}{\partial \theta_{k,i}}, \quad u_{k,ij} = \frac{\partial^2 u_k}{\partial \theta_{k,ij}}.$$

Calculations show that

$$\mathcal{H}'_k(\phi_k)(u_k) = \frac{\partial \mathcal{H}_k}{\partial \phi_k} u_k + \sum_{i=1}^2 \frac{\partial \mathcal{H}_k}{\partial \phi_{k,i}} u_{k,i} + \sum_{i,j=1}^2 \frac{\mathcal{H}_k}{\partial \phi_{k,ij}} u_{k,ij} \quad (4.28)$$

$$\begin{aligned} \mathcal{A}'_k(\phi_k)(u_k)(\theta_k) &= \frac{\gamma}{9} \int_{S^2} \frac{u_k(\omega_k) d\omega_k}{4\pi|(r_k^3 + \phi_k(\theta_k))^{1/3}\theta_k - (r_k^3 + \phi_k(\omega_k))^{1/3}\omega_k|} \\ &\quad - \frac{\gamma u_k(\theta_k)}{9(r_k^3 + \phi_k(\theta_k))^{2/3}} \int_{E_{\phi_k}} \frac{((r_k^3 + \phi_k(\theta_k))^{1/3}\theta_k - y) \cdot \theta_k}{4\pi|(r_k^3 + \phi_k(\theta_k))^{1/3}\theta_k - y|^3} dy. \end{aligned} \quad (4.29)$$

$$\begin{aligned} \mathcal{B}'_k(\phi_k)(u_k)(\theta_k) &= \frac{\gamma}{9} \int_{S^2} u_k(\omega_k) R(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3}\theta_k, \xi_k + (r_k^3 + \phi_k(\omega_k))^{1/3}\omega_k) d\omega_k \\ &\quad + \frac{\gamma u_k(\theta_k)}{9(r_k^3 + \phi_k(\theta_k))^{2/3}} \int_{E_{\phi_k}} \nabla R(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3}\theta_k, y) \cdot \theta_k dy. \end{aligned} \quad (4.30)$$

$$\begin{aligned} \mathcal{C}'_{kl}(\phi_k, \phi_l)(u_k, u_l)(\theta_k) &= \frac{\gamma}{9} \int_{S^2} u_l(\omega_l) G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3}\theta_k, \xi_l + (r_l^3 + \phi_l(\omega_l))^{1/3}\omega_l) d\omega_l \\ &\quad + \frac{\gamma u_k(\theta_k)}{9(r_k^3 + \phi_k(\theta_k))^{2/3}} \int_{E_{\phi_l}} \nabla G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3}\theta_k, y) \cdot \theta_k dy. \end{aligned} \quad (4.31)$$

The derivative

$$\lambda'_\xi(\phi_1, \phi_2, \dots, \phi_k)(u_1, u_2, \dots, u_k) \quad (4.32)$$

is so chosen that

$$\sum_{k=1}^K \overline{\mathcal{S}'_k(u)} = 0. \quad (4.33)$$

## 5 A linear operator

Let  $\mathcal{L}$  be the linearized operator of  $\mathcal{S}$  at  $\phi = 0$ , i.e.

$$\mathcal{L} = \mathcal{S}'(0). \quad (5.1)$$

Going back to (4.28), (4.29), (4.30) and (4.31) we find that

$$\begin{aligned} \mathcal{H}'_k(0)(u_k) &= -\frac{1}{9r_k^4} \left[ \frac{1}{\sin^2 \theta_{k,2}} \frac{\partial^2 u_k}{\partial \theta_{k,1}^2} + \frac{\partial u_k}{\partial \theta_{k,2}^2} + \cot \theta_{k,2} \frac{\partial u_k}{\partial \theta_{k,2}} \right] - \frac{2}{9r_k^4} u \\ \mathcal{A}'_k(0)(u_k)(\theta_k) &= \frac{\gamma}{9r_k} \int_{S^2} \frac{u_k(\omega_k) d\omega_k}{4\pi|\theta_k - \omega_k|} - \frac{\gamma u_k(\theta_k)}{27r_k} \\ \mathcal{B}'_k(0)(u_k)(\theta_k) &= \frac{\gamma}{9} \int_{S^2} u_k(\omega_k) R(\xi_k + r_k \theta_k, \xi_k + r_k \omega_k) d\omega_k \\ &\quad + \frac{\gamma u_k(\theta_k)}{9r_k^2} \int_{B_k} \nabla R(\xi_k + r_k \theta_k, y) \cdot \theta_k dy \\ \mathcal{C}'_{kl}(0,0)(u_k, u_l)(\theta_k) &= \frac{\gamma}{9} \int_{S^2} u_l(\omega_l) G(\xi_k + r_k \theta_k, \xi_l + r_l \omega_l) d\omega_l \\ &\quad + \frac{\gamma u_k(\theta_k)}{9r_k^2} \int_{B_l} \nabla G(\xi_k + r_k \theta_k, y) \cdot \theta_k dy. \end{aligned}$$

The derivation of  $\mathcal{A}'_k(0)$  is explained in more detail in Appendix A.

Let us separate  $\mathcal{L}$  to a dominant part  $\mathcal{L}_1$  and a minor part  $\mathcal{L}_2$ . We define  $\mathcal{L}_{1,k}$ , the  $k$ -th component of  $\mathcal{L}_1$ , to be

$$\mathcal{L}_{1,k}(u)(\theta_k) = \mathcal{H}'_k(0)(u_k)(\theta_k) + \mathcal{A}'_k(0)(u_k)(\theta_k) + l_1(u).$$

The real valued linear operator  $l_1$  is independent of  $k$ . It is so chosen that  $\mathcal{L}_1$  maps from  $\mathcal{X}$  to  $\mathcal{Y}$ . The rest of  $\mathcal{L}$  is denoted by  $\mathcal{L}_2$ .

We are more interested in the operators  $\Pi\mathcal{L}$  and  $\Pi\mathcal{L}_1$  where  $\Pi$  is the orthogonal projection operator from  $\mathcal{Y}$  to

$$\mathcal{Y}_* = \{q = (q_1, \dots, q_K) \in \mathcal{Y} : q_k \perp H_1, q_k \perp 1, k = 1, \dots, K\}. \quad (5.2)$$

Here  $H_1$  is the space of spherical harmonics of degree 1. See for instance [10] for more on spherical harmonics. The operator  $\Pi\mathcal{L}$  is defined on

$$\mathcal{X}_* = \{q = (q_1, \dots, q_K) \in \mathcal{X} : q_k \perp H_1, q_k \perp 1, k = 1, \dots, K\}. \quad (5.3)$$

We use the same  $\Pi$  to denote the orthogonal projection from

$$L^2(S^2) \text{ to } \{q_k \in L^2(S^2) : q_k \perp H_1, q_k \perp 1\}. \quad (5.4)$$

**Lemma 5.1** Consider  $\Pi\mathcal{L}_1$  as an operator from  $\mathcal{X}_*$  to  $\mathcal{Y}_*$ . The eigenvalues of  $\Pi\mathcal{L}_1$  are

$$\lambda_{k,n} = \frac{(n-1)(n+2)}{9r_k^4} - \frac{\gamma}{9r_k} \left[ \frac{2(n-1)}{3(2n+1)} \right], \quad k = 1, 2, \dots, K, \quad n = 2, 3, 4, \dots \quad (5.5)$$

whose multiplicity is  $2n+1$ . The corresponding eigenvectors are the spherical harmonics of degree  $n$ , i.e.  $H_n$  is the eigen-space associated with  $\lambda_{k,n}$ .

*Proof.* In  $\mathcal{X}_*$ ,  $\mathcal{L}_1$  is simplified to

$$\mathcal{L}_{1,k}(u) = -\frac{1}{9r_k^4} \left[ \frac{1}{\sin^2 \theta_{k,2}} \frac{\partial^2 u_k}{\partial \theta_{k,1}^2} + \frac{\partial u_k}{\partial \theta_{k,2}^2} + \cot \theta_{k,2} \frac{\partial u_k}{\partial \theta_{k,2}} \right] - \frac{2u_k}{9r_k^4} + \frac{\gamma}{9r_k} \int_{S^2} \frac{u_k(\omega_k) d\omega_k}{4\pi|\theta_k - \omega_k|} - \frac{\gamma u_k(\theta_k)}{27r_k},$$

for each  $k$ . This is a diagonalized operator. Note that in  $\mathcal{X}_*$ ,  $\Pi\mathcal{L}_1 = \mathcal{L}_1$ . To find the spectrum of  $\mathcal{L}_1$  in  $\mathcal{X}_*$  we consider the effect of  $\mathcal{L}_1$  on the spherical harmonics  $h \in H_n$  of degree  $n$ . Since

$$\frac{1}{\sin^2 \theta_{k,2}} \frac{\partial^2}{\partial \theta_{k,1}^2} + \frac{\partial^2}{\partial \theta_{k,2}^2} + \cot \theta_{k,2} \frac{\partial}{\partial \theta_{k,2}} := \Delta_{S^2} \quad (5.6)$$

is the Laplacian-Beltrami operator on the sphere,

$$-\left[ \frac{1}{\sin^2 \theta_{k,2}} \frac{\partial^2 h}{\partial \theta_{k,1}^2} + \frac{\partial^2 h}{\partial \theta_{k,2}^2} + \cot \theta_{k,2} \frac{\partial h}{\partial \theta_{k,2}} \right] = n(n+1)h. \quad (5.7)$$

In Appendix B we find that

$$\int_{S^2} \frac{h(\omega) d\omega}{4\pi|\theta - \omega|} = \frac{h(\theta)}{2n+1}. \quad (5.8)$$

Following (5.7) and (5.8) one deduces that

$$\mathcal{L}_{1,k}(h) = \left[ \frac{n(n+1)-2}{9r_k^4} + \frac{\gamma}{9r_k} \left( \frac{1}{2n+1} - \frac{1}{3} \right) \right] h. \quad (5.9)$$

This proves the lemma.  $\square$

The second part of  $\mathcal{L}$  is minor.

**Lemma 5.2** *There exists  $C > 0$  in dependent of  $\xi$ ,  $r$ ,  $\rho$ , and  $\gamma$  such that*

$$\|\mathcal{L}_2(u)\|_{L^p} \leq \frac{C}{\rho^2} \|u\|_{L^p}$$

for all  $u \in \mathcal{Y}_*$ . A similar estimate holds if the two  $p$ 's above are replaced by 2.

*Proof.* Let  $\mathcal{L}_{2,k}$  be the  $k$ -th component of  $\mathcal{L}_2$ . Then

$$\begin{aligned} \mathcal{L}_{2,k}(u)(\theta_k) &= \frac{\gamma}{9} \int_{S^2} u_k(\omega_k) (R(\xi_k + r_k \theta_k, \xi_k + r_k \omega_k) - R(\xi_k, \xi_k)) d\omega_k \\ &\quad + \frac{\gamma u_k(\theta_k)}{9r_k^2} \int_{B_k} \nabla R(\xi_k + r_k \theta_k, y) \cdot \theta_k dy \\ &\quad + \sum_{l \neq k} \frac{\gamma}{9} \int_{S^2} u_l(\omega_l) (G(\xi_k + r_k \theta_k, \xi_l + r_l \omega_l) - G(\xi_k, \xi_l)) d\omega_l \\ &\quad + \sum_{l \neq k} \frac{\gamma u_k(\theta_k)}{9r_k^2} \int_{B_l} \nabla G(\xi_k + r_k \theta_k, y) \cdot \theta_k dy \\ &\quad + l_2(u) \end{aligned}$$

where  $l_2(u)$  is real valued and independent of  $k$ . It is included so that  $\mathcal{L}_2(u)$  is in  $\mathcal{Y}_*$ .

Because

$$\begin{aligned} R(\xi_k + r_k \theta_k, \xi_k + r_k \omega_k) - R(\xi_k, \xi_k) &= O(\rho), \\ G(\xi_k + r_k \theta_k, \xi_l + r_l \omega_l) - G(\xi_k, \xi_l) &= O(\rho), \end{aligned}$$

we obtain that

$$\begin{aligned} \left\| \frac{\gamma}{9} \int_{S^2} u_k(\omega_k) (R(\xi_k + r_k \theta_k, \xi_k + r_k \omega_k) - R(\xi_k, \xi_k)) d\omega_k \right\|_{L^p} &\leq C\gamma\rho \|u\|_{L^p} \\ \left\| \frac{\gamma}{9} \int_{S^2} u_l(\omega_l) (G(\xi_k + r_k \theta_k, \xi_l + r_l \omega_l) - G(\xi_k, \xi_l)) d\omega_l \right\|_{L^p} &\leq C\gamma\rho \|u_k\|_{L^p}. \end{aligned}$$

Since the volume of  $B_k$  is  $\frac{4\pi r_k^3}{3}$ ,

$$\begin{aligned} \left\| \frac{\gamma u_k(\theta_k)}{9r_k^2} \int_{B_k} \nabla R(\xi_k + r_k \theta_k, y) \cdot \theta_k dy \right\|_{L^p} &\leq C\gamma\rho \|u_k\|_{L^p} \\ \left\| \frac{\gamma u_k(\theta_k)}{9r_k^3} \int_{B_l} \nabla G(\xi_k + r_k \theta_k, y) \cdot \theta_k dy \right\|_{L^p} &\leq C\gamma\rho \|u_k\|_{L^p}. \end{aligned}$$

The condition

$$\sum_{k=1}^K \overline{\mathcal{L}_{2,k}(u)(\theta_k)} = 0$$

implies that

$$|l_2(u)| \leq C\gamma\rho\|u\|_{L^p}.$$

The lemma then follows, with the help of (2.10).  $\square$

**Lemma 5.3** 1. For  $u \in \mathcal{X}_*$

$$\|u\|_{W^{2,p}} \leq C\rho^4\|\Pi\mathcal{L}u\|_{L^p}.$$

2. The operator  $\Pi\mathcal{L}$  is invertible from  $\mathcal{X}_*$  to  $\mathcal{Y}_*$ .

3. If (2.8) holds,

$$\|u\|_{W^{1,2}}^2 \leq C\rho^4\langle \Pi\mathcal{L}u, u \rangle.$$

*Proof.* From Lemma 5.1 we have

$$\frac{|\lambda_{k,n}|}{n^2} = \frac{n-1}{9r_k^4 n} \left| \frac{n+2}{n} - \frac{2\gamma r_k^3}{3(2n+1)n} \right| > \frac{n-1}{18r_k^4 n} \left| \frac{n+2}{n} - \frac{2\gamma\rho^3}{3(2n+1)n} \right|$$

if  $\delta_2$  in the definition (3.1) of  $U_2$  is small enough. Then (2.6) implies that

$$\frac{|\lambda_{k,n}|}{n^2} > \frac{(n-1)}{18r_k^4 n} \frac{2\epsilon n}{3(2n+1)} \geq \frac{C}{\rho^4}, \quad n = 2, 3, \dots$$

If we expand  $u_k$  by spherical harmonics

$$u_k = \sum_{n=2}^{\infty} \sum_{l=1}^{2n+1} c_{n,l} h_{n,l}$$

where  $h_{n,l}$ ,  $l = 1, \dots, 2n+1$ , form an orthonormal basis in  $H_n$ , then

$$-\Delta_{S^2} u_k = \sum_{n=2}^{\infty} \sum_{l=1}^{2n+1} n(n+1) c_{n,l} h_{n,l}, \quad \mathcal{L}_{1,k} u_k = \sum_{n=2}^{\infty} \sum_{l=1}^{2n+1} \lambda_{k,n} c_{n,l} h_{n,l}.$$

Our estimate on  $|\lambda_{k,n}|$  shows that

$$\|\Delta_{S^2} u_k\|_{L^2}^2 = \sum_{n=2}^{\infty} \sum_{l=1}^{2n+1} n^2(n+1)^2 c_{n,l}^2 \leq C\rho^8 \sum_{n=2}^{\infty} \sum_{l=1}^{2n+1} \lambda_{k,n}^2 c_{n,l}^2 = C\rho^8 \|\mathcal{L}_{1,k} u_k\|_{L^2}^2.$$

The standard elliptic theory implies that

$$\|u\|_{W^{2,2}} \leq C\|\Delta_{S^2} u\|_{L^2} \leq C\rho^4\|\Pi\mathcal{L}_1(u)\|_{L^2}. \quad (5.10)$$

To prove Part 1. we divide  $\Pi\mathcal{L}_1$  into

$$\Pi\mathcal{L}_{1,k} = -\frac{1}{9r_k^4} \Delta_{S^2} + \mathcal{M}_k. \quad (5.11)$$

where  $\Delta_{S^2}$  is defined in (5.6), and  $\mathcal{M}$  is defined by (5.11). The standard elliptic estimate asserts that

$$\|u_k\|_{W^{2,p}} \leq C\|\Delta_{S^2}u_k\|_{L^p},$$

which by (5.11) is turned to

$$\begin{aligned} \|u_k\|_{W^{2,p}} &\leq C\|9r_k^4\mathcal{M}_k u - 9r_k^4\Pi\mathcal{L}_{1,k}u\|_{L^p} \\ &\leq C\rho^4(\|\mathcal{M}_k u\|_{L^p} + \|\Pi\mathcal{L}_{1,k}u\|_{L^p}). \end{aligned}$$

One observes that

$$\|\mathcal{M}u\|_{L^p} \leq \frac{C}{\rho^4}\|u\|_{L^p} \leq \frac{C}{\rho^4}\|u\|_{W^{2,2}}$$

where the last inequality comes from the Sobolev Embedding  $W^{2,2}(S^2) \rightarrow W^{1,p}(S^2) \subset L^p(S^2)$  for any  $p \geq 1$ . Hence when  $p > 2$ , by (5.10) we deduce that

$$\begin{aligned} \|u_k\|_{W^{2,p}} &\leq C\rho^4(\rho^{-4}\|u\|_{W^{2,2}} + \|\Pi\mathcal{L}_{1,k}u\|_{L^p}) \\ &\leq C\rho^4(\|\Pi\mathcal{L}_{1,k}u\|_{L^2} + \|\Pi\mathcal{L}_{1,k}u\|_{L^p}) \\ &\leq C\rho^4\|\Pi\mathcal{L}_{1,k}u\|_{L^p}. \end{aligned}$$

Lemma 5.2 implies that

$$\|\Pi\mathcal{L}u\|_{L^p} \geq \|\Pi\mathcal{L}_1u\|_{L^p} - \|\Pi\mathcal{L}_2u\|_{L^p} \geq \frac{C}{\rho^4}\|u\|_{W^{2,p}} - \frac{C}{\rho^2}\|u\|_{L^p} \geq \frac{C}{\rho^4}\|u\|_{W^{2,p}}$$

for small  $\rho$ . This proves Part 1.

Part 2 follows from the Fredholm Alternative.

When (2.8) holds,

$$\frac{\lambda_{k,n}}{n^2} = \frac{n-1}{9r_k^4 n} \left( \frac{n+2}{n} - \frac{2\gamma r_k^3}{3(2n+1)n} \right) > \frac{n-1}{18r_k^4 n} \frac{2\epsilon n}{3(2n+1)} \geq \frac{C}{\rho^4}, \quad n = 2, 3, \dots,$$

if  $\delta_2$  in (3.1) is small. This implies that, with the help of expansion by spherical harmonics,

$$\begin{aligned} \langle \Pi\mathcal{L}_{1,k}(u_k), u_k \rangle &= \sum_{n=2}^{\infty} \sum_{l=1}^{2n+1} \lambda_{k,n} c_{n,l}^2 \geq \frac{C}{\rho^4} \sum_{n=2}^{\infty} \sum_{l=1}^{2n+1} n(n+1) c_{n,l}^2 \\ &= \frac{C}{\rho^4} \langle -\Delta_{S^2}u_k, u_k \rangle = \frac{C}{\rho^4} \langle \nabla u_k, \nabla u_k \rangle \geq \frac{C}{\rho^4} \|u\|_{W^{1,2}}^2. \end{aligned}$$

Using the estimate of Lemma 5.2 with  $p$  replaced by 2, we find that

$$\langle \Pi\mathcal{L}(u), u \rangle = \langle \Pi\mathcal{L}_1(u), u \rangle - \langle \Pi\mathcal{L}_2(u), u \rangle \geq \frac{C}{\rho^4} \|u\|_{W^{1,2}}^2 - \frac{C}{\rho^2} \|u\|_{L^2}^2 \geq \frac{C}{\rho^4} \|u\|_{W^{1,2}}^2.$$

This proves Part 3.  $\square$

## 6 The Second Fréchet derivative

**Lemma 6.1** *Suppose that  $\|\phi\|_{W^{2,p}} \leq c\rho^3$  where  $c$  is sufficiently small. The following estimates hold.*

1.  $\|\mathcal{H}_k''(\phi_k)(u_k, v_k)\|_{L^p} \leq \frac{C}{\rho^7} \|u_k\|_{W^{2,p}} \|v_k\|_{W^{2,p}}.$
2.  $\|\mathcal{A}_k''(\phi_k)(u_k, v_k)\|_{L^p} \leq \frac{C}{\rho^7} \|u_k\|_{W^{1,p}} \|v_k\|_{W^{1,p}}.$
3.  $\|\mathcal{B}_k''(\phi_k)(u_k, v_k)\|_{L^p} \leq \frac{C}{\rho^5} \|u_k\|_{W^{1,p}} \|v_k\|_{W^{1,p}}.$
4.  $\|\mathcal{C}_{kl}''(\phi_k, \phi_l)(u_k, u_l)(v_k, v_l)\|_{L^p} \leq \frac{C}{\rho^5} (\|u_k\|_{W^{1,p}} + \|u_l\|_{W^{1,p}}) (\|v_k\|_{W^{1,p}} + \|v_l\|_{W^{1,p}}).$
5.  $|\lambda''(\phi)(u, v)| \leq \frac{C}{\rho^7} \|u\|_{W^{2,p}} \|v\|_{W^{2,p}}.$

*Proof.* Note that by taking  $c$  small, we keep  $\rho^3 + \phi$  positive, so  $\partial E_\phi$  is a perturbed sphere.

The mean curvature operator  $\mathcal{H}_k$  is elliptic and quasilinear. Its second Fréchet derivative is calculated from (4.28):

$$\begin{aligned} & \mathcal{H}_k''(\phi_k, D\phi_k, D^2\phi_k)(u_k, v_k) \\ &= \frac{\partial^2 \mathcal{H}_k}{\partial \phi_k^2} u_k v_k + \sum_{i=1}^2 \frac{\partial^2 \mathcal{H}_k}{\partial \phi_k \partial \phi_{k,i}} (u_k v_{k,i} + u_{k,i} v_k) + \sum_{i,j=1}^2 \frac{\partial^2 \mathcal{H}_k}{\partial \phi_{k,i} \partial \phi_{k,j}} (u_{k,i} v_{k,j} + u_{k,j} v_{k,i}) \\ &+ \sum_{l,m=1}^2 \frac{\partial^2 \mathcal{H}_k}{\partial \phi_k \partial \phi_{k,lm}} (u_k v_{k,lm} + u_{k,lm} v_k) + \sum_{i,l,m=1}^2 \frac{\partial^2 \mathcal{H}_k}{\partial \phi_{k,i} \partial \phi_{k,lm}} (u_{k,i} v_{k,lm} + u_{k,lm} v_{k,i}). \end{aligned}$$

It is important to note that because  $\mathcal{H}_k$  is quasilinear, i.e. it is linear in  $D^2\phi_k$ , the term

$$\sum_{i,j,l,m=1}^2 \frac{\partial^2 \mathcal{H}_k}{\partial \phi_{k,ij} \partial \phi_{k,lm}} (u_{k,ij} v_{k,lm} + u_{k,lm} v_{k,ij})$$

is 0 and hence absent in  $\mathcal{H}_k''$ . The Sobolev embedding  $W^{1,p} \rightarrow L^\infty$  and  $\|\phi_k\|_{W^{2,p}} \leq c\rho^3$  for a small  $c$  imply that  $|\phi_k| \leq C\rho^3$  and  $|D\phi_k| \leq C\rho^3$ . From the definition (4.12) of  $\mathcal{H}_k$  we have the pointwise estimate

$$\begin{aligned} & |\mathcal{H}_k''(\phi_k, D\phi_k, D^2\phi_k)(u_k, v_k)| \\ & \leq \frac{C}{\rho^7} (|\frac{D^2\phi_k}{r_k^3}| |u_k| |v_k| + |\frac{D^2\phi_k}{r_k^3}| |u_k| |Dv_k| + |\frac{D^2\phi_k}{r_k^3}| |Du_k| |v_k| + |\frac{D^2\phi_k}{r_k^3}| |Du_k| |Dv_k| \\ & + |u_k| |D^2v_k| + |D^2u_k| |v_k| + |Du_k| |D^2v_k| + |D^2u_k| |Dv_k|), \end{aligned}$$

when  $\theta_k$  is some distance away from the two poles (where  $\theta_{k,2} = 0$  or  $\pi$ ) of  $S^2$ . Near the two poles one can use a different parametrization of  $S^2$  so that the same pointwise estimate holds. The same Sobolev embedding implies that

$$\|\mathcal{H}_k''(\phi)(u_k, v_k)\|_{L^p} \leq \frac{C}{\rho^7} \|u_k\|_{W^{2,p}} \|v_k\|_{W^{2,p}}. \quad (6.1)$$

This proves Part 1.

We now turn to Part 2. In our estimation of  $\mathcal{A}_k''$  and  $\mathcal{B}_k''$  we drop the subscript  $k$  in most quantities. The second Fréchet derivative of  $\mathcal{A}_k$  is calculated from (4.29):

$$\mathcal{A}_k''(\phi)(u, v) = A_1(\phi)(u, v) + A_2(\phi)(u, v) + A_3(\phi)(u, v) + A_4(\phi)(u, v) + A_5(\phi)(u, v) \quad (6.2)$$

where

$$\begin{aligned} A_1(\phi)(u, v) &= -\frac{\gamma v(\theta)\theta}{108\pi(r^3 + \phi(\theta))^{2/3}} \cdot \int_{S^2} K(\theta, \omega)u(\omega) d\omega \\ A_2(\phi)(u, v) &= -\frac{\gamma u(\theta)\theta}{108\pi(r^3 + \phi(\theta))^{2/3}} \cdot \int_{S^2} K(\theta, \omega)v(\omega) d\omega \\ A_3(\phi)(u, v) &= \frac{\gamma}{108\pi} \int_{S^2} K(\theta, \omega) \cdot \omega \frac{u(\omega)v(\omega)}{(r^3 + \phi(\omega))^{2/3}} d\omega \\ A_4(\phi)(u, v) &= -\frac{\gamma u(\theta)v(\theta)}{108\pi(r^3 + \phi(\theta))^{4/3}} \int_{E_{\phi_k}} \frac{|(r^3 + \phi(\theta))^{1/3}\theta - y|^2 - 3((r^3 + \phi(\theta))^{1/3} - \theta \cdot y)^2}{|(r^3 + \phi(\theta))^{1/3}\theta - y|^5} dy \\ A_5(\phi)(u, v) &= \frac{2\gamma u(\theta)v(\theta)}{108\pi(r^3 + \phi(\theta))^{5/3}} \int_{E_{\phi_k}} \frac{(r^3 + \phi(\theta))^{1/3}\theta - y}{|(r^3 + \phi(\theta))^{1/3}\theta - y|^3} \cdot \theta dy. \end{aligned}$$

The kernel  $K$  is

$$K(\theta, \omega) = \frac{(r^3 + \phi(\theta))^{1/3}\theta - (r^3 + \phi(\omega))^{1/3}\omega}{|(r^3 + \phi(\theta))^{1/3}\theta - (r^3 + \phi(\omega))^{1/3}\omega|^3}. \quad (6.3)$$

Here we encounter a singular integral operator

$$\mathcal{K}(u)(\theta) = \int_{S^2} K(\theta, \omega)u(\omega) d\omega. \quad (6.4)$$

A variant of the Calderon-Zygmund estimate [32, Theorem 1] is applicable to this operator:

$$\|\mathcal{K}(u)\|_q \leq \frac{C}{\rho^2} \|u\|_{L^q}$$

for any  $q \in (1, \infty)$ . In [32] the kernel takes the form  $K(x - y)$ . To meet this requirement, we can transform (6.4) to an integral on the perturbed sphere  $\partial E_{\phi_k}$ , then  $K(\theta, \omega)$  becomes  $\frac{x-y}{|x-y|^3}$  where  $x, y \in \partial E_{\phi_k}$ .

For  $\|\phi\|_{W^{2,p}} \leq c\rho^3$  with a small  $c$ , we consider

$$\|\mathcal{A}_k''(\phi)(u, v)\|_{L^p} \leq \sum_{i=1}^5 \|A_i(\phi)(u, v)\|_{L^p}.$$

For sufficiently large  $q$

$$\|A_1(\phi)(u, v)\|_{L^p} \leq \frac{C}{\rho^7} \|v_k\|_{L^q} \|\mathcal{K}(u_k)\|_{L^q} \leq \frac{C}{\rho^7} \|v_k\|_{L^q} \|u_k\|_{L^q} \leq \frac{C}{\rho^7} \|u_k\|_{W^{1,p}} \|v_k\|_{W^{1,p}}.$$

Similarly

$$\|A_2(\phi)(u, v)\|_{L^p} \leq \frac{C}{\rho^7} \|u\|_{W^{1,p}} \|v_k\|_{W^{1,p}}.$$

Regarding  $A_3$  we have, using the Calderon-Zygmund estimate in  $L^p$  and the Sobolev Embedding theory,

$$\|A_3(\phi)(u, v)\|_{L^p} \leq \frac{C}{\rho^7} \|uv\|_{L^p} \leq \frac{C}{\rho^7} \|u\|_{W^{1,p}} \|v\|_{W^{1,p}}.$$

For  $A_4$ , the integral

$$\int_{E_{\phi_k}} \frac{|(r^3 + \phi(\theta))^{1/3}\theta - y|^2 - 3((r^3 + \phi(\theta))^{1/3} - \theta \cdot y)^2}{|(r^3 + \phi(\theta))^{1/3}\theta - y|^5} dy$$

is a convergent improper integral defined by its principal part. It is of order 1 and uniformly bounded with respect to  $\theta$ . In the case of  $\Phi$  equal to 0, it may be explicitly computed. (See Appendix C.) Therefore

$$\|A_4(\phi)(u, v)\|_{L^p} \leq \frac{C}{\rho^7} \|uv\|_{L^p} \leq \frac{C}{\rho^7} \|u\|_{W^{1,p}} \|v\|_{W^{1,p}}.$$

For  $A_5$ , because of the mild singularity, we easily find that

$$\|A_5(\phi)(u, v)\|_{L^p} \leq \frac{C}{\rho^7} \|u\|_{W^{1,p}} \|v\|_{W^{1,p}}.$$

Now we have

$$\|\mathcal{A}_k''(\phi)(u_k, v_k)\|_{L^p} \leq \frac{C}{\rho^7} \|u_k\|_{W^{1,p}} \|v_k\|_{W^{1,p}}.$$

This proves Part 2.

The kernel  $R$  in  $\mathcal{B}_k$  is a smooth function. Calculations from (4.30) show that

$$\begin{aligned} & \mathcal{B}_k''(\phi)(u, v)(\theta) \\ &= \frac{\gamma v(\theta)}{27(r^3 + \phi(\theta))^{2/3}} \int_{S^2} u(\omega) D_1 R(\xi + (r^3 + \phi(\theta))^{1/3}\theta, \xi + (r^3 + \phi(\omega))^{1/3}\omega) \cdot \theta d\omega \\ & \quad - \frac{\gamma u(\theta)}{27(r^3 + \phi(\theta))^{2/3}} \int_{S^2} v(\omega) D_1 R(\xi + (r^3 + \phi(\theta))^{1/3}\theta, \xi + (r^3 + \phi(\omega))^{1/3}\omega) \cdot \theta d\omega \\ & \quad + \frac{\gamma}{27} \int_{S^2} \frac{u(\omega)v(\omega)}{(r^3 + \phi(\omega))^{2/3}} D_2 R(\xi + (r^3 + \phi(\theta))^{1/3}\theta, \xi + (r^3 + \phi(\omega))^{1/3}\omega) \cdot \theta d\omega \\ & \quad + \frac{\gamma u(\theta)v(\theta)}{27(r^3 + \phi(\theta))^{4/3}} \int_{E_{\phi_k}} D_1^2 R(\xi + (r^3 + \phi(\theta))^{1/3}\theta, y) \theta \cdot \theta dy \\ & \quad - \frac{2\gamma u(\theta)v(\theta)}{27(r^3 + \phi(\theta))^{5/3}} \int_{E_{\phi_k}} D_1 R(\xi + (r^3 + \phi(\theta))^{1/3}\theta, y) \cdot \theta dy \end{aligned}$$

where  $D_1$  and  $D_2$  refer to the derivatives of  $R$  with respect to its first and second arguments respectively.  $D_1^2 R$  is the second derivative matrix of  $R$  with respect to the first argument of  $R$ . Part 3 is now proved easily.

The function  $G$  is also smooth in  $\mathcal{C}$ . We restore subscripts in the rest of this section. Similar to  $\mathcal{B}_k''$ , we find from (4.31) that

$$\mathcal{C}_{kl}''(\phi_k, \phi_l)(u_k, u_l)(v_k, v_l)(\theta_k)$$

$$\begin{aligned}
&= \frac{\gamma v_k(\theta_k)}{27(r_k^3 + \phi_k(\theta_k))^{2/3}} \int_{S^2} u_l(\omega_l) D_1 G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, \xi_l + (r_l^3 + \phi_l(\omega_l))^{1/3} \omega_l) \cdot \theta_k d\omega_l \\
&+ \frac{\gamma u_k(\theta_k)}{27(r_k^3 + \phi_k(\theta_k))^{2/3}} \int_{S^2} v_l(\omega_l) D_1 G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, \xi_l + (r_l^3 + \phi_l(\omega_l))^{1/3} \omega_l) \cdot \theta_k d\omega_l \\
&+ \frac{\gamma}{27} \int_{S^2} \frac{u_l(\omega_l) v_l(\omega_l)}{(r_l^3 + \phi_l(\omega_l))^{2/3}} D_2 G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, \xi_l + (r_l^3 + \phi_l(\omega_l))^{1/3} \omega_l) \cdot \omega_l d\omega_l \\
&+ \frac{\gamma u_k(\theta_k) v_k(\theta_k)}{27(r_k^3 + \phi_k(\theta_k))^{4/3}} \int_{E_{\phi_l}} D_1^2 G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) \theta_k \cdot \theta_k dy \\
&- \frac{2\gamma u_k(\theta_k) v_k(\theta_k)}{27(r_k^3 + \phi_k(\theta_k))^{5/2}} \int_{E_{\phi_l}} D_1 G(\xi_k + (r_k^3 + \phi_k(\theta_k))^{1/3} \theta_k, y) \cdot \theta_k dy.
\end{aligned}$$

Part 4 then follows.

Part 5 follows from Parts 1-4 and the fact that

$$\begin{aligned}
0 &= \sum_k \overline{S_k''(\phi)(u, v)} \\
&= \sum_k \overline{\mathcal{H}_k''(\phi_k)(u_k, v_k)} + \sum_k \overline{\mathcal{A}_k''(\phi_k)(u_k, v_k)} + \sum_k \overline{\mathcal{B}_k''(\phi_k)(u_k, v_k)} \\
&\quad + \sum_k \overline{\mathcal{C}_k''(\phi)(u)} + K \lambda_\xi''(\phi)(u, v). \quad \square
\end{aligned}$$

## 7 Reduction to $4K - 1$ dimensions

We view  $\mathcal{S}$  as a nonlinear operator from  $\mathcal{X}$  to  $\mathcal{Y}$ . In this section it will be proved that, for each  $(\xi, r) \in U$ , a  $\varphi(\cdot, \xi, r)$  exists such that  $\varphi(\cdot, \xi, r) \in \mathcal{X}_*$  and

$$\mathcal{S}_k(\varphi)(\theta_k) = A_{k,1} \cos \theta_{k,1} \sin \theta_{k,2} + A_{k,2} \sin \theta_k \sin \theta_{k,2} + A_{k,3} \cos \theta_{k,2} + A_k, \quad k = 1, 2, \dots, K \quad (7.1)$$

for some numbers  $A_{k,1}, A_{k,2}, A_{k,3}, A_k$ . Note that  $\varphi$  is sought in  $\mathcal{X}_*$ . Each  $\phi \in \mathcal{X}_*$  satisfies

$$\int_{S^2} \phi_k(\theta_k) d\theta_k = 0, \quad k = 1, 2, \dots, K \quad (7.2)$$

$$\int_{S^2} \phi_k(\theta_k) \cos \theta_{k,1} \sin \theta_{k,2} d\theta_k = 0, \quad k = 1, 2, \dots, K \quad (7.3)$$

$$\int_{S^2} \phi_k(\theta_k) \sin \theta_{k,1} \sin \theta_{k,2} d\theta_k = 0, \quad k = 1, 2, \dots, K \quad (7.4)$$

$$\int_{S^2} \phi_k(\theta_k) \cos \theta_{k,2} d\theta_k = 0, \quad k = 1, 2, \dots, K. \quad (7.5)$$

The condition (7.2) means that  $\phi_k \perp H_0$ , the space of spherical harmonics of degree 0, and the conditions (7.3-7.5) state that  $\phi_k \perp H_1$ .

Write the equation (7.1) as

$$\Pi \mathcal{S}(\varphi) = 0 \quad (7.6)$$

where  $\Pi$  is the orthogonal projection operator from  $\mathcal{Y}$  to  $\mathcal{Y}_*$ . In the next section we will find a particular  $(\xi, r)$ , say  $(\zeta, s)$  at which  $A_{k,1} = A_{k,2} = A_{k,3} = A_k = 0$ , i.e.  $\mathcal{S}(\varphi(\cdot, \zeta, s)) = 0$ . This means

that by finding  $\varphi$  we reduce the original problem (1.1) to a problem of finding a  $(\zeta, s)$  in a  $4K - 1$  dimensional set  $U$ .

Recall  $\mathcal{L}$ , the linearized operator of  $\mathcal{S}$  at  $\phi = 0$ . Expand  $\mathcal{S}(\phi)$  as

$$\mathcal{S}(\phi) = \mathcal{S}(0) + \mathcal{L}(\phi) + \mathcal{N}(\phi) \quad (7.7)$$

where  $\mathcal{N}$  is a higher order term defined by (7.7). Turn (7.6) to a fixed point form:

$$\phi = -(\Pi\mathcal{L})^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)). \quad (7.8)$$

**Lemma 7.1** *There exists  $\varphi = \varphi(\theta, \xi, r)$  such that for every  $(\xi, r) \in U$ ,  $\varphi(\cdot, \xi, r) \in \mathcal{X}_*$  solves (7.8) and  $\|\varphi\|_{W^{2,p}} \leq c\rho^5$  where  $c$  is a sufficiently large constant independent of  $\xi, r, \rho$  and  $\gamma$ .*

*Proof.* To use the Contraction Mapping Principle, let

$$\mathcal{T}(\phi) = -(\Pi\mathcal{L})^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)) \quad (7.9)$$

be an operator defined on

$$D(\mathcal{T}) = \{\phi \in \mathcal{X}_* : \|\phi\|_{W^{2,p}} \leq c\rho^5\} \quad (7.10)$$

where the constant  $c$  is sufficiently large and will be determined shortly.

Lemma 3.1 shows that

$$\mathcal{S}_k(0)(\theta_k) - \lambda(0) = \frac{1}{3r_k} + \frac{\gamma}{3} \left[ \frac{r_k^2}{3} + \frac{4\pi r_k^3}{3} R(\xi_k, \xi_k) + \sum_{l \neq k} \frac{4\pi r_l^3}{3} G(\xi_k, \xi_l) \right] + O(\rho).$$

Each  $\mathcal{S}_k(0)$  is sum of a number independent of  $\theta_k$  and a quantity of order  $O(\rho)$ . After we apply the projection operator  $\Pi$  the number vanishes and

$$\|\Pi\mathcal{S}(0)\|_{L^p} = O(\rho). \quad (7.11)$$

By Lemma 5.3 we find

$$\|(\Pi\mathcal{L})^{-1}\Pi\mathcal{S}(0)\|_{W^{2,p}} \leq C\rho^5. \quad (7.12)$$

For  $\mathcal{N}(\phi)$  we decompose it into three parts. The first is  $\mathcal{N}_1$  whose  $k$ -th component is

$$\mathcal{N}_{1,k}(\phi_k) = \mathcal{H}_k(\phi_k) - \mathcal{H}_k(0) - \mathcal{H}'_k(0)(\phi_k) \quad (7.13)$$

which is  $\mathcal{H}_k(\phi)$  minus its linear approximation at 0. Lemma 6.1, Part 1, shows that

$$\|\mathcal{N}_1(\phi)\|_{L^p} \leq \frac{C}{\rho^7} \|\phi\|_{W^{2,p}}^2. \quad (7.14)$$

The second part of  $\mathcal{N}$ , denoted by  $\mathcal{N}_2$ , is  $\mathcal{A}(\phi) + \mathcal{B}(\phi) + \mathcal{C}(\phi)$  minus its linear approximation, i.e.

$$\mathcal{N}_2(\phi) = \mathcal{A}(\phi) - \mathcal{A}(0) - \mathcal{A}'(0)(\phi) + \mathcal{B}(\phi) - \mathcal{B}(0) - \mathcal{B}'(0)(\phi) + \mathcal{C}(\phi) - \mathcal{C}(0) - \mathcal{C}'(0)(\phi). \quad (7.15)$$

Lemma 6.1, Parts 2, 3, and 4, implies that

$$\|\mathcal{N}_2(\phi)\|_{L^p} \leq \frac{C}{\rho^7} \|\phi\|_{W^{1,p}}^2. \quad (7.16)$$

The third part of  $\mathcal{N}$ , which is denoted by  $\mathcal{N}_3$ , merely gives a constant so that

$$\sum_k \overline{\mathcal{N}_k(\phi)} = \sum_k \overline{\mathcal{N}_{1,k}(\phi)} + \sum_k \overline{\mathcal{N}_{2,k}(\phi)} + K\mathcal{N}_3(\phi) = 0.$$

It follows that

$$|\mathcal{N}_3(\phi)| \leq \frac{C}{\rho^7} \|\phi\|_{W^{2,p}}^2. \quad (7.17)$$

Therefore we deduce, from (7.14), (7.16), (7.17) and with the help of Lemma 5.3, that

$$\|\mathcal{N}(\phi)\|_{L^p} \leq \frac{C}{\rho^7} \|\phi\|_{W^{2,p}}^2 \quad (7.18)$$

$$\|(\Pi\mathcal{L})^{-1}\Pi\mathcal{N}(\phi)\|_{W^{2,p}} \leq \frac{C}{\rho^3} \|\phi\|_{W^{2,p}}^2. \quad (7.19)$$

Using (2.10), (7.12), (7.10), and (7.19) we find

$$\|\mathcal{T}(\phi)\|_{W^{2,p}} \leq C\rho^5 + Cc^2\rho^7 \leq c\rho^5$$

if  $c$  is sufficiently large and  $\rho$  sufficiently small. Therefore  $\mathcal{T}$  is a map from  $D(\mathcal{T})$  into itself.

Next we show that  $\mathcal{T}$  is a contraction. For  $\mathcal{N}_1$  we note that

$$\mathcal{N}_1(\phi_1) - \mathcal{N}_1(\phi_2) = \mathcal{H}(\phi_1) - \mathcal{H}(\phi_2) - \mathcal{H}'(0)(\phi_1 - \phi_2)$$

Therefore using Lemma 6.1, Part 1, we obtain

$$\begin{aligned} & \|\mathcal{H}(\phi_1) - \mathcal{H}(\phi_2) - \mathcal{H}'(0)(\phi_1 - \phi_2)\|_{L^p} \\ & \leq \|\mathcal{H}'(\phi_2)(\phi_1 - \phi_2) - \mathcal{H}'(0)(\phi_1 - \phi_2)\|_{L^p} + \frac{C}{\rho^7} \|\phi_1 - \phi_2\|_{W^{2,p}}^2 \\ & \leq \frac{C}{\rho^7} \|\phi_2\|_{W^{2,p}} \|\phi_1 - \phi_2\|_{W^{2,p}} + \frac{C}{\rho^7} \|\phi_1 - \phi_2\|_{W^{2,p}}^2 \\ & \leq \frac{C}{\rho^7} (\|\phi_1\|_{W^{2,p}} + \|\phi_2\|_{W^{2,p}}) \|\phi_1 - \phi_2\|_{W^{2,p}}. \end{aligned}$$

This shows that

$$\|\mathcal{N}_1(\phi_1) - \mathcal{N}_2(\phi_2)\|_{L^p} \leq \frac{C}{\rho^2} \|\phi_1 - \phi_2\|_{W^{2,p}}. \quad (7.20)$$

For  $\mathcal{N}_2$  we note that

$$\begin{aligned} \mathcal{N}_2(\phi_1) - \mathcal{N}_2(\phi_2) &= \mathcal{A}(\phi_1) - \mathcal{A}(\phi_2) - \mathcal{A}'(0)(\phi_1 - \phi_2) + \mathcal{B}(\phi_1) - \mathcal{B}(\phi_2) - \mathcal{B}'(0)(\phi_1 - \phi_2) \\ &\quad + \mathcal{C}(\phi_1) - \mathcal{C}(\phi_2) - \mathcal{C}'(0)(\phi_1 - \phi_2). \end{aligned} \quad (7.21)$$

Therefore using Lemma 6.1, Part 2, we obtain

$$\begin{aligned} & \|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2) - \mathcal{A}'(0)(\phi_1 - \phi_2)\|_{L^p} \\ & \leq \|\mathcal{A}'(\phi_2)(\phi_1 - \phi_2) - \mathcal{A}'(0)(\phi_1 - \phi_2)\|_{L^p} + \frac{C}{\rho^7} \|\phi_1 - \phi_2\|_{W^{1,p}}^2 \\ & \leq \frac{C}{\rho^7} \|\phi_2\|_{W^{1,p}} \|\phi_1 - \phi_2\|_{W^{1,p}} + \frac{C}{\rho^7} \|\phi_1 - \phi_2\|_{W^{1,p}}^2 \\ & \leq \frac{C}{\rho^7} (\|\phi_1\|_{W^{1,p}} + \|\phi_2\|_{W^{1,p}}) \|\phi_1 - \phi_2\|_{W^{1,p}}. \end{aligned}$$

Similarly using Lemma 6.1, Parts 3 and 4, we deduce

$$\begin{aligned}\|\mathcal{B}(\phi_1) - \mathcal{B}(\phi_2) - \mathcal{B}'(0)(\phi_1 - \phi_2)\|_{L^p} &\leq \frac{C}{\rho^5}(\|\phi_1\|_{W^{1,p}} + \|\phi_2\|_{W^{1,p}})\|\phi_1 - \phi_2\|_{W^{1,p}} \\ \|\mathcal{C}(\phi_1) - \mathcal{C}(\phi_2) - \mathcal{C}'(0)(\phi_1 - \phi_2)\|_{L^p} &\leq \frac{C}{\rho^5}(\|\phi_1\|_{W^{1,p}} + \|\phi_2\|_{W^{1,p}})\|\phi_1 - \phi_2\|_{W^{1,p}}.\end{aligned}$$

From (7.21) we conclude that

$$\|\mathcal{N}_2(\phi_1) - \mathcal{N}_2(\phi_2)\|_{L^p} \leq \frac{C}{\rho^7}(\|\phi_1\|_{W^{1,p}} + \|\phi_2\|_{W^{1,p}})\|\phi_1 - \phi_2\|_{W^{1,p}} \leq \frac{C}{\rho^2}\|\phi_1 - \phi_2\|_{W^{1,p}}. \quad (7.22)$$

We also have

$$\|\mathcal{N}_3(\phi_1) - \mathcal{N}_3(\phi_2)\|_{L^p} \leq \frac{C}{\rho^2}\|\phi_1 - \phi_2\|_{W^{2,p}}. \quad (7.23)$$

Hence, following (7.20), (7.22), and (7.23), we find that

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_{W^{2,p}} = \|(\Pi\mathcal{L})^{-1}\Pi\mathcal{N}(\phi_1) - (\Pi\mathcal{L})^{-1}\Pi\mathcal{N}(\phi_2)\|_{W^{2,p}} \leq C\rho^2\|\phi_1 - \phi_2\|_{W^{2,p}}, \quad (7.24)$$

i.e. that  $\mathcal{T}$  is a contraction map if  $\rho$  is sufficiently small. A fixed point  $\varphi$  exists.  $\square$

Since  $\varphi$  satisfies  $\|\varphi\|_{W^{2,p}} \leq c\rho^5$ , by taking  $\rho$  small we see that  $r_k^3 + \varphi_k$  remains positive.  $\partial E_{\varphi_k}$  is a perturbed sphere.

Denote  $\mathcal{S}'(\varphi)$  by  $\tilde{\mathcal{L}}$ . We derive a lemma for  $\tilde{\mathcal{L}}$  similar to Lemma 5.3.

**Lemma 7.2** *Let  $\Pi$  be the same projection operator from  $\mathcal{X}$  to  $\mathcal{X}_*$ .*

1. *There exists  $C > 0$  such that for all  $u \in \mathcal{X}_*$*

$$\|u\|_{W^{2,p}} \leq C\rho^4\|\Pi\tilde{\mathcal{L}}(u)\|_{L^p}$$

2. *If (2.8) holds,*

$$\|u\|_{W^{1,2}}^2 \leq C\rho^4\langle \Pi\tilde{\mathcal{L}}(u), u \rangle.$$

*Proof.* By Lemma 5.3, Part 1, Lemma 6.1 and the fact  $\|\varphi\|_{W^{2,p}} = O(\rho^5)$ , we deduce

$$\begin{aligned}\|\Pi\tilde{\mathcal{L}}(u)\|_{L^p} &\geq \|\Pi\mathcal{L}(u)\|_{L^p} - \|\Pi(\tilde{\mathcal{L}} - \mathcal{L})(u)\|_{L^p} \\ &\geq \frac{C}{\rho^4}\|u\|_{W^{2,p}} - \frac{C}{\rho^7}\|\varphi\|_{W^{2,p}}\|u\|_{W^{2,p}} \\ &\geq \frac{C}{\rho^4}\|u\|_{W^{2,p}} - \frac{C}{\rho^2}\|u\|_{W^{2,p}} \geq \frac{C}{\rho^4}\|u\|_{W^{2,p}}\end{aligned}$$

when  $\rho$  is small. This proves part 1.

Write  $\tilde{\mathcal{L}} = \mathcal{H}'(\varphi) + \mathcal{A}'(\varphi) + \mathcal{B}'(\varphi) + \mathcal{C}'(\varphi) + \lambda'(\varphi)$ . Then, according to (4.7),

$$\langle \mathcal{H}'_k(\varphi_k)(u_k), u_k \rangle = \int_{S^2} \left[ \frac{\partial^2 L_k}{\partial \phi_k^2} u_k^2 + 2 \sum_{i=1}^2 \frac{\partial^2 L_k}{\partial \phi_k \partial \phi_{k,i}} u_k u_{k,i} + \sum_{i,j=1}^2 \frac{\partial^2 L_k}{\partial \phi_{k,i} \partial \phi_{k,j}} u_{k,i} u_{k,j} \right] d\theta_k.$$

and a similar expression holds if we replace  $\varphi_k$  and  $\varphi_{k,i}$  by 0 in the last formula.

With  $\|\varphi\|_{W^{2,p}} = O(\rho^5)$  calculations show that

$$\begin{aligned}
& | \langle (\mathcal{H}'_k(\varphi_k) - \mathcal{H}'_k(0))u_k, u_k \rangle | \\
& \leq \left| \int_{S^2} \left( \frac{\partial^2 L_k(\varphi_k)}{\partial \phi_k^2} - \frac{\partial^2 L_k(0)}{\partial \phi_k^2} \right) u_k^2 d\theta_k \right| + 2 \sum_{i=1}^2 \left| \int_{S^2} \left( \frac{\partial^2 L_k(\varphi_k)}{\partial \phi_k \partial \phi_{k,i}} - \frac{\partial^2 L_k(0)}{\partial \phi_k \partial \phi_{k,i}} \right) u_k u_{k,i} d\theta_k \right| \\
& \quad + \sum_{i,j=1}^2 \left| \int_{S^2} \left( \frac{\partial^2 L_k(\varphi_k)}{\partial \phi_{k,i} \partial \phi_{k,j}} - \frac{\partial^2 L_k(0)}{\partial \phi_{k,i} \partial \phi_{k,j}} \right) u_{k,i} u_{k,j} d\theta_k \right| \\
& \leq \frac{C}{\rho^2} \|u\|_{L^2}^2 + \frac{C}{\rho^2} \|u\|_{L^2} \|Du\|_{L^2} + \frac{C}{\rho^2} \|Du\|_{L^2}^2 \leq \frac{C}{\rho^2} \|u\|_{W^{1,2}}^2. \tag{7.25}
\end{aligned}$$

Next we estimate  $\|(\mathcal{A}'_k(\varphi_k) - \mathcal{A}'_k(0))u_k\|_{L^2}$ . We re-visit  $\mathcal{A}''_k$ . Argue as in the proof of Lemma 6.1, Part 2, we deduce that

$$\|\mathcal{A}''_k(\phi)(u_k, v_k)\|_{L^2} \leq \frac{C}{\rho^7} \|u_k\|_{W^{1,2}} \|v_k\|_{W^{1,2}}.$$

This implies that in this lemma

$$\|(\mathcal{A}'_k(\varphi) - \mathcal{A}'_k(0))u_k\|_{L^2} \leq \frac{C}{\rho^7} C\rho^5 \|u_k\|_{W^{1,2}} \leq \frac{C}{\rho^2} \|u_k\|_{W^{1,2}}.$$

Simpler arguments show that

$$\|(\mathcal{B}'_k(\varphi) - \mathcal{B}'_k(0))u_k\|_{L^2} \leq \frac{C}{\rho^2} \|u_k\|_{W^{1,2}}, \quad \|(\mathcal{C}'(\varphi) - \mathcal{C}'(0))u\|_{L^2} \leq \frac{C}{\rho^2} \|u\|_{W^{1,2}}.$$

We obtain that

$$\|(\mathcal{A}'(\varphi) + \mathcal{B}'(\varphi) + \mathcal{C}'(\varphi) - \mathcal{A}'(0) - \mathcal{B}'(0) - \mathcal{C}'(0))u\|_{L^2} \leq \frac{C}{\rho^2} \|u\|_{W^{1,2}}. \tag{7.26}$$

If (2.8) holds, we combine Lemma 5.3, Part 3, (7.25), and (7.26) to deduce that

$$\langle \Pi \tilde{\mathcal{L}}(u), u \rangle = \langle \Pi \mathcal{L}(u), u \rangle + \langle \Pi(\tilde{\mathcal{L}} - \mathcal{L})u, u \rangle \geq \frac{C}{\rho^4} \|u\|_{W^{1,2}}^2 - \frac{C}{\rho^2} \|u\|_{W^{1,2}}^2 \geq \frac{C}{\rho^4} \|u\|_{W^{1,2}}^2,$$

proving the second part.  $\square$

One consequence of Lemma 7.2, Part 1, is an estimate of  $\frac{\partial \varphi}{\partial \xi_{l,j}}$ .

**Lemma 7.3** *The fixed point  $\varphi$  satisfies  $\|\frac{\partial \varphi}{\partial \xi_{l,j}}\|_{W^{2,p}} = O(\rho^4)$ ,  $l = 1, 2, \dots, K$ ,  $j = 1, 2, 3$ .*

*Proof.* We prove this lemma by the Implicit Function Theorem. Fix  $l \in \{1, 2, \dots, K\}$  and  $j \in \{1, 2, 3\}$ . Differentiating  $\Pi \mathcal{S}_\xi(\varphi)$  with respect to  $\xi_{l,j}$  finds that, for  $k = 1, 2, \dots, K$ , if  $k = l$ , then

$$\begin{aligned}
\frac{\partial \Pi \mathcal{S}_l(\varphi)}{\partial \xi_{l,j}} &= \Pi \tilde{\mathcal{L}}_l \left( \frac{\partial \varphi}{\partial \xi_{l,j}} \right) \\
&+ \Pi \frac{\gamma}{3} \int_{E_{\varphi_l}} \left[ \frac{\partial R(\xi_l + (r_l^3 + \varphi_l(\theta_l))^{1/3} \theta_l, y)}{\partial x_j} + \frac{\partial R(\xi_l + (r_l^3 + \varphi_l(\theta_l))^{1/3} \theta_l, y)}{\partial y_j} \right] dy \\
&+ \sum_{m \neq l} \Pi \frac{\gamma}{3} \int_{E_{\varphi_m}} \frac{\partial G(\xi_l + (r_l^3 + \varphi_l(\theta_l))^{1/3} \theta_l, y)}{\partial x_j} dy,
\end{aligned}$$

and if  $k \neq l$ ,

$$\frac{\partial \Pi \mathcal{S}_k(\varphi)}{\partial \xi_{l,j}} = \Pi \tilde{\mathcal{L}}_k \left( \frac{\partial \varphi}{\partial \xi_{l,j}} \right) + \Pi \frac{\gamma}{3} \int_{E_{\varphi_l}} \frac{\partial G(\xi_k + (r_k^3 + \varphi_k(\theta_k))^{1/3} \theta_k, y)}{\partial y_j} dy.$$

Here  $R = R(x, y)$  and  $G = G(x, y)$ . It is clear that

$$\begin{aligned} \left\| \frac{\gamma}{3} \int_{E_{\varphi_l}} \left[ \frac{\partial R(\xi_l + (r_l^3 + \varphi_l(\theta_l))^{1/3} \theta_l, y)}{\partial x_j} + \frac{\partial R(\xi_l + (r_l^3 + \varphi_l(\theta_l))^{1/3} \theta_l, y)}{\partial y_j} \right] dy \right\|_{L^p} &= O(\gamma \rho^3), \\ \left\| \frac{\gamma}{3} \int_{E_{\varphi_m}} \frac{\partial G(\xi_l + (r_l^3 + \varphi_l(\theta_l))^{1/3} \theta_l, y)}{\partial x_j} dy \right\|_{L^p} &= O(\gamma \rho^3), \\ \left\| \frac{\gamma}{3} \int_{E_{\varphi_l}} \frac{\partial G(\xi_k + (r_k^3 + \varphi_k(\theta_k))^{1/3} \theta_k, y)}{\partial y_j} dy \right\|_{L^p} &= O(\gamma \rho^3). \end{aligned}$$

Therefore

$$\frac{\partial \Pi \mathcal{S}(\varphi)}{\partial \xi_{l,j}} = \Pi \tilde{\mathcal{L}}_\xi \left( \frac{\partial \varphi}{\partial \xi_{l,j}} \right) + W, \quad \text{where } \|W\|_{L^p} = O(\gamma \rho^3) = O(1).$$

On the other hand

$$\frac{\partial \Pi \mathcal{S}(\varphi)}{\partial \xi_{l,j}} = 0, \quad \text{since } \Pi \mathcal{S}(\varphi) = 0.$$

By Lemma 7.2 we deduce that

$$\left\| \frac{\partial \varphi}{\partial \xi_{l,j}} \right\|_{W^{2,p}} \leq C \rho^4 O(1) \leq C \rho^4. \quad \square$$

## 8 Solving the reduced problem

We now turn to solve  $\mathcal{S}(\phi) = 0$ .

**Lemma 8.1**  $J(E_\varphi) = J(B) + O(\rho^6)$ . *More explicitly*

$$J(E_\varphi) = \sum_{k=1}^K 4\pi r_k^2 + \frac{\gamma}{2} \left\{ \sum_{k=1}^K \left[ \frac{8\pi r_k^5}{15} + \left( \frac{4\pi}{3} \right)^2 r_k^6 R(\xi_k, \xi_k) \right] + \sum_{k=1}^K \sum_{l=1, l \neq k}^K \left( \frac{4\pi}{3} \right)^2 r_k^3 r_l^3 G(\xi_k, \xi_l) \right\} + O(\rho^5).$$

Here  $J(E_\varphi) = J(E_{\varphi(\cdot, \xi, r)})$  can be considered as a function of  $(\xi, r)$ .

*Proof.* Expanding  $J(E_\varphi)$  yields

$$J(E_\varphi) = J(B) + \sum_k \int_{S^2} \mathcal{S}_k(0) \varphi_k d\theta_k + \frac{1}{2} \sum_k \int_{S^2} \mathcal{L}_k(\varphi) \varphi_k d\theta_k + O(\rho^8). \quad (8.1)$$

The error term  $O(\rho^8)$  in (8.1) is obtained in the same way that (7.18) is derived.

On the other hand  $\Pi \mathcal{S}(\varphi) = 0$  implies that

$$\Pi(\mathcal{S}_k(0) + \mathcal{L}_k(\varphi) + \mathcal{N}_k(\varphi)) = 0$$

where  $\mathcal{N}$  is given in (7.7) and estimated in (7.18). We multiply the last equation by  $\varphi_k$  and integrate to derive

$$\int_{S^2} \mathcal{S}_k(0)\varphi_k d\theta_k + \int_{S^2} \mathcal{L}(\varphi_k)\varphi_k d\theta_k = O(\rho^8).$$

We can now rewrite (8.1) as

$$J(E_\varphi) = J(B) + \frac{1}{2} \sum_k \int_{S^2} \mathcal{S}_k(0)\varphi_k d\theta_k + O(\rho^8).$$

Note that  $\mathcal{S}_k(0)$  is the sum of a number independent of  $\theta_k$  and a quantity of order  $\rho$  by Lemma 3.1. Since  $\varphi_k$  satisfies (7.2), the inner product of the number and  $\varphi_k$  is zero and hence

$$\int_{S^2} \mathcal{S}_k(0)\varphi_k d\theta = O(\rho^6).$$

Therefore

$$J(E_\varphi) = J(B) + O(\rho^6).$$

Lemma 3.2 implies that

$$\begin{aligned} J(E_\varphi) &= \sum_{k=1}^K 4\pi r_k^2 + \frac{\gamma}{2} \left\{ \sum_{k=1}^K \left[ \frac{8\pi r_k^5}{15} + \left(\frac{4\pi}{3}\right)^2 r_k^6 R(\xi_k, \xi_k) \right] + \sum_{k=1}^K \sum_{l=1, l \neq k}^K \left(\frac{4\pi}{3}\right)^2 r_k^3 r_l^3 G(\xi_k, \xi_l) \right. \\ &\quad \left. + \sum_{k=1}^K \sum_{l=1}^K \left(\frac{4\pi}{3}\right)^2 \left( \frac{r_k^3 r_l^5}{10|D|} + \frac{r_k^5 r_l^3}{10|D|} \right) \right\} + O(\rho^6) \\ &= \sum_{k=1}^K 4\pi r_k^2 + \frac{\gamma}{2} \left\{ \sum_{k=1}^K \left[ \frac{8\pi r_k^5}{15} + \left(\frac{4\pi}{3}\right)^2 r_k^6 R(\xi_k, \xi_k) \right] \right. \\ &\quad \left. + \sum_{k=1}^K \sum_{l=1, l \neq k}^K \left(\frac{4\pi}{3}\right)^2 r_k^3 r_l^3 G(\xi_k, \xi_l) \right\} + O(\rho^5). \end{aligned}$$

This proves the lemma.  $\square$

**Lemma 8.2** *When  $\rho$  is sufficiently small,  $J(E_{\varphi(\cdot, \xi, r)})$  is minimized at some  $(\xi, r) = (\zeta, s) \in U$ . As  $\rho \rightarrow 0$ ,  $\frac{s}{\rho} \rightarrow (1, 1, \dots, 1)$ , and  $\zeta \rightarrow \zeta_0$  along a subsequence where  $\zeta_0 \in U_1$  is a global minimum of  $F$ .*

*Proof.* Let us re-scale the problem with

$$R = \frac{r}{\rho}, \quad \tilde{J}(\xi, R) = \frac{2}{\gamma \rho^5} J(E_{\varphi(\cdot, \xi, r)}), \quad (\xi, R) \in U_1 \times \tilde{U}_2$$

where

$$\tilde{U}_2 = \{(R_1, R_2, \dots, R_K) : 1 - \delta_2 < R_k < 1 + \delta_2, \sum_{k=1}^K R_k^3 = K\}$$

is a scaled version of  $U_2$ . Note that by (2.5) and Lemma 8.1,

$$\tilde{J}(\xi, R) = \frac{8\pi}{\gamma \rho^3} \sum_{k=1}^K R_k^2 + \sum_{k=1}^K \frac{8\pi R_k^5}{15} + \rho \left(\frac{4\pi}{3}\right)^2 \left[ \sum_{k=1}^K (R_k^6 R(\xi_k, \xi_k)) + \sum_{k=1}^K \sum_{l \neq k}^K R_k^3 R_l^3 G(\xi_k, \xi_l) \right] + O(\rho^3).$$

Again by (2.5) we may assume that along a subsequence

$$\frac{8\pi}{\gamma\rho^3} \rightarrow b_0 \leq \frac{8\pi}{(3+\epsilon)\pi}, \text{ as } \rho \rightarrow 0. \quad (8.2)$$

Let  $(\zeta, S)$  be the global minimum of  $\tilde{J}$  on the closure of  $U_1 \times \tilde{U}_2$ . Here  $S = \frac{\epsilon}{\rho}$ . Let  $(\zeta, S) \rightarrow (\zeta_0, S_0)$  along a subsequence as  $\rho$  tends to 0. First we claim that  $S_0 = (1, 1, \dots, 1)$ . Suppose that this is false, i.e.  $S_0 \neq (1, 1, \dots, 1)$ . Then as  $\rho$  tends to 0,

$$\begin{aligned} \tilde{J}(\zeta, (1, \dots, 1)) - \tilde{J}(\zeta, S) &= \sum_k \frac{8\pi}{\gamma\rho^3} + \sum_k \frac{8\pi}{15} - \sum_k \frac{8\pi S_k^2}{\gamma\rho^3} - \sum_k \frac{8\pi S_k^5}{15} + O(\rho) \\ &\rightarrow \sum_k b_0 + \sum_k \frac{8\pi}{15} - \sum_k b_0 S_{0,k}^2 - \sum_k \frac{8\pi S_{0,k}^5}{15}. \end{aligned}$$

Because of (8.2) and the constraint  $\sum_k S_{0,k}^3 = K$ , it is easy to show that the last line is negative if  $\delta_2$  in (3.1) is small enough, depending on  $\epsilon$ . For, under (8.2), the function

$$x \rightarrow b_0 x^{2/3} + \frac{8\pi}{15} x^{5/3}$$

is convex when  $x$  is near 1. The last assertion then follows from the Jensen's inequality, when  $x$  takes values  $S_{0,k}^3$ . This is a contradiction to that  $(\zeta, S)$  is a minimum of  $\tilde{J}$ .

Next we claim that  $\zeta_0$  minimizes  $F$  in  $U_1$ . Suppose that this is false. Let  $\eta$  be a minimum of  $F$  in  $U_1$ . Then  $F(\eta) < F(\zeta_0)$ . Consider

$$\begin{aligned} \frac{1}{\rho} \left(\frac{3}{4\pi}\right)^2 (\tilde{J}(\eta, S) - \tilde{J}(\zeta_0, S)) &= \sum_{k=1}^K S_k^6 R(\eta_k, \eta_k) + \sum_{k=1}^K \sum_{l \neq k} S_k^3 S_l^3 G(\eta_k, \eta_l) \\ &\quad - \sum_{k=1}^K S_k^6 R(\zeta_{0,k}, \zeta_{0,k}) - \sum_{k=1}^K \sum_{l \neq k} S_k^3 S_l^3 G(\zeta_{0,k}, \zeta_{0,l}) + O(\rho^2) \\ &\rightarrow F(\eta) - F(\zeta_0) < 0, \text{ as } \rho \rightarrow 0, \end{aligned}$$

another contradiction to that  $(\zeta, S)$  minimizes  $\tilde{J}$ . Note that  $(\zeta, S) \in U_1 \times \tilde{U}_2$  when  $\rho$  is small, since  $(\zeta_0, S_0) \in U_1 \times \tilde{U}_2$ .  $\square$

We show that  $\varphi(\cdot, \zeta, s)$  is an exact solution of (1.1) in the next two lemmas. The first shows that  $A_k = 0$  in (7.1) at  $\xi = \zeta$  and  $r = s$ .

**Lemma 8.3** *At  $\xi = \zeta$  and  $r = s$ ,  $\mathcal{S}_k(\varphi(\cdot, \zeta, s))(\theta_k) = A_{k,1} \cos \theta_{k,1} \sin \theta_{k,2} + A_{k,2} \sin \theta_{k,1} \sin \theta_{k,2} + A_{k,3} \cos \theta_{k,2}$ .*

*Proof.* At each  $(\xi, r) \in U$ , let

$$p_k = r_k^3, \quad q_k = s_k^3. \quad (8.3)$$

Calculations show that

$$\frac{\partial J(E_\varphi)}{\partial p_k} = \int_{S^2} [\mathcal{H}_k(\varphi_k) + \mathcal{A}_k(\varphi_k) + \mathcal{B}_k(\varphi_k) + \sum_{l \neq k} \mathcal{C}_{kl}(\varphi)] \frac{\partial(p_k + \varphi_k)}{\partial p_k} d\theta_k$$

$$\begin{aligned}
&= \int_{S^2} [\mathcal{S}(\varphi) - \lambda(\varphi)] \left(1 + \frac{\partial \varphi_k}{\partial p_k}\right) d\theta_k \\
&= \int_{S^2} \mathcal{S}(\varphi) \left(1 + \frac{\partial \varphi_k}{\partial p_k}\right) d\theta_k - \lambda(\varphi) \int_{S^2} \left(1 + \frac{\partial \varphi_k}{\partial p_k}\right) d\theta_k \\
&= \int_{S^2} (A_{k,1} \cos \theta_{k,1} \sin \theta_{k,2} + A_{k,2} \sin \theta_{k,1} \sin \theta_{k,2} + A_{k,3} \cos \theta_{k,2} + A_k) \left(1 + \frac{\partial \varphi_k}{\partial p_k}\right) d\theta_k \\
&\quad - 4\pi \lambda(\varphi) \\
&= 4\pi A_k - 4\pi \lambda(\varphi).
\end{aligned}$$

Here we have used the facts that

$$\frac{\partial \varphi_k}{\partial p_k} \perp \cos \theta_{k,1} \sin \theta_{k,2}, \sin \theta_{k,1} \sin \theta_{k,2}, \cos \theta_{k,2}, 1$$

which follow from  $\varphi \in \mathcal{X}_*$ .

On the other hand at the minimum  $p = q$  and  $\xi = \zeta$  with respect to  $p$ , we must have

$$\frac{\partial J(E_\varphi)}{\partial p_k} \Big|_{\xi=\zeta, p=q} = \mu$$

for all  $k = 1, 2, \dots, K$ . Here  $\mu$  is a Lagrange multiplier coming from the constraint

$$\sum_{k=1}^K p_k = \frac{3a|D|}{4\pi}.$$

Therefore we deduce that

$$A_k = \frac{\mu}{4\pi} + \lambda$$

which is independent of  $k$ . By (4.20) we derive that  $\sum_{k=1}^K A_k = 0$  and then we conclude that each  $A_k$  must be 0.  $\square$

Next we show that  $A_{k,1}$ ,  $A_{k,2}$  and  $A_{k,3}$  in (7.1) are 0 at  $\xi = \zeta$  and  $r = s$ . The proof uses a tricky re-parametrization technique.

**Lemma 8.4** *At  $\xi = \zeta$  and  $r = s$ ,  $\mathcal{S}(\varphi(\cdot, \zeta, s)) = 0$ .*

*Proof.* To simplify notations in this proof, we do not explicitly indicate the dependence of  $\varphi$  on  $r$ , i.e. we write  $\varphi(\cdot, \xi)$  instead of  $\varphi(\cdot, \xi, r)$ . For each  $\xi_k = (\xi_{k,1}, \xi_{k,2}, \xi_{k,3})$  near  $\zeta_k$  we re-parametrize  $\partial_D E_{\varphi_k(\cdot, \xi)}$ . Let  $\zeta_k$  be the center of a new polar coordinates,  $r_k^3 + \psi_k$  the new radius cube and  $\eta_k$  the new angle. A point on  $\partial_D E_{\varphi_k(\cdot, \xi)}$  is described as  $\zeta_k + (r_k^3 + \psi_k)^{1/3} \eta_k$ . It is related to the old polar coordinates via

$$\zeta_k + (r_k^3 + \psi_k)^{1/3} \eta_k = \xi_k + (r_k^3 + \varphi_k)^{1/3} \theta_k. \quad (8.4)$$

In the new coordinates  $E_{\varphi_k}$  becomes  $E_{\psi_k}$ . It is viewed as a perturbation of the ball centered at  $\zeta_k$  with radius  $r_k$ . The perturbation is described by  $\psi_k$  which is a function of  $\eta_k$  and  $\xi$ .

The main effect of the new coordinates is to “freeze” the center. The center of the new polar system is  $\zeta_k$  which is fixed while the center of the old polar system is  $\xi_k$  which varies in  $D$ .

We now consider the derivative of  $J(E_{\varphi(\cdot, \xi)}) = J(E_{\psi(\cdot, \xi)})$  with respect to  $\xi_k$ . On one hand, at  $\xi = \zeta$  and  $r = s$ ,

$$\frac{\partial J(E_{\psi(\cdot, \xi)})}{\partial \xi_{k,j}} \Big|_{\xi=\zeta} = \frac{\partial J(E_{\varphi(\cdot, \xi)})}{\partial \xi_{k,j}} \Big|_{\xi=\zeta} = 0, \quad j = 1, 2, 3, \quad (8.5)$$

since  $\zeta$  is a minimum.

On the other hand calculations show that

$$\frac{\partial J(E_{\psi(\cdot, \xi)})}{\partial \xi_{k,j}} = \sum_{l=1}^K \int_{S^2} \mathcal{S}_l(\psi(\cdot, \xi))(\eta_l) \frac{\partial \psi_l}{\partial \xi_{k,j}} d\eta_l. \quad (8.6)$$

We emphasize that (8.6) is obtained under the re-parametrized coordinates, in which the dependence of  $J(E_{\psi(\cdot, \xi)})$  on  $\xi$  is only reflected in the dependence of  $\psi$  on  $\xi$ . Had we calculated in the original coordinates,  $\xi$  would have appeared also in the nonlocal part of  $J$  through  $R(\xi_l + \dots, \xi_l + \dots)$  and  $G(\xi_k + \dots, \xi_l + \dots)$ . The result would have been very different from (8.6). See the proof of Lemma 7.3 which involves differentiation with respect to  $\xi$  in the original coordinates. In the derivation of (8.6) we have used the fact that  $\sum_l \int_{S^2} \psi_l d\eta_l = 0$  which implies that  $\sum_l \int_{S^2} \frac{\partial \psi_l}{\partial \xi_{k,j}} d\eta_l = 0$ , so that  $\sum_l \int_{S^2} \lambda(\psi) \frac{\partial \psi_l}{\partial \xi_{k,j}} d\eta_l = 0$  where  $\lambda(\psi)$  is part of

$$\mathcal{S}_l(\psi) = \mathcal{H}_l(\psi) + \mathcal{A}_l(\psi) + \mathcal{B}_l(\psi) + \mathcal{C}_l(\psi) + \lambda(\psi),$$

and we can reach the right side of (8.6).

The expression  $\mathcal{S}(\phi)$  is invariant under re-parametrization, i.e.

$$\mathcal{S}_l(\varphi(\cdot, \xi))(\theta_l) = \mathcal{S}_l(\psi(\cdot, \xi))(\eta_l). \quad (8.7)$$

Now we return to the original coordinate system and integrate with respect to  $\theta_l$  in (8.6). Then

$$\frac{\partial J(E_{\psi(\cdot, \xi)})}{\partial \xi_{k,j}} = \sum_{l=1}^K \int_{S^2} \mathcal{S}_l(\varphi(\cdot, \xi))(\theta_l) \frac{\partial \psi_l(\eta_l(\theta_l, \xi), \xi)}{\partial \xi_{k,j}} \Big|_{\frac{\partial(\eta_{l,1}, \eta_{l,2})}{\partial(\theta_{l,1}, \theta_{l,2})}} \frac{\sin \eta_{l,2}}{\sin \theta_{l,2}} d\theta_l. \quad (8.8)$$

There are two cases:  $l = k$  and  $l \neq k$ . We start with the first case. Recall that  $\psi_k$  and  $\eta_k$  are defined implicitly as functions of  $\theta_k$  and  $\xi$  by (8.4). Let us agree that  $\psi_k = \psi_k(\eta_k, \xi)$  is a function of  $\eta_k$  and  $\xi$ . Set  $\Psi_k(\theta_k, \xi) = \psi_k(\eta_k(\theta_k, \xi), \xi)$ . To simplify notations let us set

$$g = (r_k^3 + \Psi_k)^{1/3}, \quad \tilde{g} = (r_k^3 + \varphi_k)^{1/3}. \quad (8.9)$$

Implicit differentiation shows that, with the help of Lemmas 7.1 and 7.3,

$$\begin{aligned} & \begin{bmatrix} \frac{\partial \eta_{k,1}}{\partial \theta_{k,1}} & \frac{\partial \eta_{k,1}}{\partial \theta_{k,2}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,1}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,2}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,3}} \\ \frac{\partial \eta_{k,2}}{\partial \theta_{k,1}} & \frac{\partial \eta_{k,2}}{\partial \theta_{k,2}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,1}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,2}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,3}} \\ \frac{\partial \Psi_k}{\partial \theta_{k,1}} & \frac{\partial \Psi_k}{\partial \theta_{k,2}} & \frac{\partial \Psi_k}{\partial \xi_{k,1}} & \frac{\partial \Psi_k}{\partial \xi_{k,2}} & \frac{\partial \Psi_k}{\partial \xi_{k,3}} \end{bmatrix} = \\ & - \begin{bmatrix} g \sin \eta_{k,1} \sin \eta_{k,2} & -g \cos \eta_{k,1} \cos \eta_{k,2} & -\frac{\cos \eta_{k,1} \sin \eta_{k,2}}{3g^2} \\ -g \cos \eta_{k,1} \sin \eta_{k,2} & -g \sin \eta_{k,1} \cos \eta_{k,2} & -\frac{\sin \eta_{k,1} \sin \eta_{k,2}}{3g^2} \\ 0 & g \sin \eta_{k,2} & -\frac{\cos \eta_{k,2}}{3g^2} \end{bmatrix}^{-1} \end{aligned}$$

$$\begin{aligned}
& \left[ \begin{array}{ccc} \frac{\cos \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \theta_{k,1}} - \tilde{g} \sin \theta_{k,1} \sin \theta_{k,2} & \frac{\sin \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \theta_{k,1}} + \tilde{g} \cos \theta_{k,1} \sin \theta_{k,2} & \frac{\cos \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \theta_{k,1}} \\ \frac{\cos \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \theta_{k,2}} + \tilde{g} \cos \theta_{k,1} \cos \theta_{k,2} & \frac{\sin \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \theta_{k,2}} + \tilde{g} \sin \theta_{k,1} \cos \theta_{k,2} & \frac{\cos \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \theta_{k,2}} - \tilde{g} \sin \theta_{k,2} \\ 1 + \frac{\cos \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,1}} & \frac{\sin \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,1}} & \frac{\cos \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,1}} \\ \frac{\cos \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,2}} & 1 + \frac{\sin \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,2}} & \frac{\cos \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,2}} \\ \frac{\cos \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,3}} & \frac{\sin \theta_{k,1} \sin \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,3}} & 1 + \frac{\cos \theta_{k,2}}{3\tilde{g}^2} \frac{\partial \varphi_k}{\partial \xi_{k,3}} \end{array} \right]^T \\
= & \frac{-1}{\sin \eta_{k,2}} \left[ \begin{array}{ccc} \frac{\sin \eta_{k,1}}{g} & -\frac{\cos \eta_{k,1}}{g} & 0 \\ -\frac{\cos \eta_{k,1} \cos \eta_{k,2} \sin \eta_{k,2}}{g} & -\frac{\sin \eta_{k,1} \cos \eta_{k,2} \sin \eta_{k,2}}{g} & \frac{\sin^2 \eta_{k,2}}{g} \\ -3g^2 \cos \eta_{k,1} \sin^2 \eta_{k,2} & -3g^2 \sin \eta_{k,1} \sin^2 \eta_{k,2} & -3g^2 \cos \eta_{k,2} \sin \eta_{k,2} \end{array} \right] \sin \theta_{k,2} \\
& \left[ \begin{array}{ccc} -\tilde{g} \sin \theta_{k,1} + O(\rho^3) & \frac{\tilde{g} \cos \theta_{k,1} \cos \theta_{k,2}}{\sin \theta_{k,2}} + O(\rho^3) & \frac{1}{\sin \theta_{k,2}} + O(\rho^2) & O(\rho^2) & O(\rho^2) \\ \tilde{g} \cos \theta_{k,1} + O(\rho^3) & \frac{\tilde{g} \sin \theta_{k,1} \cos \theta_{k,2}}{\sin \theta_{k,2}} + O(\rho^3) & O(\rho^2) & \frac{1}{\sin \theta_{k,2}} + O(\rho^2) & O(\rho^2) \\ \frac{O(\rho^3)}{\sin \theta_{k,2}} & -\tilde{g} + \frac{O(\rho^3)}{\sin \theta_{k,2}} & \frac{O(\rho^2)}{\sin \theta_{k,2}} & \frac{O(\rho^2)}{\sin \theta_{k,2}} & \frac{1}{\sin \theta_{k,2}} + \frac{O(\rho^2)}{\sin \theta_{k,2}} \end{array} \right].
\end{aligned}$$

To fill the large matrix above in this page, we have to awkwardly write its transpose. At  $\xi = \zeta$ ,  $\eta = \theta$ ,  $\Psi = \varphi$  and the above becomes

$$\begin{aligned}
& \left[ \begin{array}{ccccc} \frac{\partial \eta_{k,1}}{\partial \theta_{k,1}} & \frac{\partial \eta_{k,1}}{\partial \theta_{k,2}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,1}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,2}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,3}} \\ \frac{\partial \eta_{k,2}}{\partial \theta_{k,1}} & \frac{\partial \eta_{k,2}}{\partial \theta_{k,2}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,1}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,2}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,3}} \\ \frac{\partial \Psi_k}{\partial \theta_{k,1}} & \frac{\partial \Psi_k}{\partial \theta_{k,2}} & \frac{\partial \Psi_k}{\partial \xi_{k,1}} & \frac{\partial \Psi_k}{\partial \xi_{k,2}} & \frac{\partial \Psi_k}{\partial \xi_{k,3}} \end{array} \right]_{\xi=\zeta} = \\
& \left[ \begin{array}{ccccc} 1 + O(\rho^2) & O(\rho^2) & -\frac{\sin \theta_{k,1}}{\sin \theta_{k,2} g} + O(\rho) & \frac{\cos \theta_{k,1}}{\sin \theta_{k,2} g} + O(\rho) & O(\rho) \\ O(\rho^2) & 1 + O(\rho^2) & \frac{\cos \theta_{k,1} \cos \theta_{k,2}}{g} + O(\rho) & \frac{\sin \theta_{k,1} \cos \theta_{k,2}}{g} + O(\rho) & -\frac{\sin \theta_{k,2}}{g} + O(\rho) \\ O(\rho^5) & O(\rho^5) & 3g^2 \cos \theta_{k,1} \sin \theta_{k,2} + O(\rho^4) & 3g^2 \sin \theta_{k,1} \sin \theta_{k,2} + O(\rho^4) & 3g^2 \cos \theta_{k,2} + O(\rho^4) \end{array} \right]. \tag{8.10}
\end{aligned}$$

We have found that at  $\xi = \zeta$ ,

$$\left( \frac{\partial \Psi_k}{\partial \xi_{k,1}}, \frac{\partial \Psi_k}{\partial \xi_{k,2}}, \frac{\partial \Psi_k}{\partial \xi_{k,3}} \right) \Big|_{\xi=\zeta} = 3r_k^2 \theta_k + O(\rho^4). \tag{8.11}$$

To compute  $\frac{\partial \psi_k}{\partial \xi_{k,j}}$ , we invert  $\eta_k = \eta_k(\xi, \theta_k)$  to express  $\theta_k = \Theta_k(\eta_k, \xi)$ . Then

$$\frac{\partial \psi_k}{\partial \xi_{k,j}} = \frac{\partial \Psi_k}{\partial \xi_{k,j}} + \frac{\partial \Psi_k}{\partial \theta_{k,1}} \frac{\partial \theta_{k,1}}{\partial \xi_{k,j}} + \frac{\partial \Psi_k}{\partial \theta_{k,2}} \frac{\partial \theta_{k,2}}{\partial \xi_{k,j}}.$$

At  $\xi = \zeta$ , since

$$\frac{\partial \Psi_k}{\partial \theta_{k,m}} \Big|_{\xi=\zeta} = O(\rho^5) \quad (8.12)$$

and

$$\begin{bmatrix} \frac{\partial \Theta_{k,1}}{\partial \xi_{k,1}} & \frac{\partial \Theta_{k,1}}{\partial \xi_{k,2}} & \frac{\partial \Theta_{k,1}}{\partial \xi_{k,3}} \\ \frac{\partial \Theta_{k,2}}{\partial \xi_{k,1}} & \frac{\partial \Theta_{k,2}}{\partial \xi_{k,2}} & \frac{\partial \Theta_{k,2}}{\partial \xi_{k,3}} \end{bmatrix} \Big|_{\xi=\zeta} = - \begin{bmatrix} \frac{\partial \eta_{k,1}}{\partial \theta_{k,1}} & \frac{\partial \eta_{k,1}}{\partial \theta_{k,2}} \\ \frac{\partial \eta_{k,2}}{\partial \theta_{k,1}} & \frac{\partial \eta_{k,2}}{\partial \theta_{k,2}} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \eta_{k,1}}{\partial \xi_{k,1}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,2}} & \frac{\partial \eta_{k,1}}{\partial \xi_{k,3}} \\ \frac{\partial \eta_{k,2}}{\partial \xi_{k,1}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,2}} & \frac{\partial \eta_{k,2}}{\partial \xi_{k,3}} \end{bmatrix} = \frac{O(\frac{1}{\rho})}{\sin \theta_{k,2}}, \quad (8.13)$$

we deduce that

$$\left( \frac{\partial \psi_k}{\partial \xi_{k,1}}, \frac{\partial \psi_k}{\partial \xi_{k,2}}, \frac{\partial \psi_k}{\partial \xi_{k,3}} \right) \Big|_{\xi=\zeta} = 3r_k^2 \theta_k + \frac{O(\rho^4)}{\sin \theta_{k,2}} (1, 1, 1). \quad (8.14)$$

The second case  $l \neq k$  is similar, for which we omit the details of our computation. At  $\xi = \zeta$ , we have

$$\left( \frac{\partial \psi_l}{\partial \xi_{k,1}}, \frac{\partial \psi_l}{\partial \xi_{k,2}}, \frac{\partial \psi_l}{\partial \xi_{k,3}} \right) \Big|_{\xi=\zeta} = \frac{O(\rho^4)}{\sin \theta_{l,2}} (1, 1, 1). \quad (8.15)$$

Following (8.14), (8.15) and the fact that  $|\frac{\partial(\eta_{l,1}, \eta_{l,2})}{\partial(\theta_{l,1}, \theta_{l,2})}|_{\xi=\zeta} = 1 + O(\rho^2)$  we find that (8.8) becomes

$$\begin{aligned} \frac{\partial J(E_{\psi(\cdot, \xi)})}{\partial \xi_{k,1}} \Big|_{\xi=\zeta} &= \int_{S^2} \mathcal{S}_k(\varphi) (3r_k^2 \cos \theta_{k,1} \sin \theta_{k,2} + \frac{O(\rho^4)}{\sin \theta_{k,2}}) d\theta_k + \sum_{l \neq k} \int_{S^2} \mathcal{S}_l(\varphi) \frac{O(\rho^4)}{\sin \theta_{l,2}} d\theta_l, \\ \frac{\partial J(E_{\psi(\cdot, \xi)})}{\partial \xi_{k,2}} \Big|_{\xi=\zeta} &= \int_{S^2} \mathcal{S}_k(\varphi) (3r_k^2 \sin \theta_{k,1} \sin \theta_{k,2} + \frac{O(\rho^4)}{\sin \theta_{k,2}}) d\theta_k + \sum_{l \neq k} \int_{S^2} \mathcal{S}_l(\varphi) \frac{O(\rho^4)}{\sin \theta_{l,2}} d\theta_l, \\ \frac{\partial J(E_{\psi(\cdot, \xi)})}{\partial \xi_{k,3}} \Big|_{\xi=\zeta} &= \int_{S^2} \mathcal{S}_k(\varphi) (3r_k^2 \cos \theta_{k,2} + \frac{O(\rho^4)}{\sin \theta_{k,2}}) d\theta_k + \sum_{l \neq k} \int_{S^2} \mathcal{S}_l(\varphi) \frac{O(\rho^4)}{\sin \theta_{l,2}} d\theta_l. \end{aligned}$$

Now we combine (7.1), (8.5) and the above to derive that at  $\xi = \zeta$  and  $r = s$ ,

$$\begin{aligned} &A_{k,1} \int_{S^2} \cos \theta_{k,1} \sin \theta_{k,2} (3r_k^2 \cos \theta_{k,1} \sin \theta_{k,2} + \frac{O(\rho^4)}{\sin \theta_{k,2}}) d\theta_k + A_{k,2} O(\rho^4) + A_{k,3} O(\rho^4) \\ &\quad + \sum_{l \neq k} A_{l,1} O(\rho^4) + \sum_{l \neq k} A_{l,2} O(\rho^4) + \sum_{l \neq k} A_{l,3} O(\rho^4) = 0 \\ &A_{k,1} O(\rho^4) + A_{k,2} \int_{S^2} \sin \theta_{k,1} \sin \theta_{k,2} (3r_k^2 \sin \theta_{k,1} \sin \theta_{k,2} + \frac{O(\rho^4)}{\sin \theta_{k,2}}) d\theta_k + A_{k,3} O(\rho^4) \\ &\quad + \sum_{l \neq k} A_{l,1} O(\rho^4) + \sum_{l \neq k} A_{l,2} O(\rho^4) + \sum_{l \neq k} A_{l,3} O(\rho^4) = 0 \\ &A_{k,1} O(\rho^4) + A_{k,2} O(\rho^4) + A_{k,3} \int_{S^2} \cos \theta_{k,2} (3r_k^2 \cos \theta_{k,2} + \frac{O(\rho^4)}{\sin \theta_{k,2}}) d\theta_k \\ &\quad + \sum_{l \neq k} A_{l,1} O(\rho^4) + \sum_{l \neq k} A_{l,2} O(\rho^4) + \sum_{l \neq k} A_{l,3} O(\rho^4) = 0 \end{aligned}$$

Writing the system in matrix form

$$\left( \begin{array}{ccccccc} 4\pi r_1^2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 4\pi r_1^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 4\pi r_1^2 & 0 & \dots & 0 & 0 \\ \dots & & & & & & \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 4\pi r_K^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 4\pi r_K^2 \end{array} \right) + O(\rho^4) \begin{bmatrix} A_{1,1} \\ A_{1,2} \\ A_{1,3} \\ \dots \\ \dots \\ A_{K,2} \\ A_{K,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ 0 \end{bmatrix} \quad (8.16)$$

we deduce, since (8.16) is non-singular when  $\rho$  is small,  $A_{k,1} = A_{k,2} = A_{k,3} = 0$ .  $\square$

The existence part of Theorem 2.1 follows from Lemma 8.4. The centers  $\zeta_k$  and radii  $s_k$  of the spheres are found in Lemma 8.2. In Lemma 7.1 we see that  $\|\varphi\|_{W^{2,p}} \leq c\rho^5$ , which implies that the radius of a sphere is approximately

$$(s_k^3 + \varphi_k(\theta_k))^{1/3} = s_k + \frac{O(|\varphi_k(\theta_k)|)}{\rho^2} = s_k + O(\rho^3).$$

By Lemma 8.2,  $\zeta$  is close to a minimum of  $F$  and  $s_k$  is close to  $\rho$ . The formula in Lemma 8.1 gives the free energy of our solution.

In Theorem 2.2, a solution is termed stable if it is a local minimizer of  $J$  in the space

$$U \times \{\phi = (\phi_1, \dots, \phi_K) : |\rho^3 + \phi_k| \geq \frac{\rho^3}{2}, \phi_k \in W^{1,2}(S^2), \phi_k \perp 1, \phi_k \perp H_1, k = 1, 2, \dots, K\}. \quad (8.17)$$

The condition  $|\rho^3 + \phi_k| \geq \frac{\rho^3}{2}$  ensures that  $J$  is well defined in this space. Under the condition (2.8) Lemma 7.2, Part 2, shows that each  $\varphi(\cdot, \xi, r)$  we found in Lemma 7.1 locally minimizes  $J$ , with fixed  $(\xi, r) \in U$ , in  $\{\phi : |\rho^3 + \phi_k| \geq \frac{\rho^3}{2}, \phi_k \in W^{1,2}(S^2), \phi_k \perp 1, \phi_k \perp H_1\}$ . On the other hand  $\varphi(\cdot, \zeta, s)$  minimizes  $J(E_{\varphi(\cdot, \xi, r)})$  with respect to  $\xi$  and  $r$ . Hence  $\varphi(\cdot, \zeta, s)$  is a local minimizer of  $J$  in (8.17).

If (2.9) holds, then we can find one eigenvalue  $\lambda_{k,n}$  of  $\mathcal{L}_1$ , Lemma 5.1, for some  $n \in \{2, 3, \dots\}$  such that

$$\lambda_{k,n} < -\frac{C}{\rho^4}, \quad \langle \mathcal{L}_1(e_{k,n}), e_{k,n} \rangle < -\frac{C}{\rho^4} \|e_{k,n}\|_{W^{1,2}}^2$$

where  $e_{k,n}$  is an eigenvector corresponding to  $\lambda_{k,n}$ . By Lemma 5.2, the last inequality implies that

$$\langle \mathcal{L}(e_{k,n}), e_{k,n} \rangle < -\frac{C}{\rho^4} \|e_{k,n}\|_{W^{1,2}}^2.$$

Then by Lemma 6.1, Parts 2, 3 and 4, and (7.25) in the proof of Lemma 7.2

$$\langle \tilde{\mathcal{L}}(e_{k,n}), e_{k,n} \rangle < -\frac{C}{\rho^4} \|e_{k,n}\|_{W^{1,2}}^2.$$

Therefore the solution is unstable.

## 9 Discussion

The functional (1.2) was derived as a  $\Gamma$ -limit from the Ohta-Kawasaki theory of diblock copolymers in [22]. One uses a function  $u$  on  $D$  to describe the density of A-monomers and  $1 - u$  to describe the density of B-monomers. The free energy of a diblock copolymer is

$$I(u) = \int_D \left[ \frac{\varepsilon^2}{2} |Du|^2 + W(u) + \frac{\sigma}{2} |(-\Delta)^{-1/2}(u - a)|^2 \right] dx \quad (9.1)$$

where  $u$  is in

$$\{u \in H^1(D) : \bar{u} = a\}. \quad (9.2)$$

The  $\varepsilon$  in (9.1) is not to be confused with the  $\epsilon$  that has appeared in this paper. The function  $W$  is a balanced double well potential such as  $W(u) = \frac{1}{4}u^2(1 - u)^2$ . There are three positive parameters in (9.1):  $\varepsilon$ ,  $\sigma$ , and  $a$ , where  $\varepsilon$  is small and  $a$  is in  $(0, 1)$ .

If we take  $\sigma$  to be of order  $\varepsilon$ , i.e. by setting

$$\sigma = \varepsilon\gamma \quad (9.3)$$

for some  $\gamma$  independent of  $\varepsilon$ . As  $\varepsilon$  tends to 0, the limiting problem of  $\varepsilon^{-1}I$  turns out to be

$$J(E) = \tau |D\chi_E|(D) + \frac{\gamma}{2} \int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2 dx \quad (9.4)$$

which is the same as the  $J$  in (1.2) except for the additional constant  $\tau$  here. This constant is known as the surface tension and is given by

$$\tau = \int_0^1 \sqrt{2W(q)} dq. \quad (9.5)$$

The functional (9.4) is defined on the same admissible set  $\Sigma$ , (1.3). In this paper we have taken  $\tau = 1$  without the loss of generality.

The theory of  $\Gamma$ -convergence was developed by De Giorgi [7], Modica and Mortola [14], Modica [13], and Kohn and Sternberg [11]. It was proved that  $\varepsilon^{-1}I$   $\Gamma$ -converges to  $J$  in the following sense.

**Proposition 9.1 (Ren and Wei [22])** 1. For every family  $\{u_\varepsilon\}$  of functions in (9.2) satisfying  $\lim_{\varepsilon \rightarrow 0} u_\varepsilon = \chi_E$  in  $L^2(D)$ ,

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^{-1}I(u_\varepsilon) \geq J(E);$$

2. For every  $E$  in  $\Sigma$ , there exists a family  $\{u_\varepsilon\}$  of functions in (9.2) such that  $\lim_{\varepsilon \rightarrow 0} u_\varepsilon = \chi_E$  in  $L^2(D)$ , and

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1}I(u_\varepsilon) \leq J(E).$$

The relationship between  $I$  and  $J$  becomes more clear when a result of Kohn and Sternberg [11] was used to show the following.

**Proposition 9.2 (Ren and Wei [22])** Let  $\delta > 0$  and  $E \in \Sigma$  be such that  $J(E) < J(F)$  for all  $\chi_F \in B_\delta(\chi_E)$  with  $F \neq E$ , where  $B_\delta(\chi_E)$  is the open ball of radius  $\delta$  centered at  $\chi_E$  in  $L^2(D)$ . Then there exists  $\varepsilon_0 > 0$  such that for all  $\varepsilon < \varepsilon_0$  there exists  $u_\varepsilon \in B_{\delta/2}(\chi_E)$  with  $I(u_\varepsilon) \leq I(u)$  for all  $u \in B_{\delta/2}(\chi_E)$ . In addition  $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - \chi_E\|_{L^2(D)} = 0$ .

The existence of a stable solution  $E_{\varphi(\cdot, \zeta)}$  to (1.1) in the sense of Theorem 2.1 does not quite imply the existence of a local minimizer, close to  $\chi_{E_{\varphi(\cdot, \zeta)}}$  in  $L^2(D)$ , of  $I$ . One must show that  $E_{\varphi(\cdot, \zeta)}$  is a strict local minimizer in the sense of Proposition 9.2. This issue requires more study.

## A Appendix

We drop the subscript  $k$  in this appendix. The derivative of  $\mathcal{A}$  at 0 has two terms according to (4.29). The first is

$$\frac{\gamma}{9r_k} \int_{S^2} \frac{u(\omega)}{4\pi|\theta - \omega|} d\omega.$$

The second is

$$-\frac{\gamma u(\theta)}{9r_k} \int_{B_1(0)} \frac{(\theta - y) \cdot \theta}{4\pi|\theta - y|^2} dy$$

for which we calculate the integral. Here  $B_1(0)$  is the unit ball. This integral is independent of  $\theta \in S^2$  so without the loss of generality we assume that  $\theta = (0, 0, 1)$ . Write  $y = (r \cos p, r \sin p, y_3)$  in the cylindrical coordinates. Then the integral becomes

$$\int_{B_1(0)} \frac{(\theta - y) \cdot \theta}{4\pi|\theta - y|^2} dy = \frac{1}{4\pi} \int_{-1}^1 \int_0^{2\pi} \int_0^{\sqrt{1-y_3^2}} \frac{(1-y_3)r dr dp dy_3}{[(1-y_3)^2 + r^2]^{3/2}} = \frac{1}{3}.$$

## B Appendix

The integral operator

$$h(\theta) \rightarrow \int_{S^2} \frac{h(\omega) d\omega}{|\theta - \omega|} \tag{B.1}$$

acts on spherical harmonics  $h \in H_n$  in a simple way. Here  $H_n$  is the space of spherical harmonics of degree  $n$  on  $S^2$ . In general one has

$$\int_{S^2} \Phi(\theta \cdot \omega) h(\omega) d\omega = \alpha_n(\Phi) h(\theta) \tag{B.2}$$

where

$$\alpha_n(\Phi) = 2\pi \int_{-1}^1 \Phi(t) P_n(t) dt. \tag{B.3}$$

See for instance [10, Theorem 3.4.1]. Here  $P_n$  is the  $n$ -th Legendre polynomial. In our case

$$\frac{1}{|\theta - \omega|} = \frac{1}{\sqrt{2 - 2\theta \cdot \omega}},$$

so we take

$$\Phi(t) = \frac{1}{\sqrt{2 - 2t}}. \tag{B.4}$$

The classical representations of Legendre polynomials in terms of generating functions ([10, Formula 3.3.39])

$$\frac{1}{(1 + r^2 - 2rt)^{1/2}} = \sum_{n=0}^{\infty} P_n(t) r^n, \quad r, t \in (-1, 1) \tag{B.5}$$

shows that

$$\int_{-1}^1 \frac{P_n(t) dt}{(1+r^2-2rt)^{1/2}} = r^n \int_{-1}^1 P_n^2(t) dt = \frac{2r^n}{2n+1}$$

where the orthogonality of the Legendre polynomials is used ([10, Formula 3.3.16]):

$$\int_{-1}^1 P_n(t)P_m(t) dt = \frac{2\delta_{nm}}{2n+1}.$$

By sending  $r \rightarrow 1$  we find that

$$\alpha_n(\Phi) = \frac{4\pi}{2n+1}. \quad (\text{B.6})$$

## C Appendix

Here we calculate the improper integral

$$\int_{B_1(0)} \frac{|\theta - y|^2 - 3(1 - \theta \cdot y)^2}{|\theta - y|^5} dy, \quad (\text{C.1})$$

where  $B_1(0)$  is the unit ball centered at 0. This integral is independent of  $\theta \in S^2$ . We take  $\theta = (0, 0, 1)$ . Let  $z = (0, 0, 1) - y$  and set  $z = (r \cos p, r \sin p, z_3)$  in cylindrical coordinates. Then

$$\begin{aligned} & \int_{B_1(0)} \frac{|\theta - y|^2 - 3(1 - \theta \cdot y)^2}{|\theta - y|^5} dy \\ &= \int_{B_1(0,0,1)} \frac{|z|^2 - 3z_3^2}{|z|^5} dz = \int_0^2 \int_0^{\sqrt{1-(1-z_3)^2}} \int_0^{2\pi} \frac{(r^2 + z_3^2) - 3z_3^2}{(r^2 + z_3^2)^{5/2}} r dp dr dz_3 = -\frac{8\pi}{3}. \end{aligned}$$

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