Solitary and Self-similar Solutions of Two-component System of Nonlinear Schrödinger Equations

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2005.12.22

Abstract

Conventionally, to learn wave collapse and optical turbulence, one must study finite-time blow-up solutions of one-component self-focusing nonlinear Schrödinger equations (NLSE). Here we consider simultaneous blow-up solutions of two-component system of self-focusing NLSE. By studying the associated self-similar solutions, we prove two components of solutions blow up at the same time. These self-similar solutions may come from solitary wave solutions with multi-bumps forming abundant geometric patterns which cannot be found in one-component self-focusing NLSE. Our results may provide the first step to investigate optical turbulence in two-component system of NLSE.

Keywords: two-component system, solitary wave, wave collapse, nonlinear optics

1 Introduction

Here we study solutions of two-component system of nonlinear Schrödinger equations given by

$$\begin{cases} i\partial_t\Phi + \triangle\Phi + \mu_1|\Phi|^2\Phi + \beta|\psi|^2\Phi = 0, \\ i\partial_t\psi + \triangle\psi + \mu_2|\psi|^2\psi + \beta|\Phi|^2\psi = 0, x\in\mathbb{R}^n, t>0, \\ \Phi = \Phi(x,t)\,, \psi = \psi(x,t) \in \mathbb{C}\,, \\ \Phi(x,t)\,, \psi(x,t) \to 0 \quad \text{as} \quad |x|\to+\infty, t>0, \end{cases} \tag{1.1}$$

with initial data

$$\Phi|_{t=0} = \Phi_0 \in H_0^s(\mathbb{R}^n;\mathbb{C}), \quad \psi|_{t=0} = \psi_0 \in H_0^s(\mathbb{R}^n;\mathbb{C}),\ s>2\,, \tag{1.2}$$

where $\mu_j > 0$'s are positive constants, $n \le 3$, and $\beta \in \mathbb{R}$ is a coupling constant. The system (1.1) has applications in many physical problems, especially in nonlinear optics. Physically, the solution (Φ,ψ) denotes the two-component beam in Kerr-like photorefractive media(cf. [1]). The positive constant μ_j is for self-focusing in the j-th component of the beam. The coupling constant β is the interaction between two components of the beam. As $\beta > 0$, the interaction is attractive, but the interaction is repulsive if $\beta < 0$. When the spatial dimension is one i.e. $n = 1$, the system (1.1) is integrable, and there are many analytical and numerical results on solitary wave solutions of the general N coupled nonlinear Schrödinger equations(cf. [2], [6], [7], [8]). However, when the spatial dimension is two and three i.e. $n = 2,3$, there are only few results on solitary wave solutions of general N coupled nonlinear Schrödinger equations. One may refer to [11] for high dimensional solitary wave solutions of three coupled nonlinear Schrödinger equations.

From physical experiment(cf. [12]), two dimensional photorefractive screening solitons and a two dimensional self-trapped beam were observed. It is natural to believe that there are two dimensional multi-component solitons and self-trapped beams. To obtain solitary wave solutions of the system (1.1), we may set $\Phi(x,t) = e^{i\,\lambda_1\,t}\,u(x)$ and $\psi(x,t) = e^{i\,\lambda_2\,t}\,v(x)$. Then we may transform the system (1.1) to steady-state two coupled nonlinear Schrödinger equations given by

$$\begin{cases} \triangle u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 \quad \text{in } \mathbb{R}^n, \\ \triangle v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 \quad \text{in } \mathbb{R}^n, \\ u, v > 0 \text{ in } \mathbb{R}^n, \quad u, v(x) \to 0, \quad \text{as } |x| \to +\infty\,, \end{cases} \tag{1.3}$$

where $\lambda_j, \mu_j > 0$ are positive constants, $n \le 3$, and β is a coupling constant. From [10], the existence of ground state (i.e. least energy) solutions of the system (1.3) may depend on the coupling constant β. When β is positive

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but sufficiently small, the system (1.3) has a ground state solution (u,v) which is radially symmetric. On the other hand, as β becomes negative, there is no ground state solution of the system (1.3). Here we show the existence and the configuration of bound state solutions if β is negative and $|\beta|$ is small enough. Moreover, bound state solutions of the problem (1.3) have multi-bumps forming abundant geometric patterns, provided the ratio λ_1/λ_2 is sufficiently small. One may refer to Theorem 1.2 and 1.3 for the details.

Conventionally, solutions of one-component self-focusing nonlinear Schrödinger equations may blow up at finite time (cf. [17]). Such a blow-up behavior may result in wave collapse and optical turbulence (cf. [4], [5] and [15]). Due to the positive sign of μ_j's, the system (1.1) is of two-component self-focusing nonlinear Schrödinger equations having an increasing tendency for the solution to be trapped in regions of highest intensity. Consequently, it is natural to believe that the system (1.1) may have blow-up solutions which may produce wave collapse and optical turbulence. Here we prove simultaneous blow-up on two components of the system (1.1) by studying the associated self-similar solutions. These self-similar solutions have multi-bumps forming abundant geometric patterns which cannot be found in one-component nonlinear Schrödinger equations. Therefore the wave collapse of the system (1.1) is more complex than that of one-component self-focusing nonlinear Schrödinger equations. This may provide the first step to investigate optical turbulence in two-component system of nonlinear Schrödinger equations.

Now we state a theorem which may support the simultaneous blow-up behavior as follows:

Theorem 1.1. *Let $\beta > -\sqrt{\mu_1\mu_2}$ if $n=3$,and β is arbitrary if $n=2$. Assume the initial condition $(\Phi_0,\psi_0)\in H^1(\mathbb{R}^n;\mathbb{C})$ satisfying $\int_{\mathbb{R}^n}|x|^2(|\Phi_0|^2+|\psi_0|^2)\,dx<\infty$ and one of the conditions as follows :*

(i) $H(\Phi_0,\psi_0)<0\,,$

(ii) $H(\Phi_0,\psi_0)=0,$*and* $\int_{\mathbb{R}^n}\sum_{j=1}^n x_j[(i\Phi_0\cdot\partial_{x_j}\Phi_0)+(i\psi_0\cdot\partial_{x_j}\psi_0)]dx<0\,,$

(iii) $H(\Phi_0,\psi_0)>0,$*and* $\int_{\mathbb{R}^n}\sum_{j=1}^n x_j[(i\Phi_0\cdot\partial_{x_j}\Phi_0)+(i\psi_0\cdot\partial_{x_j}\psi_0)]dx\le-\sqrt{H(\Phi_0,\psi_0)}(\int_{\mathbb{R}^n}|x|^2(|\Phi_0|^2+|\psi_0|^2)dx)^{\frac12}\,,$

where H is the Hamiltonian of (1.1) defined by

$$\begin{aligned}H(\Phi,\psi) &= \int_{\mathbb{R}^n}(|\nabla\Phi|^2+|\nabla\psi|^2)dx-\frac12\int_{\mathbb{R}^n}(\mu_1|\Phi|^4+\mu_2|\psi|^4)dx\\ &\quad -\beta\int_{\mathbb{R}^n}|\Phi|^2|\psi|^2dx\,.\end{aligned}\tag{1.4}$$

Besides, the dot "$\cdot$" denotes complex inner product defined by $(a\cdot b)=\frac12(\bar a b+a\bar b)\in\mathbb{R}$ for $a,b\in\mathbb{C}$,where $\bar a$ is the complex conjugate of a. Then there exists a time $t_*<\infty$ such that

$$\lim_{t\uparrow t_*}\|\nabla\Phi\|_{L^2(\mathbb{R}^n)}=\lim_{t\uparrow t_*}\|\nabla\psi\|_{L^2(\mathbb{R}^n)}=\infty\,.\tag{1.5}$$

Furthermore, either $\lim_{t\uparrow t_*}\|\Phi\|_{L^\infty(\mathbb{R}^n)}=\infty$ or $\lim_{t\uparrow t_*}\|\psi\|_{L^\infty(\mathbb{R}^n)}=\infty$, provided $\|\Phi_0\|_{L^2(\mathbb{R}^n)}$ and $\|\psi_0\|_{L^2(\mathbb{R}^n)}$ are strictly positive.

Here we have assumed that neither Φ nor ψ may blow up earlier than the other. From Theorem 1.1, the system (1.1) may have a solution (Φ,ψ) such that both $\|\nabla\Phi\|_{L^2(\mathbb{R}^n)}$ and $\|\nabla\psi\|_{L^2(\mathbb{R}^n)}$ blow up at the same time but we don't know whether $\|\Phi\|_{L^\infty(\mathbb{R}^n)}$ and $\|\psi\|_{L^\infty(\mathbb{R}^n)}$ blow up simultaneously.

To get synchronous blow-up for both $\|\Phi\|_{L^\infty(\mathbb{R}^n)}$ and $\|\psi\|_{L^\infty(\mathbb{R}^n)}$, we study self-similar solutions of the system (1.1) in the critical case $n=2$. We may generalize the idea of Rozanova (cf. [16]) to the system (1.1) by setting

$$\Phi(x,t)=A_1(x,t)\,e^{i\phi_1(x,t)}\,,\quad \psi(x,t)=A_2(x,t)\,e^{i\phi_2(x,t)}\,,\tag{1.6}$$

where

$$A_1(x,t)=u(\xi)\,\exp\left(-\int_0^t a(\tau)d\tau\right)\,,\quad A_2(x,t)=v(\xi)\,\exp\left(-\int_0^t a(\tau)d\tau\right)\,,\tag{1.7}$$

and

$$\phi_j(x,t)=a(t)\frac{|x|^2}{4}+\gamma_j(t)\,,\quad \gamma_j'(t)=\lambda_j\exp\left(-2\int_0^t a(\tau)d\tau\right)\,,\quad j=1,2\,.\tag{1.8}$$

Here u and v are real-valued functions, λ_j's are positive constants, $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ is defined by

$$\xi = x \exp\left(-\int_0^t a(\tau)d\tau\right), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \tag{1.9}$$

and $a(\cdot)$ is defined by an ordinary differential equation given by

$$a'(t) + a^2(t) = 0, \quad \forall\, t > 0, \tag{1.10}$$

with initial data

$$a(0) = a_0 < 0. \tag{1.11}$$

By (1.6)-(1.10), we may transform the system (1.1) into

$$\begin{cases} \triangle u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 & \text{in } \mathbb{R}^n, \\ \triangle v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.12}$$

where $\triangle$ is the Laplacian corresponding to ξ-coordinates denoted as $\triangle = \sum_{j=1}^n \partial^2_{\xi_j}$. Moreover, (1.10) and (1.11) imply

$$a(t) = \frac{a_0}{a_0 t + 1} \to -\infty \quad \text{as } t \uparrow t_* = -1/a_0. \tag{1.13}$$

Hence by (1.6), (1.7) and (1.13), we obtain a simultaneous blow-up solution with the blow-up time $t_* = -1/a_0$. The configuration of such a solution is governed by the system (1.12) which is equivalent to the system (1.3).

To solve the system (1.3) and (1.12), we study the following problem:

$$\begin{cases} \triangle u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 & \text{in } \mathbb{R}^n, \\ \triangle v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^n, \\ u, v > 0 \text{ in } \mathbb{R}^n, \ u, v \in H^1(\mathbb{R}^n), \end{cases} \tag{1.14}$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ and $\beta < 0$. For self-similar solutions of the system (1.1), we are particularly interested in the case of $n = 2$. The energy functional of the problem (1.14) is defined by

$$E[u,v] = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + \frac{\lambda_1}{2}\int_{\mathbb{R}^n} u^2 - \frac{\mu_1}{4}\int_{\mathbb{R}^n} u^4 + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 + \frac{\lambda_2}{2}\int_{\mathbb{R}^n} v^2 - \frac{\mu_2}{4}\int_{\mathbb{R}^n} v^4 - \frac{\beta}{2}\int_{\mathbb{R}^n} u^2 v^2. \tag{1.15}$$

To find a least energy (ground state) solution of the problem (1.14), we consider the following minimization problem:

$$C = \inf_{\substack{(u,v)\in N \\ u,v\geq 0,\, uv\neq 0}} E[u,v], \tag{1.16}$$

where N is the associated Nehari manifold given by

$$N = \left\{ (u,v) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \Big| \int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_1 \int_{\mathbb{R}^n} u^2 = \mu_1 \int_{\mathbb{R}^n} u^4 + \beta \int_{\mathbb{R}^n} u^2 v^2, \right.$$
$$\left. \int_{\mathbb{R}^n} |\nabla v|^2 + \lambda_2 \int_{\mathbb{R}^n} v^2 = \mu_2 \int_{\mathbb{R}^n} v^4 + \beta \int_{\mathbb{R}^n} u^2 v^2 \right\}. \tag{1.17}$$

In [10], we proved

Theorem A. *There exists $\beta_0 \in (0, \sqrt{\mu_1 \mu_2})$ such that for $\beta < 0$, the minimum C of (1.16) is not attained. However, for $\beta \in (0, \beta_0)$, the minimum C of (1.16) is attained.*

A natural question is : are there another bound state solutions of the problem (1.14) for $\beta < 0$? In this paper, we shall show amazing rich structures of bound state solutions for $\beta < 0$. Without loss of generality, we assume that $\lambda_1 \neq \lambda_2$. Note that when $\lambda_1 = \lambda_2$, $\beta_0 \in (-\sqrt{\mu_1 \mu_2}, 0)$, a radially symmetric bound state of the type $\binom{c_1}{c_2} w_{\lambda_1, 1}$ exists. Hereafter, $w_{\lambda,\mu}$ denotes the unique solution of

$$\begin{cases} \triangle w - \lambda w + \mu w^3 = 0, \\ w(0) = \max_{y\in\mathbb{R}^n} w(y), \\ w > 0, \ w \in H^1(\mathbb{R}^n). \end{cases} \tag{1.18}$$

It is obvious that $w_{\lambda,\mu}(y) = \sqrt{\frac{\lambda}{\mu}} w_{1,1}(\sqrt{\lambda} y)$. Hence from now on, we may assume that

$$\lambda_1 < \lambda_2. \tag{1.19}$$

Now we state our main result as follows:

Theorem 1.2. *Let the spatial dimension $n = 2$. Assume that there exists a positive integer $k \geq 2$ such that*

$$\frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}} < \sin \frac{\pi}{k}. \tag{1.20}$$

Then there exists $\beta_0 < 0$ such that for $\beta_0 < \beta < 0$, problem (1.14) has a bound state solution (u_β, v_β) satisfying

(1)

$$\begin{cases} u_\beta(ye^{\frac{2\pi}{k}i}) = u_\beta(y), \ u_\beta(\overline{y}) = u_\beta(y), \\ v_\beta(ye^{\frac{2\pi}{k}i}) = v_\beta(y), \ v_\beta(\overline{y}) = v_\beta(y), \end{cases} \tag{1.21}$$

where $i = \sqrt{-1}$, $y = (y_1, y_2)$ and $\overline{y} = (y_1, -y_2)$.

(2) As $\beta \to 0-$,

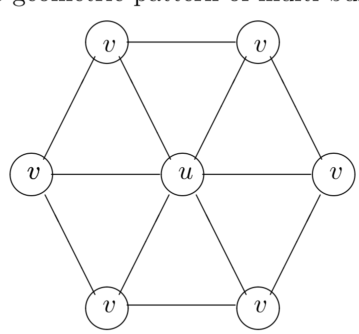
$$u_\beta(y) = w_{\lambda_1,\mu_1}(y) + O(|\beta|^\tau),$$
$$v_\beta(y) = \sum_{j=0}^{k-1} w_{\lambda_2,\mu_2}(y - \xi_j^\beta) + O(|\beta|^\tau),$$

where $\langle \xi_0^\beta, ..., \xi_{k-1}^\beta \rangle$ forms a regular $k-$polygon and

$$|\xi_j^\beta| \sim \log \frac{1}{|\beta|} \cdot \frac{1}{2\left(\sqrt{\lambda_2} \sin(\pi/k) - \sqrt{\lambda_1}\right)} \tag{1.22}$$

and τ is a positive number.

For any positive integer $k \geq 2$, one may find a bound state solution (u_β, v_β) such that v_β has k bumps forming a regular $k-$polygon around the single bump of u_β by reducing the ratio $\sqrt{\lambda_1}/\sqrt{\lambda_2}$ and $|\beta|$. This may provide abundant geometric patterns for multi-bumps of solitary and self-similar solutions of the system (1.1). In particular, as $k = 6$, the geometric pattern of multi-bumps can be sketched below:



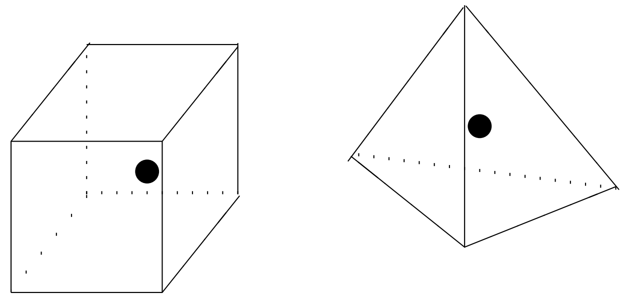
Remark.

(1) Theorem 1.2 can be regarded as results for "bifurcation from $\beta = 0$". If $\lambda_1 = \lambda_2$, computations show that bifurcation point for β may be a finite number.

(2) More complicated patterns, such as concentric polygons, honeycombs, etc can also be constructed by similar arguments with specific symmetry properties.

Now we may use Theorem 1.2 to observe simultaneous blow-up solutions of the system (1.1). As the spatial dimension $n = 2$, we may set $x_j^\beta(t) = \xi_j^\beta \exp\left(\int_0^t a(\tau)\, d\tau\right)$, for $j = 0, 1, \cdots, k-1$, where ξ_j^β's are obtained in Theorem 1.2 and $a(\cdot)$ is defined in (1.13). Then by (1.6)-(1.9), we have $\|\psi\|_{L^\infty(\mathbb{R}^n)} \sim |\psi(x_j^\beta(t), t)| = v(\xi_j^\beta)/(a_0 t + 1) \to \infty$ and $\|\Phi\|_{L^\infty(\mathbb{R}^n)} \sim |\Phi(0, t)| = u(0)/(a_0 t + 1) \to \infty$ as $t \uparrow t_* = -1/a_0$. This may provide simultaneous blow-up solutions of the system (1.1). Here we have used the fact from Theorem 1.2 that $u(0)$ and $v(\xi_j^\beta)$'s are strictly positive numbers, provided $\beta < 0$ and $|\beta|$ is small enough.

Theorem 1.2 can also be extended to $n = 3$. When $n = 3$, the geometric patterns are very important. We only consider two geometric structures: cube and tetrahedra



Theorem 1.3. *Let $n = 3$, and*

$$\sqrt{\frac{\lambda_1}{\lambda_2}} < \begin{cases} \frac{\sqrt{3}}{3} & \text{for the cube,} \\ \frac{\sqrt{3}}{2} & \text{for the tetrehedra.} \end{cases}$$

Then for $\beta < 0$ and $|\beta|$ small, problem (1.14) has a solution (u_β, v_β) such that

$$\begin{aligned} u_\beta &\approx w_{\lambda_1,\mu_1}(y) + O(|\beta|^\tau), \\ v_\beta &\approx \sum_{j=1}^{k} w_{\lambda_2,\mu_2}(y - \xi_j) + O(|\beta|^\tau), \end{aligned} \tag{1.23}$$

where $\langle \xi_1, ..., \xi_k \rangle$ forms regular cube or tetrahedra.

Remark. *It is natural to believe that solutions with multi-bumps forming geometric patterns like octahedron, dodecahedron, and icosahedron (i.e. the other three regular polyhedra) can also be constructed by similar methods.*

The rest of this paper is organized as follows: In Section 2, we provide the proof of Theorem 1.1. Theorem 1.2 is proved in Sections 3-5, and Theorem 1.3 is proved in Section 6.

Acknowledgements: The research of the first author is partially supported by a research Grant (No. NSC 94-2115-M-002-019) from NSC of Taiwan. The research of the second author is partially supported by an Earmarked Grant from RGC of HK.

2 Proof of Theorem 1.1

Here we may generalize ideas for single scalar nonlinear Schrödinger equations (cf. [17]) to the system (1.1). To prove Theorem 1.1, we need the following lemma:

Lemma 2.1. *Let $V(t) = \int_{\mathbb{R}^n} |x|^2(|\Phi|^2 + |\psi|^2)dx, \forall t \geq 0$, where$(\Phi, \psi)$ is the regular solution of (1.1). Then $\forall t > 0$,*

$$\frac{d^2}{dt^2}V(t) = 8H + 8\beta(1 - \frac{n}{2}) \int_{\mathbb{R}^n} |\Phi|^2|\psi|^2 dx - 2(n-2) \int_{\mathbb{R}^n} (\mu_1|\Phi|^4 + \mu_2|\psi|^4)dx, \tag{2.1}$$

where H is the Hamiltonian of (1.1) defined by (1.4).

Proof. By direct calculation and (1.1),it is easy to check that

$$-\frac{d}{dt}V(t) = -4 \int_{\mathbb{R}^n} \sum_{j=1}^{n} x_j[(i\Phi \cdot \partial_{x_j}\Phi) + (i\psi \cdot \partial_{x_j}\psi)]dx\,. \tag{2.2}$$

Here we have used integration by parts. Besides,the dot "$\cdot$" denotes complex inner product defined by $(a \cdot b) = \frac{1}{2}(\bar{a}b + a\bar{b})$ for $a, b \in \mathbb{C}$,where $\bar{a}$ is the complex conjugate of a.
Moreover,by (2.2),

$$\begin{aligned} -\tfrac{d^2}{dt^2}V(t) = \quad & -4 \int_{\mathbb{R}^n} \sum_{j=1}^{d} x_j[(i\partial_t\Phi \cdot \partial_{x_j}\Phi) + (i\Phi \cdot \partial_{x_j}\partial_t\Phi)]dx \\ & -4 \int_{\mathbb{R}^n} \sum_{j=1}^{n} x_j[(i\partial_t\psi \cdot \partial_{x_j}\psi) + (i\psi \cdot \partial_{x_j}\partial_t\psi)]dx\,. \end{aligned} \tag{2.3}$$

Hence by (1.1), (2.3) and integration by parts,

$$
\begin{aligned}
-\frac{d^2}{dt^2}V(t) &= -4\int_{\mathbb{R}^n}\sum_{j=1}^n x_j\left[\Phi\cdot\partial_{x_j}(\triangle\Phi+\mu_1|\Phi|^2\Phi+\beta|\psi|^2\Phi)\right]dx\\
&+4\int_{\mathbb{R}^n}\sum_{j=1}^n x_j\left[(\triangle\Phi\cdot\partial_{x_j}\Phi)+\mu_1(\Phi\cdot\partial_{x_j}\Phi)|\Phi|^2+\beta(\Phi\cdot\partial_{x_j}\Phi)|\psi|^2\right]dx\\
&+4\int_{\mathbb{R}^n}\sum_{j=1}^n x_j\left[(\triangle\psi\cdot\partial_{x_j}\psi)+\mu_2(\psi\cdot\partial_{x_j}\psi)|\psi|^2+\beta(\psi\cdot\partial_{x_j}\psi)|\Phi|^2\right]dx\\
&-4\int_{\mathbb{R}^n}\sum_{j=1}^n x_j[\psi\cdot\partial_{x_j}(\triangle\psi+\mu_2|\psi|^2\psi+\beta|\Phi|^2\psi)]dx\\
&=4\int_{\mathbb{R}^n}\sum_{j=1}^n x_j\left[(\triangle\Phi\cdot\partial_{x_j}\Phi)+\mu_1(\Phi\cdot\partial_{x_j}\Phi)|\Phi|^2+\beta(\Phi\cdot\partial_{x_j}\Phi)|\psi|^2\right]dx\\
&+4\int_{\mathbb{R}^n}\sum_{j=1}^n \partial_{x_j}(x_j\Phi)\cdot(\triangle\Phi+\mu_1|\Phi|^2\Phi+\beta|\psi|^2\Phi)dx\\
&+4\int_{\mathbb{R}^n}\sum_{j=1}^n x_j\left[(\triangle\psi\cdot\partial_{x_j}\psi)+\mu_2(\psi\cdot\partial_{x_j}\psi)|\psi|^2+\beta(\psi\cdot\partial_{x_j}\psi)|\Phi|^2\right]dx\\
&+4\int_{\mathbb{R}^n}\sum_{j=1}^n \partial_{x_j}(x_j\psi)\cdot(\triangle\psi+\mu_2|\psi|^2\psi+\beta|\Phi|^2\psi)dx\\
&=8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j\left[(\triangle\Phi\cdot\partial_{x_j}\Phi)+\mu_1(\Phi\cdot\partial_{x_j}\Phi)|\Phi|^2+\beta(\Phi\cdot\partial_{x_j}\Phi)|\psi|^2\right]dx\\
&+4n\int_{\mathbb{R}^n}\Phi\cdot(\triangle\Phi+\mu_1|\Phi|^2\Phi+\beta|\psi|^2\Phi)dx\\
&+8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j\left[(\triangle\psi\cdot\partial_{x_j}\psi)+\mu_2(\psi\cdot\partial_{x_j}\psi)|\psi|^2+\beta(\psi\cdot\partial_{x_j}\psi)|\Phi|^2\right]dx\\
&+4n\int_{\mathbb{R}^n}\psi\cdot(\triangle\psi+\mu_2|\psi|^2\psi+\beta|\Phi|^2\psi)dx
\end{aligned}
$$

i.e.

$$
\begin{aligned}
-\tfrac{d^2}{dt^2}V(t)= \quad & 8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j[(\triangle\Phi\cdot\partial_{x_j}\Phi)+\mu_1(\Phi\cdot\partial_{x_j}\Phi)|\Phi|^2\\
&+\beta(\Phi\cdot\partial_{x_j}\Phi)|\psi|^2]dx\\
&+4n\int_{\mathbb{R}^n}\Phi\cdot(\triangle\Phi+\mu_1|\Phi|^2\Phi+\beta|\psi|^2\Phi)dx\\
&+8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j[(\triangle\psi\cdot\partial_{x_j}\psi)+\mu_2(\psi\cdot\partial_{x_j}\psi)|\psi|^2\\
&+\beta(\psi\cdot\partial_{x_j}\psi)|\Phi|^2]dx\\
&+4n\int_{\mathbb{R}^n}\psi\cdot(\triangle\psi+\mu_2|\psi|^2\psi+\beta|\Phi|^2\psi)dx
\end{aligned}
\tag{2.4}
$$

We may rewrite the first integral of (2.4) as

$$
8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j(\triangle\Phi\cdot\partial_{x_j}\Phi)dx+8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j(\frac{\mu_1}{4}\partial_{x_j}|\Phi|^4+\frac{\beta}{2}|\psi|^2\partial_{x_j}|\Phi|^2)dx \tag{2.5}
$$

Similarly, the third integral of (2.4) can be written as

$$
8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j(\triangle\psi\cdot\partial_{x_j}\psi)dx+8\int_{\mathbb{R}^n}\sum_{j=1}^n x_j(\frac{\mu_2}{4}\partial_{x_j}|\psi|^4+\frac{\beta}{2}|\Phi|^2\partial_{x_j}|\psi|^2)dx \tag{2.6}
$$

Combining the second integral of (2.5) and (2.6),we obtain

$$8\int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j(\frac{\mu_1}{4}\partial_{x_j}|\Phi|^4+\frac{\mu_2}{4}\partial_{x_j}|\psi|^4)dx+4\beta\int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j\partial_{x_j}(|\Phi|^2|\psi|^2)dx \tag{2.7}$$

Using integration by parts, (2.7) becomes

$$-2\mu_1 n\int_{\mathbb{R}^n}|\Phi|^4dx-2\mu_2 n\int_{\mathbb{R}^n}|\psi|^4dx-4\beta n\int_{\mathbb{R}^n}|\Phi|^2|\psi|^2dx\,. \tag{2.8}$$

For the first integral of (2.5) and (2.6), we use integration by parts as follow:

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j(\triangle\Phi\cdot\partial_{x_j}\Phi)dx &= \int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j(\sum_{k=1}^{n}\partial^2_{x_k}\Phi\cdot\partial_{x_j}\Phi)dx\\ &= -\int_{\mathbb{R}^n}\sum_{j=1}^{n}\sum_{k=1}^{n}\partial_{x_k}(x_j\partial_{x_j}\Phi)\cdot\partial_{x_k}\Phi dx\end{aligned}$$

i.e.

$$\int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j(\triangle\Phi\cdot\partial_{x_j}\Phi)dx=-\int_{\mathbb{R}^n}\sum_{j,k=1}^{n}\partial_{x_k}(x_j\partial_{x_j}\Phi)\cdot\partial_{x_k}\Phi dx\,. \tag{2.9}$$

Similarly,

$$\int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j(\triangle\psi\cdot\partial_{x_j}\psi)dx=-\int_{\mathbb{R}^n}\sum_{j,k=1}^{n}\partial_{x_k}(x_j\partial_{x_j}\psi)\cdot\partial_{x_k}\psi dx\,. \tag{2.10}$$

For the second integral of right side of (2.9) and (2.10),

$$\begin{aligned}-\int_{\mathbb{R}^n}\sum_{j,k=1}^{n}\partial_{x_k}(x_j\partial_{x_j}\Phi)\cdot\partial_{x_k}\Phi dx &= -\int_{\mathbb{R}^n}\sum_{j,k=1}^{n}(\delta_{jk}\partial_{x_j}\Phi+x_j\partial_{x_j}\partial_{x_k}\Phi)\cdot\partial_{x_k}\Phi dx\\ &= -\int_{\mathbb{R}^n}|\nabla\Phi|^2dx-\int_{\mathbb{R}^n}\frac{1}{2}\sum_{j,k=1}^{n}x_j\partial_{x_j}|\partial_{x_k}\Phi|^2dx\\ &= -\int_{\mathbb{R}^n}|\nabla\Phi|^2dx-\int_{\mathbb{R}^n}\frac{1}{2}\sum_{j=1}^{n}x_j\partial_{x_j}|\nabla\Phi|^2dx\\ &= (\frac{n}{2}-1)\int_{\mathbb{R}^n}|\nabla\Phi|^2dx\end{aligned}$$

i.e.

$$-\int_{\mathbb{R}^n}\sum_{j,k=1}^{n}\partial_{x_k}(x_j\partial_{x_j}\Phi)\cdot\partial_{x_k}\Phi dx=(\frac{n}{2}-1)\int_{\mathbb{R}^n}|\nabla\Phi|^2dx\,. \tag{2.11}$$

Similarly

$$-\int_{\mathbb{R}^n}\sum_{j,k=1}^{n}\partial_{x_k}(x_j\partial_{x_j}\psi)\cdot\partial_{x_k}\psi dx=(\frac{n}{2}-1)\int_{\mathbb{R}^n}|\nabla\psi|^2dx\,. \tag{2.12}$$

Here we have used integration by parts.

Now we put (2.11) and (2.12) into (2.9) and (2.10). Then

$$\int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j(\triangle\Phi\cdot\partial_{x_j}\Phi)dx=(\frac{n}{2}-1)\int_{\mathbb{R}^n}|\nabla\Phi|^2dx\,, \tag{2.13}$$

and

$$\int_{\mathbb{R}^n}\sum_{j=1}^{n}x_j(\triangle\psi\cdot\partial_{x_j}\psi)dx=(\frac{n}{2}-1)\int_{\mathbb{R}^n}|\nabla\psi|^2dx\,. \tag{2.14}$$

Moreover,we may put (2.7), (2.8), (2.13) and (2.14) into (2.4) and obtain

$$\begin{aligned}-\tfrac{d^2}{dt^2}V(t)= \quad &(4n-8)\int_{\mathbb{R}^n}(|\nabla\Phi|^2+|\nabla\psi|^2)dx-2\mu_1 n\int_{\mathbb{R}^n}|\Phi|^4dx\\ &-2\mu_2 n\int_{\mathbb{R}^n}|\psi|^4dx-4\beta n\int_{\mathbb{R}^n}|\Phi|^2|\psi|^2dx\\ &+4n\int_{\mathbb{R}^n}\Phi\cdot(\triangle\Phi+\mu_1|\Phi|^2\Phi+\beta|\psi|^2\Phi)dx\\ &+4n\int_{\mathbb{R}^n}\psi\cdot(\triangle\psi+\mu_2|\psi|^2\psi+\beta|\Phi|^2\psi)dx\end{aligned} \tag{2.15}$$

Using integration by parts,we have

$$4n\int_{\mathbb{R}^n}\Phi\cdot(\triangle\Phi+\mu_1|\Phi|^2\Phi+\beta|\psi|^2\Phi)dx=-4n\int_{\mathbb{R}^n}\left(|\nabla\Phi|^2-\mu_1|\Phi|^4-\beta|\Phi|^2|\psi|^2\right)dx\,. \tag{2.16}$$

and

$$4n\int_{\mathbb{R}^n}\psi\cdot(\triangle\psi+\mu_2|\psi|^2\psi+\beta|\Phi|^2\psi)dx=-4n\int_{\mathbb{R}^n}\left(|\nabla\psi|^2dx-\mu_2|\psi|^4-\beta|\Phi|^2|\psi|^2\right)dx\,. \tag{2.17}$$

Combining (2.15)-(2.17), one may get

$$\begin{aligned}-\tfrac{d^2}{dt^2}V(t)\quad &=-8\int_{\mathbb{R}^n}(|\nabla\Phi|^2+|\nabla\psi|^2)dx+2n\int_{\mathbb{R}^n}(\mu_1|\Phi|^4+\mu_2|\psi|^4)dx+4\beta n\int_{\mathbb{R}^n}|\Phi|^2|\psi|^2dx\\ &=-8H-8\beta(1-\tfrac{n}{2})\int_{\mathbb{R}^n}|\Phi|^2|\psi|^2dx+2(n-2)\int_{\mathbb{R}^n}(\mu_1|\Phi|^4+\mu_2|\psi|^4)dx\,,\end{aligned} \tag{2.18}$$

where H is the Hamiltonian of (1.1) defined by

$$H=\int_{\mathbb{R}^n}(|\nabla\Phi|^2+|\nabla\psi|^2)dx-\frac{1}{2}\int_{\mathbb{R}^n}(\mu_1|\Phi|^4+\mu_2|\psi|^4)dx-\beta\int_{\mathbb{R}^n}|\Phi|^2|\psi|^2dx\,. \tag{2.19}$$

Therefore by (2.18) and (2.19), we may complete the proof of Lemma 2.1. □

Now we want to prove Theorem 1.1 as follows:
Firstly, we claim that the Hamiltonian H is independent of time t i.e.

$$H=H(\Phi,\psi)=H(\Phi_0,\psi_0),\quad \forall t>0\,. \tag{2.20}$$

One may multiply the equation of Φ in (1.1) by $\partial_t\bar{\Phi}$ and integrate the resulting equation over $\mathbb{R}^n$, where $\bar{\Phi}$ is the complex conjugate of Φ. Then using integration by parts, we obtain

$$i\int_{\mathbb{R}^n}|\partial_t\Phi|^2dx-\int_{\mathbb{R}^n}\sum_{j=1}^{n}\partial_{x_j}\Phi\partial_{x_j}\partial_t\bar{\Phi}dx+\int_{\mathbb{R}^n}(\mu_1|\Phi|^2\Phi+\beta|\psi|^2\Phi)\partial_t\bar{\Phi}dx=0 \tag{2.21}$$

Take complex conjugate on (2.21) and we have

$$-i\int_{\mathbb{R}^n}|\partial_t\Phi|^2dx-\int_{\mathbb{R}^n}\sum_{j=1}^{d}\partial_{x_j}\bar{\Phi}\partial_{x_j}\partial_t\Phi dx+\int_{\mathbb{R}^n}(\mu_1|\Phi|^2\bar{\Phi}+\beta|\psi|^2\bar{\Phi})\partial_t\Phi dx=0 \tag{2.22}$$

Adding (2.21) and (2.22) together may give

$$\frac{d}{dt}\int_{\mathbb{R}^n}(|\nabla\Phi|^2-\frac{\mu_1}{2}|\Phi|^4)dx-\beta\int_{\mathbb{R}^n}|\psi|^2\partial_t|\Phi|^2dx=0 \tag{2.23}$$

As for (2.23), we may use the equation of ψ in (1.1) to derive

$$\frac{d}{dt}\int_{\mathbb{R}^n}(|\nabla\psi|^2-\frac{\mu_2}{2}|\psi|^4)dx-\beta\int_{\mathbb{R}^n}|\Phi|^2\partial_t|\psi|^2dx=0 \tag{2.24}$$

Hence by adding (2.23) and (2.24), we obtain

$$\frac{d}{dt}H(\Phi,\psi)=0,\quad \forall t>0\,,$$

where $H(\Phi,\psi)$ is defined in (1.4). This may imply (2.20).

Secondly, we use Lemma 2.1 to prove Theorem 1.1. Suppose $n=2$. Then (2.1) implies

$$\frac{d^2}{dt^2}V(t) \le 8H \quad \text{for } t>0\,. \tag{2.25}$$

On the other hand, if n=3, then (2.1) becomes

$$\frac{d^2}{dt^2}V(t) \le 8H - 4\beta\int_{\mathbb{R}^n} |\Phi|^2|\psi|^2dx - 2\int_{\mathbb{R}^n}(\mu_1|\Phi|^4+\mu_2|\psi|^4)dx\,. \tag{2.26}$$

Hence by (2.26), (2.25) still holds if $n=3$ and $\beta > -\sqrt{\mu_1\mu_2}$. By (2.20) and (2.25),

$$\frac{d^2}{dt^2}V(t) \le 8H(\Phi_0,\psi_0) \quad \text{for } \ t>0\,. \tag{2.27}$$

Consequently, by (2.2) and (2.27),

$$V(t) \le 4H(\Phi_0,\psi_0)t^2 + V'(0)t + V(0) \quad \text{for } \ t>0\,, \tag{2.28}$$

where

$$V'(0) = 4\int_{\mathbb{R}^n}\sum_{j=1}^n x_j[(i\Phi_0\cdot\partial_{x_j}\Phi_0)+(i\psi_0\cdot\partial_{x_j}\psi_0)]dx, \tag{2.29}$$

and

$$V(0) = \int_{\mathbb{R}^n}|x|^2(|\Phi_0|^2+|\psi_0|^2)dx\,. \tag{2.30}$$

Under any of the hypotheses (i)–(iii) in Theorem 1.1, there exists a time t_0 such that the right-hand side of (2.28) vanishes,and thus also $t_1 \le t_0$ such that

$$\lim_{t\uparrow t_1} V(t) = 0\,. \tag{2.31}$$

Furthermore, from the equality,

$$\begin{aligned}\int_{\mathbb{R}^n}|f|^2dx &= \frac{1}{n}\int_{\mathbb{R}^n}(\nabla\cdot x)|f|^2dx \\ &= -\frac{1}{n}\int_{\mathbb{R}^n} x\cdot\nabla(|f|^2)dx, \quad \forall f\in H^1(\mathbb{R}^n)\,,\end{aligned}$$

one may get the following inequality

$$\|f\|^2_{L^2(\mathbb{R}^n)} \le \frac{2}{n}\|\nabla f\|_{L^2(\mathbb{R}^n)}\|xf\|_{L^2(\mathbb{R}^n)}, \quad \forall f\in H^1(\mathbb{R}^n)\,. \tag{2.32}$$

Consequently, by (2.32),

$$\|\Phi\|^2_{L^2(\mathbb{R}^n)} \le \frac{2}{n}\|\nabla\Phi\|_{L^2(\mathbb{R}^n)}\|x\Phi\|_{L^2(\mathbb{R}^n)} \le \frac{2}{n}\|\nabla\Phi\|_{L^2(\mathbb{R}^n)}\sqrt{V(t)}\,, \tag{2.33}$$

and

$$\|\psi\|^2_{L^2(\mathbb{R}^n)} \le \frac{2}{n}\|\nabla\psi\|_{L^2(\mathbb{R}^n)}\|x\psi\|_{L^2(\mathbb{R}^n)} \le \frac{2}{n}\|\nabla\psi\|_{L^2(\mathbb{R}^n)}\sqrt{V(t)}\,, \tag{2.34}$$

On the other hand,by (1.1), it is easy to check that

$$\frac{d}{dt}\|\Phi\|^2_{L^2(\mathbb{R}^n)} = \frac{d}{dt}\|\psi\|^2_{L^2(\mathbb{R}^n)} = 0, \quad \forall t>0\,.$$

Thus

$$\|\Phi\|^2_{L^2(\mathbb{R}^n)} = \|\Phi_0\|^2_{L^2(\mathbb{R}^n)}\,, \quad \|\psi\|^2_{L^2(\mathbb{R}^n)} = \|\psi_0\|^2_{L^2(\mathbb{R}^n)}\,. \tag{2.35}$$

Hence by (2.31), (2.33), (2.34) and (2.35), there exists a time $t_* \le t_1$ such that (1.5) holds.

Finally, we want to prove either $\lim\limits_{t\uparrow t_*}\int_{\mathbb{R}^n}|\Phi|^4dx = \infty$ or $\lim\limits_{t\uparrow t_*}\int_{\mathbb{R}^n}|\psi|^4dx = \infty$. Suppose neither one of them holds i.e. both $\lim\limits_{t\uparrow t_*}\int_{\mathbb{R}^n}|\Phi|^4dx < \infty$ and $\lim\limits_{t\uparrow t_*}\int_{\mathbb{R}^n}|\psi|^4dx < \infty$. Then by Hölder inequality, we obtain

$$\lim_{t\uparrow t_*}\left|\frac{1}{2}\int_{\mathbb{R}^n}(\mu_1|\Phi|^4+\mu_2|\psi|^4)dx + \beta\int_{\mathbb{R}^n}|\Phi|^2|\psi|^2dx\right| < \infty\,. \tag{2.36}$$

Thus by (1.4), (1.5) and (2.36), we have

$$\lim_{t\uparrow t_*} H(\Phi,\psi) = \infty\,. \tag{2.37}$$

However, (2.37) may contradict with (2.20) so either $\lim\limits_{t\uparrow t_*}\int_{\mathbb{R}^n}|\Phi|^4dx = \infty$ or $\lim\limits_{t\uparrow t_*}\int_{\mathbb{R}^n}|\psi|^4dx = \infty$. Therefore we may complete the proof of Theorem 1.1.

3 Symmetry and Approximate solutions

In this section, we introduce function spaces with specific symmetry properties for the proof of Theorem 1.2. These spaces are defined by

$$\begin{cases} X = \{u \in H^2(\mathbb{C}) | u(y) = u(ye^{i\frac{2\pi}{k}}); u(\bar{y}) = u(y)\}, \\ Y = \{u \in L^2(\mathbb{C}) | u(y) = u(ye^{i\frac{2\pi}{k}}); u(\bar{y}) = u(y)\}, \end{cases} \tag{3.1}$$

where $y = (y_1, y_2)$ and $\bar{y} = (y_1, -y_2)$. In X(or Y), we use the standard H^2-norm (or L^2-norm). Note that equation (1.14) is invariant under the maps

$$(u, v) \to \left(u\left(ye^{i\frac{2\pi}{k}}\right), v\left(ye^{i\frac{2\pi}{k}}\right) \right),$$

and

$$(u, v) \to \left(u(y_1, -y_2), v(y_1, -y_2) \right),$$

where $i = \sqrt{-1} \in \mathbb{C}$.

Let $\xi_0 = (l, 0)$, $\xi_j = \xi_0 e^{i\frac{2j\pi}{k}}$, $j = 1, 2, ..., k-1$, where $c_1 \log \frac{1}{|\beta|} \le l \le c_2 \log \frac{1}{|\beta|}$ and c_1, c_2 are positive constants to be determined. To approximate the solution of (1.14) with specific symmetry properties, we may define a vector-valued function by

$$\begin{pmatrix} u_l \\ v_l \end{pmatrix} = \begin{pmatrix} w_{\lambda_1,\mu_1}(y) \\ \sum\limits_{j=0}^{k-1} w_{\lambda_2,\mu_2}(y - \xi_j) \end{pmatrix}, \tag{3.2}$$

called the approximate solution of (1.14). Note that $u_l \in X, v_l \in X$. For notation convenience, we set $w_j(y) = w_{\lambda_j,\mu_j}(y)$ and $w_{2,\xi_j}(y) = w_{\lambda_2,\mu_2}(y - \xi_j)$ for $j = 0, 1, ..., k-1$. Let

$$S \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \triangle u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 \\ \triangle v - \lambda_2 v + \mu_1 v^3 + \beta u^2 v \end{pmatrix}.$$

We first state the following lemma on the properties of $w_j(y)$:

Lemma 3.1. *(1) As $|y| \to +\infty$, we have*

$$w_j(|y|) = A_j(1 + O(\frac{1}{|y|}))|y|^{-\frac{n-1}{2}} e^{-\sqrt{\lambda_j}|y|}, w_j'(|y|) = -A_j(1 + O(\frac{1}{|y|}))|y|^{-\frac{n-1}{2}} e^{-\sqrt{\lambda_j}|y|}, \tag{3.3}$$

where $A_j > 0$ is a positive generic constant, j=1,2.

(2) w_j is nondegenerate, i.e.

$$Kernel(\Delta - \lambda_j + 3\mu_j w_j^2) \cap H^2(\mathbb{R}^n) = span \left\{ \frac{\partial w_j}{\partial y_1}, ..., \frac{\partial w_j}{\partial y_n} \right\}. \tag{3.4}$$

(3) If $0 < \sqrt{\lambda_1}\alpha_1 < \sqrt{\lambda_2}\alpha_2$, then we have for $|x_1 - x_2| >> 1$,

$$\int_{\mathbb{R}^n} w_{\lambda_1,\mu_1}^{\alpha_1}(y - x_1) w_{\lambda_2,\mu_2}^{\alpha_2}(y - x_2) \approx w_1^{\alpha_1}(|x_1 - x_2|) \int_{\mathbb{R}^n} w_2^{\alpha_2}(y) e^{-\sqrt{\lambda_1}\alpha_1 y_1} dy, \tag{3.5}$$

$$w_{\lambda_1,\mu_1}^{\alpha_1}(y - x_1) w_{\lambda_2,\mu_2}^{\alpha_2}(y - x_2) \le w_1^{\alpha_1}(|x_1 - x_2|) e^{(\sqrt{\lambda_1}\alpha_1 - \sqrt{\lambda_2}\alpha_2)|y - x_2|}. \tag{3.6}$$

Proof. (1) is well-known. (2) follows from the uniqueness of w_j (cf.[14]).

By (3.3), we have for $|x_1 - x_2| >> 1, z = y - x_2$,

$$w_{\lambda_1,\mu_1}^{\alpha_1}(y - x_1) = w_{\lambda_1,\mu_1}^{\alpha_1}(z + x_2 - x_1) = (A_1 + o(1))^{\alpha_1} (|z + x_2 - x_1|)^{-\frac{n-1}{2}} e^{-\sqrt{\lambda_1}\alpha_1 |z + x_2 - x_1|}$$

$$= w_{\lambda_1,\mu_1}^{\alpha_1}(|x_2 - x_1|) e^{-\sqrt{\lambda_1}\alpha_1 < z, \frac{x_1 - x_2}{|x_1 - x_2|} > + o(|z|)}.$$

Using (3.3) and Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_{\mathbb{R}^n} w_{\lambda_1,\mu_1}^{\alpha_1}(y - x_1) w_{\lambda_2,\mu_2}^{\alpha_2}(y - x_2) \approx w_1^{\alpha_1}(|x_1 - x_2|) \int_{\mathbb{R}^n} w_2^{\alpha_2}(y) e^{-\sqrt{\lambda_1}\alpha_1 < y, \frac{x_1 - x_2}{|x_1 - x_2|} >} dy$$

$$\approx w_1^{\alpha_1}(|x_1 - x_2|) \int_{\mathbb{R}^n} w_2^{\alpha_2}(y) e^{-\sqrt{\lambda_1}\alpha_1 y_1} dy.$$

The proof of (3.6) is similar.

□

Now we may apply Lemma 3.1 to prove

Lemma 3.2.

$$\left\| S\begin{pmatrix} u \\ v \end{pmatrix} \right\|_{Y\times Y} \le c\Big(|\beta| w_{\lambda_1,\mu_1}(l) + w_2\Big(2l\sin\frac{\pi}{k}\Big)\Big). \tag{3.7}$$

Proof.

$$\begin{aligned}\triangle u_l - \lambda_1 u_l + \mu_1 u_l^3 + \beta u_l v_l^2 &= \beta w_1 \Big(\sum_{j=0}^{k-1} w_{2,\xi_j}\Big)^2 \\ &= \beta w_1 \Big(\sum_j w_{2,\xi_j}^2 + \sum_{j\neq m} w_{2,\xi_j} w_{2,\xi_m}\Big).\end{aligned}$$

By Lemma 3.1, and the fact that $\sqrt{\lambda_1} < \sqrt{\lambda_2}$, $|a|+|b| \ge |a\pm b|$, we have

$$\begin{aligned}\|w_1 w_{2,\xi_j}^2\|_{L^2(\mathbb{R})} &\le c w_1(l) \\ \|w_1 w_{2,\xi_j}^2 w_{2,\xi_m}^2\|_{L^\infty(\mathbb{R})} &\le c e^{-|y|\sqrt{\lambda_1}} \cdot e^{-(|y-\xi_j|+|y-\xi_m|)\sqrt{\lambda_2}} \\ &\le c e^{-|y|\sqrt{\lambda_1}} \cdot e^{-(|y-\xi_j|+|y-\xi_m|)\sqrt{\lambda_1}} \cdot e^{(\sqrt{\lambda_1}-\sqrt{\lambda_2})|\xi_j-\xi_m|} \\ &\le c e^{-\sqrt{\lambda_1}(|l|+|y-\xi_m|)} \cdot e^{(\sqrt{\lambda_1}-\sqrt{\lambda_2})|\xi_j-\xi_m|}.\end{aligned}$$

Hence

$$\begin{aligned}\|\triangle u_l - \lambda_1 u_l + \mu_1 u_l^3 + \beta u_l v_l^2\|_{L^2(\mathbb{R}^n)} &\le c|\beta| w_1(l) + c w_1(|l|)\cdot e^{-2(\sqrt{\lambda_2}-\sqrt{\lambda_1})|l||\sin \pi/k|} \\ &\le c(|\beta| + e^{-\sigma l})|w_1(l)|,\end{aligned} \tag{3.8}$$

for $0<\sigma<2(\sqrt{\lambda_2}-\sqrt{\lambda_1})|\sin\frac{\pi}{k}|$. Similarly,

$$\begin{aligned}\triangle v_l - \lambda_2 v_l + \mu_2 v_l^3 + \beta u_l^2 v_l &= \mu_2\Big[\Big(\sum_{j=0}^{k-1} w_{2,\xi_j}\Big)^3 - \sum_{j=0}^{k-1} w_{2,\xi_j}^3\Big] + \beta w_1^2\Big(\sum_j w_{2,\xi_j}\Big) \\ &= O\Big(\sum_{j\neq m} w_{2,\xi_j}^2 w_{2,\xi_m}\Big) + O\Big(|\beta|\sum_j w_{2,\xi_j} w_1^2\Big).\end{aligned}$$

So

$$\begin{aligned}\|\triangle v_l - \lambda_2 v_l + \mu_2 v_l^3 + \beta u_l^2 v_l\|_{L^2(\mathbb{R}^n)} &\le c\sum_{j\neq m} w_2(|\xi_j-\xi_m|) + c|\beta| w_1(l) \\ &\le c\sum_{j\neq m} w_2(2l|\sin\frac{\pi}{k}|) + c|\beta| w_1(l),\end{aligned} \tag{3.9}$$

since $|\xi_j-\xi_m| = 2|\sin\frac{\pi(m-j)}{k}|l \ge 2l\sin\frac{\pi}{k}$.
Therefore by (3.8) and (3.9), we obtain (3.7) and complete the proof of Lemma 3.2. □

Now we want to estimate $E[u_l,v_l]$ as follows:

Lemma 3.3. *For $l >> 1$,*

$$\begin{aligned}E[u_l,v_l] =& \lambda_1^{\frac{4-n}{2}}\mu_1^{-1} I[w] + k\lambda_2^{\frac{4-n}{2}}\mu_2^{-1} I[w] - \frac{\mu_2}{2}\sum_{j\neq m}\int_{\mathbb{R}^n} w_{2,\xi_j}^2 w_{2,\xi_m} \\ &- \frac{\beta}{2}\sum_j \int_{\mathbb{R}^n} w_1^2 w_{2,\xi_j}^2 + O\big(e^{-2\sqrt{\lambda_2}l\sin\frac{\pi}{k}} + |\beta| e^{-2\sqrt{\lambda_1}l}\big)\cdot e^{-\sigma l},\end{aligned} \tag{3.10}$$

for some $\sigma > 0$.

Proof. We may use Lemma 3.1 to compute the energy $E[u_l, v_l]$. By (1.15), we have

$$\begin{aligned}
E[u_l, v_l] =&\lambda_1^{\frac{4-n}{2}} \mu_1^{-1} I[w] + \frac{1}{2}\int_{\mathbb{R}^n} \left|\nabla \sum_j w_{\lambda_2,\mu_2,\xi_j}\right|^2 + \frac{\lambda_2}{2}\int_{\mathbb{R}^n}\left(\sum_j w_{\lambda_2,\mu_2,\xi_j}\right)^2 \\
&- \frac{\mu_2}{4}\int_{\mathbb{R}^n}\left(\sum_j w_{\lambda_2,\mu_2,\xi_j}\right)^4 - \frac{\beta}{2}\int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1}\left(\sum_j w_{\lambda_2,\mu_2,\xi_j}\right)^2 \\
=&\lambda_1^{\frac{4-n}{2}} \mu_1^{-1} I[w] + k\lambda_2^{\frac{4-n}{2}} \mu_2^{-1} I[w] + \frac{1}{2}\sum_{j\neq m}\mu_2\int_{\mathbb{R}^n} w_{\lambda_2,\mu_2,\xi_j} w^3_{\lambda_2,\mu_2,\xi_m} \\
&- \frac{\mu_2}{4}\int_{\mathbb{R}^n}\left[\left(\sum_j w_{\lambda_2,\mu_2,\xi_j}\right)^4 - \sum_j w^4_{\lambda_2,\mu_2,\xi_j}\right] \\
&- \frac{\beta}{2}\int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1}\left(\sum_j w^2_{\lambda_2,\mu_2,\xi_j} + \sum_{j\neq m} w_{\lambda_2,\mu_2,\xi_j} w_{\lambda_2,\mu_2,\xi_m}\right) \\
=&\lambda_1^{\frac{4-n}{2}} \mu_1^{-1} I[w] + k\lambda_2^{\frac{4-n}{2}} \mu_2^{-1} I[w] - \frac{1}{2}\sum_{j\neq m}\mu_2\int_{\mathbb{R}^n} w_{\lambda_2,\mu_2,\xi_j} w^3_{\lambda_2,\mu_2,\xi_m} \\
&- \frac{\mu_2}{4}\int_{\mathbb{R}^n}\left[\left(\sum_j w_{\lambda_2,\mu_2,\xi_j}\right)^4 - \sum_j w^4_{\lambda_2,\mu_2,\xi_j} - 4\sum_{j\neq m} w_{\lambda_2,\mu_2,\xi_j} w^3_{\lambda_2,\mu_2,\xi_m}\right] \\
&- \frac{\beta}{2}\sum_j\int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1} w^2_{\lambda_2,\mu_2,\xi_j} - \frac{\beta}{2}\int_{\mathbb{R}^n} w^2_{\lambda_1,\mu_1}\sum_{j\neq m} w_{\lambda_2,\mu_2,\xi_j} w_{\lambda_2,\mu_2,\xi_m}. \qquad (3.11)
\end{aligned}$$

By Lemma 3.1,

$$\begin{aligned}
\int_{\mathbb{R}^n} w_{2,\xi_j} w^3_{2,\xi_j} &\approx w_2(|\xi_i - \xi_j|), \\
\int_{\mathbb{R}^n}\left|\left(\sum_j w_{2,\xi_j}\right)^4 - \sum_j w^4_{2,\xi_j} - 4\sum_{j\neq m} w_{2,j} w^3_{2,m}\right| & \\
&\leq C\sum_{j\neq m}\int_{\mathbb{R}^n} w^2_{2,\xi_j} w^2_{2,\xi_m} \\
&\leq C\sum_{j\neq m}\left(w_2(|\xi_j - \xi_m|)\right)^{3/2} \\
&\leq Ce^{-3l\sqrt{\lambda_2}|\sin\frac{\pi}{k}|}, \qquad (3.12) \\
\int_{\mathbb{R}^n} w_1^2 w_{2,\xi_j} w_{2,\xi_m} &\leq \int_{\mathbb{R}^n} w_1^2 e^{-\sqrt{\lambda_2}(|y-\xi_j|+|y-\xi_m|)} \\
&\leq \int_{\mathbb{R}^n} w_1^2 e^{-\sqrt{\lambda_1}(|y-\xi_j|+|y-\xi_m|)}\cdot e^{(\sqrt{\lambda_1}-\sqrt{\lambda_2})|\xi_j-\xi_m|} \\
&\leq e^{-2\sqrt{\lambda_1}l}\cdot e^{2l(\sqrt{\lambda_1}-\sqrt{\lambda_2})\sin\frac{\pi}{k}}, \qquad (3.13)
\end{aligned}$$

where C is a universal positive constant. Consequently, (3.10) follows from (3.11), (3.12) and (3.13). Therefore we may complete the proof of Lemma 3.3. □

4 Localized Energy Method

In this section, we use the so-called "Localized Energy Method" to reduce the problem to a finite-dimensional one. Similar method has been used in the proof of Theorem 4 of [11]. For background and references on this method, we refer to [3], [9], [11] and [13].

In this reduction process, the symmetry assumption plays an important role so we focus on two spatial dimension case i.e. $n = 2$. Let

$$\begin{aligned}
L\begin{pmatrix}\phi\\ \psi\end{pmatrix} &= S'\begin{pmatrix}u_l\\ v_l\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \\
&= \begin{pmatrix}\triangle\phi - \lambda_1\phi + 3\mu_1 u_l^2\phi + \beta v_l^2\phi + 2\beta u_l v_l\psi \\ \triangle\psi - \lambda_2\psi + 3\mu_2 v_l^2\psi + \beta u_l^2\psi + 2\beta u_l v_l\phi\end{pmatrix},
\end{aligned}$$

for $\binom{\phi}{\psi} \in X \times X$. Let

$$K_l = \text{span}\left\{ \begin{pmatrix} 0 \\ \frac{\partial v_l}{\partial l} \end{pmatrix} \right\}. \tag{4.1}$$

Recall that $v_l = \sum_{j=0}^{k-1} w_2\left(y - le^{i2\pi\frac{j}{k}}\right)$ and $i = \sqrt{-1}$. Now we consider the following linear problem:
Given $f, g \in Y$, find $\phi, \psi \in X$ and $c \in \mathbb{R}$ such that

$$\begin{cases} L\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} + c\begin{pmatrix} 0 \\ \frac{\partial v_l}{\partial l} \end{pmatrix}, \\ \displaystyle\int_{\mathbb{R}^2} \phi \frac{\partial v_l}{\partial l} = 0\,. \end{cases} \tag{4.2}$$

For the problem (4.2), we have the following crucial a-priori estimates

Lemma 4.1. *There exist $\beta_0 > 0$ such that for $|\beta| < \beta_0$, if $\binom{\phi}{\psi}$ satisfies (4.2), then*

$$\|\phi\|_{H^2} + \|\psi\|_{H^2} \le C(\|f\|_{L^2} + \|g\|_{L^2})\,. \tag{4.3}$$

Proof. We prove this by contradiction. Suppose there exist $|\beta_n| \to 0$, f_n, g_n, ϕ_n and ψ_n satisfying (4.2), such that

$$\|f_n\|_{L^2} + \|g_n\|_{L^2} \to 0 \ \text{ as } \ n \to \infty\,, \qquad \text{and} \qquad \|\phi_n\|_{H^2} + \|\psi_n\|_{H^2} = 1 \text{ for } n \in \mathbb{N}\,. \tag{4.4}$$

To avoid clumsy notation, we omit the index n. Firstly, we derive the estimate for c. Multiplying the equation of ψ by $\frac{\partial v_l}{\partial l}$, we obtain

$$\int_{\mathbb{R}^2} \left(\triangle\psi - \lambda_2\psi + 3\mu_2 v_l^2\psi + \beta u_l^2\psi + 2\beta u_l v_l \phi\right)\frac{\partial v_l}{\partial l} = \int_{\mathbb{R}^2} g\frac{\partial v_l}{\partial l} + c\int_{\mathbb{R}^2}\left(\frac{\partial v_l}{\partial l}\right)^2. \tag{4.5}$$

Since $\triangle\left(\frac{\partial v_l}{\partial l}\right) - \lambda_2\frac{\partial v_l}{\partial l} + 3\mu_2 v_l^2 \cdot \frac{\partial v_l}{\partial l} = o(1)$ in L^2, it is easy to check that

$$c = O(\|g\|_{L^2}) + O(\|\phi\|_{H^2} + \|\psi\|_{H^2})|\beta| = o(1). \tag{4.6}$$

Now we claim that the operator

$$L_{01}\phi = \triangle\phi - \lambda_1\phi + 3\mu_1 u_l^2\phi \tag{4.7}$$

is invertible from X to Y. In fact, if $L_{01}\phi = 0$ and $\phi \in \cap X$, then $\phi = \sum_{j=1}^{2} c_j \frac{\partial w_{\lambda_1,\mu_1}}{\partial y_j}$. Since $\phi(\bar{y}) = \phi(y)$, we obtain that $c_2 = 0$ and $\phi = c_1\frac{\partial w_{\lambda_1,\mu_1}}{\partial y_1} = c_1 w'_{\lambda_1,\mu_1}(|y|)\frac{y_1}{|y|}$. On the other hand, since $\phi(ye^{i2\pi/k}) = \phi(y)$, we also obtain $c_1 = 0$. Hence $\phi \equiv 0$. Thus L_{01} is invertible. Moreover, since $\beta \to 0$, the operator

$$L_{11}\phi = \triangle\phi - \lambda_1\phi + 3\mu_1 u_l^2\phi + \beta v_l^2\phi$$

can be regarded as a small perturbation of L_{01} so the operator L_{11} is also invertible. Hence, we may write

$$\phi = L_{11}^{-1}(-2\beta u_l v_l \psi + f)\,,$$

and obtain

$$\|\phi\|_{H^2(\mathbb{R}^2)} \le c\| - 2\beta u_l v_l \psi + f\|_{L^2(\mathbb{R}^2)} \to 0. \tag{4.8}$$

As for the operator L_{01} on ϕ, we may define another operator on ψ by

$$L_{02}\psi = \triangle\psi - \lambda_2\psi + 3\mu_2 v_l^2\psi\,.$$

Then we have

$$L_{02}\psi = -\beta u_l^2\psi - \beta u_l v_l \phi + g + c\frac{\partial v_l}{\partial l} =: g_2.$$

Hence $\|g_2\|_{L^2(\mathbb{R}^2)} = o(1)$. By (4.8),

$$\|\psi\|_{H^2(\mathbb{R}^2)} = 1 - \|\phi\|_{H^2(\mathbb{R}^2)} \ge \frac{1}{2}\,. \tag{4.9}$$

Now we set

$$\psi_0(y) \equiv \psi(\xi_0 + y) \to \widetilde{\psi}_0(y) \text{ in } \mathcal{C}^1_{loc}(\mathbb{R}^2) \quad \text{ as } \ l \to +\infty\,.$$

Then as $l \to +\infty$, ψ_0 satisfies

$$\begin{aligned} &\triangle\psi_0 - \lambda_2\psi_0 + 3\mu_2 w^2_{\lambda_2,\mu_2,\xi_0}(\xi_0+y)\psi_0 \\ &+3\mu_2\sum_{j\neq m}\left(w_{\lambda_2,\mu_2,\xi_j}w_{\lambda_2,\mu_2,\xi_m}\right)(\xi_0+y)\psi_0 + 3\mu_2\sum_{j\neq 0} w^2_{\lambda_2,\mu_2,\xi_j}(\xi_0+y)\psi_0 \\ &= g_2(\xi_0+y)\,, \end{aligned}$$

i.e.

$$\triangle\widetilde{\psi}_0 - \lambda_2\widetilde{\psi}_0 + 3\mu_2 w^2_{\lambda_2,\mu_2}\widetilde{\psi_0} = 0. \tag{4.10}$$

Here we have used the fact that $\|g_2\|_{L^2(\mathbb{R}^2)} = o(1)$, $w_{\lambda_2,\mu_2,\xi_0}(\xi_0+y) = w_{\lambda_2,\mu_2}(y)$ and $w^2_{\lambda_2,\mu_2,\xi_j}(\xi_0+y) = w^2_{\lambda_2,\mu_2}(\xi_0-\xi_j+y) \to 0$ as $l \to +\infty$ for $j \neq 0$. Hence

$$\widetilde{\psi}_0 = c_1\frac{\partial w_{\lambda_2,\mu_2}}{\partial y_1} + c_2\frac{\partial w_{\lambda_2,\mu_2}}{\partial y_2}. \tag{4.11}$$

Moreover,

$$0 = \int_{\mathbb{R}^2}\psi(y+\xi_0)\frac{\partial v_l}{\partial l}(y+\xi_0) = \int_{\mathbb{R}^2}\psi_0(y)\frac{\partial v_l}{\partial l}(y+\xi_0) \to k\int_{\mathbb{R}^2}\widetilde{\psi}_0(y)\frac{\partial w_{\lambda_2,\mu_2}}{\partial y_1}(y)\,.$$

Here we have used the k-symmetry property of $\widetilde{\psi}_0$. Thus it is obvious that $c_1 = 0$ and

$$\widetilde{\psi}_0 = c_2\frac{\partial w_{\lambda_2,\mu_2}}{\partial y_2} = c_2 w'_{\lambda_2,\mu_2}(|y|)\frac{y_2}{|y|}. \tag{4.12}$$

Notice that $\psi_0(\bar{y}) = \psi(y)$ i.e. $\widetilde{\psi}_0(\bar{y}) = \widetilde{\psi}_0(y)$ so we obtain $c_2 = 0$. Hence $\widetilde{\psi}_0 \equiv 0$ and we have

$$\int_{\mathbb{R}^2} v_l^4(y)\psi^2(y) = \int_{\mathbb{R}^2} v_l^4(y+\xi_0)\psi^2(y+\xi_0) = \int_{\mathbb{R}^2} v_l^4(y+\xi_0)\psi_0^2(y) \to 0\,.$$

Thus by the equation of ψ, we obtain $\|\psi\|_{H^2} \leq \|\triangle\psi - \lambda_2\psi\|_{L^2} \to 0$ which may contradict with (4.9). Therefore we may complete the proof of Lemma 4.1. □

Remark. *The proof of Lemma 4.1 also explains how we use specific symmetry properties to show the kernel of L is one dimensional, and hence the problem becomes one dimensional, too.*

Once Lemma 4.1 is proved, we have the following lemmas. We refer to Lemma 8, Proposition 1 and Lemma 10 of [11] for similar proofs.

Lemma 4.2. *Given* $\binom{f}{g} \in Y \times Y$, *there exists unique* $\binom{\phi}{\psi} \in X \times X$ *and* c *satisfying (4.2).*

Lemma 4.3. *The following nonlinear problem has a unique solution:*

$$\begin{cases} S\begin{pmatrix} u_l + \phi_l \\ v_l + \psi_l \end{pmatrix} = c\begin{pmatrix} 0 \\ \frac{\partial v_l}{\partial l} \end{pmatrix}, \\ \\ \displaystyle\int_{\mathbb{R}^2}\phi_l\frac{\partial v_l}{\partial l} = 0, \quad \begin{pmatrix} \phi_l \\ \psi_l \end{pmatrix} \in (H^2(\mathbb{R}^2)\cap X)^2. \end{cases} \tag{4.13}$$

Moreover, we have

$$\left\|\begin{pmatrix} \phi_l \\ \psi_l \end{pmatrix}\right\|_{H^2(\mathbb{R}^2)} \leq c\left\|S\begin{pmatrix} u_l \\ v_l \end{pmatrix}\right\|_{L^2(\mathbb{R}^2)} \tag{4.14}$$

and the map $l \to \phi_l$ *is* $\mathcal{C}^1$.

Finally, we define $M(l) =: E[u_l + \phi_l, v_l + \psi_l]$. Then we have

Lemma 4.4. *If* $M(l)$ *has a critical point* $l = l_0$, *then* $(u_{l_0} + \phi_{l_0}, v_{l_0} + \psi_{l_0})$ *satisfies (1.14) and the properties listed in Theorem 1.2.*

5 Reduced problem

We first compute $M(l)$:

Lemma 5.1. *For $0<|\beta|<<1$, we have*

$$M(l)=A_0-\alpha(l)-\frac{\beta}{2}\gamma(l)+O\big(e^{-2\sqrt{\lambda_2}\sin\pi l/k}+|\beta|e^{-2\sqrt{\lambda_1}l}\big)e^{-\sigma l}.$$

where

$$A_0=\lambda_1^{\frac{4-n}{2}}\mu_1^{-1}I[w]+k\lambda_2^{\frac{4-n}{2}}\mu_2^{-1}I[w]\,,\ \alpha(l)=k\mu_2\int_{\mathbb{R}^2}w_{2,\xi_0}^3w_{2,\xi_1}\,,\ \gamma(l)=k\int_{\mathbb{R}^2}w_1^2w_{2,\xi_0}^2.$$

Proof. It is easy to check that

$$\begin{aligned}M(l)=&E[u_l+\phi_l,v_l+\psi_l]\\=&E[u_l,v_l]+\int_{\mathbb{R}^2}(\triangle u_l-\lambda_1u_l+\mu_1u_l^3+\beta u_lv_l^2)\phi_l\\&+\int_{\mathbb{R}^2}(\triangle v_l-\lambda_2v_l+\mu_2v_l^3+\beta u_l^2v_l)\psi_l+\int_{\mathbb{R}^2}\big(|\phi_l|^2+|\psi_l|^2\big)\\=&E[u_l,v_l]+O\big(e^{-2\sqrt{\lambda_2}\sin\frac{\pi l}{k}}+|\beta|e^{-2\sqrt{\lambda_1}l}\big)e^{-\sigma l}.\end{aligned}$$

Then by Lemma 3.3, we may complete the proof of Lemma 5.1. □

Let $c_1<\dfrac{1}{2(\sqrt{\lambda_2}\sin\frac{\pi}{k}-\sqrt{\lambda_1})}<c_2$. Then we have

Lemma 5.2. *The following problem*

$$\max\Big\{M(l)\Big|c_1\log\frac{1}{|\beta|}\le l\le c_2\log\frac{1}{|\beta|}\Big\}$$

has a critical point l_β, i.e. $M'(l_\beta)=0$.

Proof. By lemma 3.1 and (3.3), it is not difficult to see that

$$\begin{aligned}\alpha(l)\approx l^{-1/2}e^{-2\sqrt{\lambda_2}l\sin\frac{\pi}{k}},\ \alpha'(l)\approx-2\sqrt{\lambda_2}\alpha(l)\sin\frac{\pi}{k};\\ \gamma(l)\approx l^{-1/2}e^{-2\sqrt{\lambda_1}l},\ \gamma'(l)\approx-2\sqrt{\lambda_1}\gamma(l).\end{aligned}$$

Now we let l_β be such that $\alpha'(l_\beta)+\frac{\beta}{2}\gamma'(l_\beta)=0$. Then we have

$$l_\beta\approx\frac{1}{2(\sqrt{\lambda_2}\sin\frac{\pi}{k}-\sqrt{\lambda_1})}\log\frac{1}{|\beta|},$$

and

$$\begin{aligned}M(l_\beta)=&A_0-\alpha(l_\beta)-\frac{\beta}{2}\gamma(l_\beta)\\=&A_0-\alpha(l_\beta)+\frac{\gamma(l_\beta)}{\gamma'(l_\beta)}\alpha'(l_\beta)\\\approx&A_0-\alpha(l_\beta)+\frac{\sqrt{\lambda_2}\sin\frac{\pi}{k}}{\sqrt{\lambda_1}}\cdot\alpha(l_\beta)\\\approx&A_0+\frac{\sqrt{\lambda_2}\sin\frac{\pi}{k}-\sqrt{\lambda_1}}{\sqrt{\lambda_1}}\cdot l_\beta^{-1/2}e^{-2\sqrt{\lambda_2}l_\beta\sin\frac{\pi}{k}}\\\approx&A_0+\frac{\sqrt{\lambda_2}\sin\frac{\pi}{k}-\sqrt{\lambda_1}}{\sqrt{\lambda_1}}\Big(\log\frac{1}{|\beta|}\Big)^{-1/2}\cdot|\beta|^{\frac{\sqrt{\lambda_2}\sin\frac{\pi}{k}}{\sqrt{\lambda_2}\sin\frac{\pi}{k}-\sqrt{\lambda_1}}}.\end{aligned}$$

Hence

$$\max M(l)\ge M(l_\beta)\ge A_0+c_0\Big(\log\frac{1}{|\beta|}\Big)^{-1/2}|\beta|^{\tau_1},\tag{5.1}$$

where $\tau_1 = \dfrac{\sqrt{\lambda_2}\sin\frac{\pi}{k}}{\sqrt{\lambda_2}\sin\frac{\pi}{k} - \sqrt{\lambda_1}}$.
Now that

$$\begin{aligned} M\Big(c_1\log\frac{1}{|\beta|}\Big) &\approx A_0 - \Big(|\beta|^{2c_1\sqrt{\lambda_2}\sin\frac{\pi}{k}} + \frac{\beta}{2}\cdot|\beta|^{2c_1\sqrt{\lambda_1}}\Big)\Big(c_1\log\frac{1}{|\beta|}\Big)^{-1/2} \\ &\approx A_0 - \Big(|\beta|^{2c_1\sqrt{\lambda_2}\sin\frac{\pi}{k}} - |\beta|^{1+2c_1\sqrt{\lambda_1}}\Big)\Big(c_1\log\frac{1}{|\beta|}\Big)^{-1/2} \\ &\leq A_0 - |\beta|^{2c_1\sqrt{\lambda_2}\sin\frac{\pi}{k}}\Big(c_1\log\frac{1}{|\beta|}\Big)^{-1/2} \qquad \text{if } 2c_1\sqrt{\lambda_2}\sin\frac{\pi}{k} < 1 + 2c_1\sqrt{\lambda_1}. \end{aligned}$$

$$\begin{aligned} M\Big(c_2\log\frac{1}{|\beta|}\Big) &\approx A_0 - \Big(|\beta|^{2c_2\sqrt{\lambda_2}\sin\frac{\pi}{k}} - |\beta|^{1+2c_2\sqrt{\lambda_1}}\Big)\Big(c_2\log\frac{1}{|\beta|}\Big)^{-1/2} \\ &\leq A_0 + |\beta|^{1+2c_2\sqrt{\lambda_1}}\Big(c_2\log\frac{1}{|\beta|}\Big)^{-1/2} \qquad \text{if } 2c_2\sqrt{\lambda_2}\sin\frac{\pi}{k} > 1 + 2c_2\sqrt{\lambda_1}. \end{aligned}$$

In any case, we have

$$\max\Big\{(M\Big(c_1\log\frac{1}{|\beta|}\Big), M\Big(c_2\log\frac{1}{|\beta|}\Big)\Big\} \leq A_0 + |\beta|^{1+2c_2\sqrt{\lambda_1}}\Big(c_2\log\frac{1}{|\beta|}\Big)^{-1/2} \qquad \text{if } 2c_2\sqrt{\lambda_2}\sin\frac{\pi}{k} > 1 + 2c_2\sqrt{\lambda_1}. \tag{5.2}$$

Suppose $M(l_\beta) \leq \max\Big\{(M\Big(c_1\log\frac{1}{|\beta|}\Big), M\Big(c_2\log\frac{1}{|\beta|}\Big)\Big\}$. Then comparing (5.1) and (5.2), we obtain

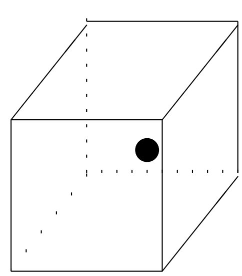
$$c_0|\beta|^{\tau_1} \leq c_2^{-1/2}|\beta|^{1+2c_2\sqrt{\lambda_1}}.$$

As long as we choose $1 + 2c_2\sqrt{\lambda_1} > \tau_1 = \dfrac{\sqrt{\lambda_2}\sin\frac{\pi}{k}}{\sqrt{\lambda_2}\sin\frac{\pi}{k} - \sqrt{\lambda_1}}$ i.e. $c_2 > \frac{1}{2(\sqrt{\lambda_2}\sin\frac{\pi}{k} - \sqrt{\lambda_1})}$, we obtain a contradiction right away. Therefore we may complete the proof of Lemma 5.2. □

Remark. *We have showed that the critical point l_β satisfying $l_\beta \sim \dfrac{1}{2(\sqrt{\lambda_2}\sin\frac{\pi}{k} - \sqrt{\lambda_1})}\log\dfrac{1}{|\beta|}$.*

6 Proof of Theorem 1.3

The proof of Theorem 1.3 depends on the choice of symmetry class. We first consider the cubic case



Assume the cube has a center at $(0,0,0)$ and eight vertices at (a,b,c)'s for $a,b,c \in \{\pm 1\}$. Let L_1 be the plane $y - z = 0$, L_2 be the plane $x - z = 0$, and L_3 be the plane $x - y = 0$. Let T_j be the transformation of reflection through the plane L_j, i.e. $T_j(x,y,z) =$ reflection of (x,y,z) to the plane L_j.
Now we set

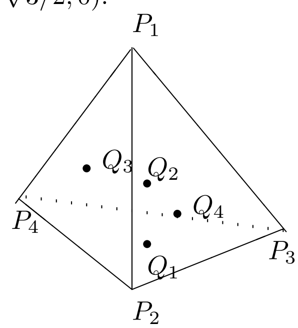
$$X = \{u \in H^2(\mathbb{R}^3) | u(x,y,z) = u(-x,y,z) = u(x,-y,z) = u(x,y,-z) = u\big(T_j(x,y,z)\big)\ ,\ j = 1,2,3\}\ ,$$

and

$$Y = \{u \in L^2(\mathbb{R}^3) | u(x,y,z) = u(-x,y,z) = u(x,-y,z) = u(x,y,-z) = u\big(T_j(x,y,z)\big)\ ,\ j = 1,2,3\}\ .$$

We wish to put one spike at the center for u and eight spikes at the vertices of a cube for v. We need to determine the locations of eight spikes. Due to specific symmetry properties $v(x, y, z) = v(\pm x, \pm y, \pm z)$, the problem of determining eight spikes can be reduced first to determine one spike in the quadrant $\{x > 0, y > 0, z > 0\}$. One may regard the problem as a three dimensional problem. Then we use the reflection symmetry $v(x, y, z) = v(T_j(x, y, z))\,, j = 1, 2, 3$ to further reduce the problem to an one dimensional problem. Thus, as for the proof of Theorem 1.2, we may set $\xi_0 = (l, l, l)$ as one vertex of the cube and $\xi_j\,, j = 1, \cdots, 7$ as the other seven vertices of the cube, where $l > 0$ is an one-dimensional parameter. Then the problem can be reduced to an one dimensional problem like the critical point problem of Lemma 4.4 and 5.2. Therefore we may apply similar arguments to complete the proof of Theorem 1.3 for this case.

In the tetrahedra case, we may assume that the four vertices are $P_1 = (0, 0, 1)$, $P_2 = (1, 0, 0)$, $P_3 = (-1/2, \sqrt{3}/2, 0)$ and $P_4 = (-1/2, -\sqrt{3}/2, 0)$.



Let Q_1 be the center of the triangle $P_2P_3P_4$. We set l_1 as the axis joining P_1 and Q_1. Let T_1 be the rotation around l_1 by angle $\frac{2\pi}{3}$. Similarly, we can define l_j, T_j, $j = 2, 3, 4$. Let L_1 be the plane containing P_1, P_2 and $(0, 0, 0)$, L_2 be the plane containing P_1, P_3 and $(0, 0, 0)$ and L_3 be the plane containing P_1, P_4 and $(0, 0, 0)$. Let T_{4+j} be the reflection through L_j for $j = 1, 2, 3$. Then it is easy to see that the tetrahedra is invariant to T_j's. Moreover, the Laplace operator is also invariant under T_j's. Now we set

$$X = \{u \in H^2(\mathbb{R}^3) | u\big(T_j(x, y, z)\big) = u(x, y, z)\,, j = 1, \cdots, 7\}\,,$$

and

$$Y = \{u \in L^2(\mathbb{R}^3) | u\big(T_j(x, y, z)\big) = u(x, y, z)\,, j = 1, \cdots, 7\}.$$

Then similar as before, the problem can be reduced to an one-dimensional problem. Therefore we may apply similar arguments to complete the proof of Theorem 1.3 for this case.

References

[1] N. Akhmediev and A. Ankiewicz, Partially coherent solitons on a finite background, *Phys. Rev. Lett.* **82**(1999) 2661(1-4).

[2] K. W. Chow, Periodic solutions for a system of four coupled nonlinear Schrodinger equations, *Phys. Lett. A* **285** (2001) 319-V326.

[3] M. del Pino, P. Felmer and M. Musso, Two-bubble solutions in t he super-critical Bahri-Coron's pronblem, *Cal. Var. PDE* **16**(2003), no. 2, 113-145.

[4] S. Dyachenco, A. C. Newell, A. Pushkarev, V. E. Zakharov, Optical turbulence: weak turbulence, condensate and collapsing filaments in the nonlinear Schrödinger equation, *Physica D* **57**(1992) 96-160.

[5] M. V. Goldman, Strong turbulence of plasma waves, *Rev. Modern Phys.* **56** no. 4 (1984) 709-733.

[6] F. T. Hioe, Solitary Waves for N Coupled Nonlinear Schrödinger Equations, *Phys. Rev. Lett.* **82**(1999) 1152-1155.

[7] F. T. Hioe, T. S. Salter, Special set and solutions of coupled nonlinear schrödinger equations, *J. Phys. A: Math. Gen.* **35** (2002) 8913-8928.

[8] T. Kanna and M. Lakshmanan, Exact soliton solutions, shape changing collisions, and partially coherent solitons in coupled nonlinear Schrödinger equations, *Phys. Rev. Lett.* **86** (2001) 5043(1-4).

[9] C. Gui and J. Wei, Multiple interior spike solutions for some singular perturbed Neumann problems, *J. Diff. Eqns* **158** (1999) 1-27.

[10] T. C. Lin and J. Wei, Spikes in two coupled nonlinear Schrödinger equations, *Ann. I. H. Poincaré* **22**(2005) 403-439.

[11] T. C. Lin and J. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n, n \leq 3$, *Comm. Math. Phys.* **255**(2005), 629-653.

[12] M. Mitchell, Z. Chen, M. Shih, and M. Segev, Self-Trapping of partially spatially incoherent light, *Phys. Rev. Lett.* **77**(1996) 490-493.

[13] W.-M. Ni, *Qualitative properties of solutions to elliptic problems.* Stationary partial differential equations. Vol. I, 157–233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[14] W.-M. Ni and I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.* **70** (1993), 247-281.

[15] P. A. Robinson, Nonlinear wave collapse and strong turbulence, *Reviews of Modern Physics*, **69**(1997) 507-573.

[16] O. S. Rozanova, Hydrodynamic approach to constructing solutions of nonlinear Schrödinger equation in the critical case, *Proc. American Math. Soc.* **133** no. 8 (2005) 2347-2358.

[17] C. Sulem, P. L. Sulem, The nonlinear Schrödinger equation. Self-focusing and wave collapse, *Applied Math. Sci.* **139** Springer-Verlag, New York (1999).