STABILITY OF SPIKES IN THE SHADOW GIERER-MEINHARDT SYSTEM WITH ROBIN BOUNDARY CONDITIONS

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Abstract. We consider the shadow system of the Gierer-Meinhardt system in a smooth bounded domain $\Omega \subset R^N$:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{\xi^q}, & x \in \Omega,\ t > 0, \\ \tau |\Omega| \xi_t = -|\Omega| \xi + \frac{1}{\xi^s} \int_\Omega A^r \, dx, & t > 0 \end{cases}$$

with Robin boundary condition

$$\epsilon \frac{\partial A}{\partial \nu} + a_A A = 0, \quad x \in \partial\Omega,$$

where $a_A > 0$, the positive reaction rates (p, q, r, s) satisfy

$$1 < \frac{qr}{(s+1)(p-1)} < +\infty, \quad 1 < p < \left(\frac{N+2}{N-2}\right)_+,$$

the diffusion constant is chosen such that $\epsilon << 1$ and the time relaxation constant such that $\tau \geq 0$.

We rigorously prove results on the stability of spiky solutions.

These results are as follows: (i) If $r = 2$ and $1 < p < 1 + 4/N$ or if $r = p + 1$ and $1 < p < \infty$ then for $a_A > 1$ and τ sufficiently small the interior spike is stable. (ii) For $N = 1$ if $r = 2$ and $1 < p \leq 3$ or if $r = p + 1$ and $1 < p < \infty$ then for $0 < a_A < 1$ the near-boundary spike, for which existence was obtained in [1], is stable. (iii) For $N = 1$ if $3 < p < 5$ and $r = 2$ then there exist $a_0 \in (0, 1)$ and $\mu_0 > 1$ such that for $a \in (a_0, 1)$ and $\mu = \frac{2q}{(s+1)(p-1)} \in (1, \mu_0)$ the near-boundary spike solution is unstable. This instability is not present for the Neumann boundary condition but only arises for Robin boundary condition. Further we show that the corresponding eigenvalue is of order $O(1)$ as $\epsilon \to 0$.

1. Introduction

Since the work of Turing [25] in 1952, a lot of models have been established and investigated to explore instability of homogeneous steady states, which is now called Turing instability. One of the most famous models in biological pattern formation is the Gierer-Meinhardt system [9], [14], [15]. It can be stated as follows:

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{H^q} & x \in \Omega,\ t > 0, \\ \tau H_t = D \Delta H - H + \frac{A^r}{H^s} & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded, smooth domain. Further, we assume that the reaction rates (p, q, r, s) are positive and satisfy

$$1 < \frac{qr}{(s+1)(p-1)} < +\infty, \quad 1 < p < \left(\frac{N+2}{N-2}\right),$$

where

$$\left(\frac{N+2}{N-2}\right)_+ = \begin{cases} +\infty & \text{for } N = 1,\ 2 \\ \frac{N+2}{N-2} & \text{for } N = 3, 4, \ldots \end{cases}$$

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We assume that the first diffusion constant satisfies $\epsilon << 1$ and we will consider the case of $D = \infty$, the so-called shadow system of the Gierer-Meinhardt system. The time relaxation constant is chosen such that $\tau \geq 0$ independent of ϵ.

This is a typical activator-inhibitor system, where A is called activator and H is called inhibitor. This model has been extensively studied in recent years, usually with Neumann boundary conditions [12], [26], [27], [28], [37], [38], [39].

In this work we consider Robin boundary conditions (also called mixed boundary conditions), which can be stated as follows:

$$\epsilon \frac{\partial A}{\partial \nu} + a_A A = 0, \quad \sqrt{D} \frac{\partial H}{\partial \nu} + a_H = 0, \quad x \in \partial\Omega, \tag{1.2}$$

where $a_A > 0$, $a_H \geq 0$.

From a biological viewpoint, such boundary conditions correspond to an impermeable membrane/barrier. While in many cases this is a realistic assumption, there are several cases, for example, in skeletal limb development, where the boundary is a source of some chemical morphogens and a sink for others. It is therefore essential that the study of these model equations is extended to incorporate more general types of boundary conditions. For example, in [3], a comparative numerical study of a reaction-diffusion system with a range of different boundary conditions revealed that certain types of boundary conditions selected particular patterning modes at the expense of others. It was also shown that the robustness of certain patterns could be greatly enhanced and the authors showed a possible application to skeletal patterns in the limb. This study answered the standard criticism of Turing patterns being too sensitive to fluctuations for the model to be viable for robust embryological patterning.

In this paper, we initiate a rigorous study of stationary spikes in (1.1), (1.2) in the shadow system case.

We now (formally) derive the shadow system. To this end, we let $D \to \infty$ and suppose that A and H remain bounded. Then

$$\Delta H \to 0 \qquad \text{in } \Omega$$

and

$$\frac{\partial H}{\partial \nu} \to 0 \qquad \text{on } \partial\Omega.$$

This implies that $H(x) \to \xi$, a constant in Ω. To derive the equation for ξ, we integrate both sides of the second equation in (1.1) over Ω. For the l.h.s we obtain

$$\tau \int_\Omega H_t(x)\, dx = \tau \left(\int_\Omega H(x)\, dx \right)_t \to \tau |\Omega| \xi_t.$$

To compute the r.h.s., we begin with

$$D \int_\Omega \Delta H(x)\, dx = D \int_{\partial\Omega} \frac{\partial H}{\partial \nu}(x)\, dS =$$

$$= \sqrt{D} \int_{\partial\Omega} (-a_H) H(x)\, dS \to -\sqrt{D} a_H |\partial\Omega| \xi,$$

where we have used (1.2). Further, we get

$$\int_\Omega H(x)\, dx \to |\Omega| \xi$$

and

$$\int_\Omega \frac{A^r(x)}{H^s(x)}\,dx \to \int_\Omega \frac{A^r(x)}{\xi^s}\,dx = \frac{\int_\Omega A^r(x)\,dx}{\xi^s}.$$

From these computations, we finally get the following so-called shadow system of (1.1):

$$\begin{cases} A_t = \epsilon^2\Delta A - A + \frac{A^p}{\xi^q}, & x\in\Omega,\ t>0,\\ \tau|\Omega|\xi_t = -(|\Omega| + \sqrt{D}a_H|\partial\Omega|)\xi + \frac{1}{\xi^s}\int_\Omega A^r\,dx, & t>0,\\ \epsilon\frac{\partial A}{\partial\nu} + a_A A = 0, & \partial\Omega,\ t>0. \end{cases} \tag{1.3}$$

Remarks. 1. Note that since $D\to\infty$, for the shadow system (1.3) to make sense, we need to assume that $\lim_{D\to\infty}\sqrt{D}a_H$ exists, i.e. we assume that, as $D\to\infty$ either (i) $a_H \sim \frac{1}{\sqrt{D}}$ or (ii) $a_H << \frac{1}{\sqrt{D}}$. We still denote this limit by "$\sqrt{D}a_H$". In Case (ii) the term "$\sqrt{D}a_H$" is omitted in the shadow system.

2. We further discuss Case (ii) in the previous remark, i.e. $a_H << \frac{1}{\sqrt{D}}$. If the term $\sqrt{D}a_H$ in (1.3) vanishes, this merely changes one constant factor in the shadow system. The qualitative behavior of solutions is not altered by this as one can compensate by a simple re-scaling of the amplitudes of both functions $A(x,t)$ and $\xi(x,t)$ by a constant factor. Thus from now on we assume that $a_H = 0$, i.e. the inhibitor of the Gierer-Meinhardt system (1.1) satisfies the Neumann boundary condition.

Let us now consider stationary solutions to the shadow system (1.3). Set $A(x) = \xi^{q/(p-1)}u(x)$, $a_A = a$. Then u satisfies

$$\begin{cases} \epsilon^2\Delta u - u + u^p = 0 & \text{for } x\in\Omega,\\ u>0 & \text{for } x\in\Omega,\\ \epsilon\frac{\partial u}{\partial\nu} + au = 0 & \text{for } x\in\partial\Omega. \end{cases} \tag{1.4}$$

For ξ we have

$$0 = -(|\Omega| + \sqrt{D}a_H|\partial\Omega|)\xi + \frac{\xi^{qr/(p-1)}}{\xi^s}\int_\Omega u^r\,dx$$

which gives

$$\xi^{1+s-qr/(p-1)} = \frac{1}{|\Omega| + \sqrt{D}a_H|\partial\Omega|}\int_\Omega u^r\,dx$$

and so

$$\xi = \left(\frac{1}{|\Omega| + \sqrt{D}a_H|\partial\Omega|}\int_\Omega u^r\,dx\right)^{-(p-1)/(qr-(p-1)(s+1))}.$$

Problem (1.4) has been studied by Berestycki and Wei in [1] and the following result has been proved:

Theorem A. *Let $1 < p < \left(\frac{N+2}{N-2}\right)_+$. Then there exists a number $a(N,p)$, where $a(1,p) = 1$ and $a(N,p) > 1$ for $N \ge 2$, such that problem (1.4) has a solution $u_{\epsilon,a}$ satisfying*

(1) $u_{\epsilon,a}$ has the least energy among all solutions to (1.4), i.e.

$$E_\epsilon[u_{\epsilon,a}] \le E_\epsilon[u] \tag{1.5}$$

for all solutions u to (1.4), where E_ϵ is the energy functional defined by

$$E_\epsilon[u] = \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2\,dx - \frac{1}{p+1}\int_\Omega u_+^{p+1}\,dx + \frac{\epsilon a}{2}\int_{\partial\Omega} u^2\,ds. \tag{1.6}$$

(2) If $0 < a < a(N,p)$, then $u_{\epsilon,a}$ has a local maximum point $x_\epsilon \in \Omega$ with

$$\frac{d(x_\epsilon, \partial\Omega)}{\epsilon} \to d_0 > 0. \tag{1.7}$$

(3) If $a > a(N,p)$, then $u_{\epsilon,a}$ has a unique local maximum point $x_\epsilon \in \Omega$ with

$$d(x_\epsilon, \partial\Omega) \to \max_{x\in\Omega} d(x, \partial\Omega). \tag{1.8}$$

Remarks. 1. The solution in part (2) of Theorem A is called a **near-boundary spike** (see Figure 1 in Section 4).

2. The solution in part (3) of Theorem A is called an **interior spike** (see Figure 2 in Section 4).

Now we consider the stability of the steady state $(A_{\epsilon,a}, \xi_{\epsilon,a})$ to the shadow system (1.3), where

$$\begin{cases} A_{\epsilon,a} = \xi_{\epsilon,a}^{q/(p-1)} u_{\epsilon,a} \\ \xi_{\epsilon,a} = \left(\frac{1}{|\Omega| + \sqrt{D} a_H |\partial\Omega|} \int_\Omega u_{\epsilon,a}^r \, dx \right)^{-(p-1)/(qr-(p-1)(s+1))} \end{cases}. \tag{1.9}$$

and $u_{\epsilon,a}$ is the minimal energy solution of (1.4) given in Theorem A.

In analogy to Theorem A we also call $(A_{\epsilon,a}, \xi_{\epsilon,a})$ a **near-boundary spike** if $0 < a < a(N,p)$ and an **interior spike** if $a > a(N,p)$.

For the Neumann boundary condition a stability result has been be obtained in [34] for

$$r = 2 \quad \text{and} \quad 1 < p < 1 + \frac{4}{N},$$

or

$$r = p + 1 \quad \text{and} \quad 1 < p < \left(\frac{N+2}{N-2} \right). \tag{1.10}$$

In this paper, for Robin boundary conditions, we can give an answer under similar but slightly more restricted conditions.

Our first result implies that if $a > a(N,p)$, then the interior spike is stable.

Theorem 1.1. *(Stability of the interior spike.) Suppose that $a > a(N,p)$. Assume that either*

$$r = 2 \quad \textit{and} \quad 1 < p < 1 + \frac{4}{N}$$

or

$$r = p + 1 \quad \textit{and} \quad 1 < p < \left(\frac{N+2}{N-2} \right)_+.$$

Then there exists $\tau_0 > 0$ such that if $0 < \epsilon << 1$ and $0 \le \tau < \tau_0$ the interior spike $(A_{\epsilon,a}, \xi_{\epsilon,a})$ is a (linearly) stable steady state to the shadow system (1.3).

Our second theorem shows that if $N = 1$, i.e. if Ω is an interval, then in particular for all $1 < p \le 3$ and $0 < a < 1$ the near-boundary is stable.

Theorem 1.2. *(Stability of the near-boundary spike.) Suppose that*

$$N = 1 \quad \textit{and} \quad 0 < a < 1. \tag{1.11}$$

Assume that either

$$r = 2 \quad \textit{and} \quad 1 < p \leq 3$$

or

$$r = p + 1 \quad \textit{and} \quad 1 < p < \infty.$$

Then there exists a $\tau_0 > 0$ *such that if* $0 < \epsilon << 1$ *and* $0 \leq \tau < \tau_0$ *the boundary spike* $(A_{\epsilon,a}, \xi_{\epsilon,a})$ *is a (linearly) stable steady state to the shadow system (1.3).*

Our third and last theorem shows that the near-boundary spike may become **unstable** if the exponent p is increased beyond 3.

Theorem 1.3. *(***Instability** *of the near-boundary spike.) Suppose that (1.11) holds. Assume that* $r = 2$ *and* $p > 3$*. Then there exist* $a_0 > 0$ *and* $\mu_0 > 0$ *such that if*

$$a_0 < a < 1 \quad \textit{and} \quad 1 < \mu := \frac{2q}{(p-1)(s+1)} < \mu_0 \tag{1.12}$$

then for $0 < \epsilon << 1$ *and all* $\tau \geq 0$ *the near-boundary spike* $(A_{\epsilon,a}, \xi_{\epsilon,a})$ *is an unstable steady state to the shadow system (1.3).*

Remarks.

1. The phenomenon described in Theorem 1.3 is new and unexpected. It is important to note that for $N = 1$ and the Neumann boundary condition the minimal energy solution analogous to Theorem A, which is a boundary spike, is stable for all p, q, s such that

$$1 < \mu := \frac{2q}{(p-1)(s+1)},$$

see [33]. This means that the instability given in Theorem 1.3 only arises for the Robin boundary condition and not for the Neumann boundary condition.

In some sense, for the Robin boundary condition the instability which for Neumann boundary conditions occurs only for $p > 5$ is shifted to the range $3 < p \leq 5$.

2. Note that we assume that both the constants $a < 1$ and $\mu := \frac{qr}{(p-1)(s+1)} > 1$ are each sufficiently close to 1.

3. Note that under the conditions (1.12) a proof similar the one for Theorem 1.1 shows that the interior spike is unstable (this proof is omitted). On the other hand, by Theorem 1.3, the near-boundary spike is unstable as well. Thus we do know about any stable spiky steady state and we conjecture that there are none. This behavior is similar to the supercritical case $\mu := \frac{qr}{(p-1)(s+1)} < 1$ for the Neumann boundary condition, see [28]. In all of these situations, due to the non-existence of stable steady states blow-up frequently occurs for the dynamical system. This effect will be shown in the simulations of the dynamical system in the final section of this paper (see Figures 3 and 4.)

Let us now outline the proof of Theorems 1.1 – 1.3 by highlighting the strategy and explaining the main difficulties.

To study the stability of the steady state, we have to linearise (1.3) at (1.9). This results in the following eigenvalue problem:

$$\begin{cases} \epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + p\frac{A_\epsilon^{p-1}}{\xi_\epsilon^q}\phi_\epsilon - q\frac{A_\epsilon^p}{\xi_\epsilon^{q+1}}\eta = \alpha_\epsilon\phi_\epsilon, \\ \frac{r}{\tau|\Omega|}\int_\Omega \frac{A_\epsilon^{r-1}\phi_\epsilon}{\xi_\epsilon^s}dx - \frac{1+s}{\tau}\eta = \alpha_\epsilon\eta, \\ \epsilon\frac{\partial\partial\phi_\epsilon}{\partial\nu} + a\phi = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.13}$$

where $(\phi_\epsilon, \eta_\epsilon)$ in $H^2_{rob}(\Omega) \times R$ and

$$H^2_{rob}(\Omega) = \{\phi \in H^2(\Omega) \,:\, a\phi' + \phi = 0 \quad \text{on } \partial\Omega\}.$$

Using (1.9), it is easy to see that for $a_H = 0$ the eigenvalues of problem (1.13) in $H^2_{rob}(\Omega) \times R$ are the same as the eigenvalues of the eigenvalue problem

$$\epsilon^2\Delta\phi_\epsilon - \phi_\epsilon + pu_\epsilon^{p-1}\phi_\epsilon - \frac{qr}{s+1+\tau\alpha_\epsilon}\frac{\int_\Omega u_\epsilon^{r-1}\phi_\epsilon\,dx}{\int_\Omega u_\epsilon^r\,dx}u_\epsilon^p = \alpha_\epsilon\phi, \quad \phi \in H^2_{rob}(\Omega). \tag{1.14}$$

in $H^2_{rob}(\Omega)$.

When $N = 1$ and $0 < a < 1$, we have $u_{\epsilon,a}(x) \sim w\left(\frac{x-x_\epsilon}{\epsilon}\right) = w\left(\frac{x}{\epsilon} - \frac{x_\epsilon}{\epsilon}\right) =: w_{x_\epsilon/\epsilon}\left(\frac{x}{\epsilon}\right)$, where w is the unique homoclinic solution of the second-order ODE

$$w'' - w + w^p = 0, \quad w > 0, \quad w = w(|y|), \quad w(y) \to 0 \text{ as } |y| \to \infty. \tag{1.15}$$

Further, by the Robin boundary condition, $\frac{x_\epsilon}{\epsilon} \to y_0$, where $y_0 > 0$ is determined by

$$w'(-y_0) = aw(-y_0). \tag{1.16}$$

By (1.9) the steady state of the shadow system (1.3) is given.

Let α_ϵ be an eigenvalue of (1.14). Then the following Lemma holds.

Lemma A.

(1) For $a > a(N,p)$ we have $\alpha_\epsilon = o(1)$ as $\epsilon \to 0$ if and only if $\alpha_\epsilon = (1+o(1))\tau_j^\epsilon$ for some $j = 1, ..., N$, where τ_j^ϵ is given in Theorem 3.4 below (interior spike case).

For $N = 1$ and $a > a(N,p)$ there are no eigenvalues $\alpha_\epsilon = o(1)$ (near boundary spike case).

(2) If $\alpha_\epsilon \to \alpha_0 \neq 0$, then all possible α_0 are given by the eigenvalues of the following eigenvalue problem

$$\Delta\phi - \phi + pw_{y_0}^{p-1}\phi - \frac{qr}{s+1+\tau\alpha_0}\frac{\int_0^\infty w_{y_0}^{r-1}\phi}{\int_0^\infty w_{y_0}^r}w_{y_0}^p = \alpha_0\phi, \tag{1.17}$$

where (i) for $a > a(N,P)$ we have $w_{y_0} = w$, $\phi \in H^1(R)$ (interior spike case) (ii) for $N = 1$ and $a < a(1,P) = 1$ we have $w_{y_0} = w(y-y_0)$, where y_0 is given by the unique solution of $aw'(y_0)+w(y_0) = 0$, and we choose $\phi \in H^2_{rob}(R^+)$ (near-boundary spike case).

Proof. When $a > a(N,p)$ the proof of Part (1) in Lemma A for the Robin boundary condition is similar to that in [34] for the Neumann boundary condition. In both cases, because interior spikes are considered which have exponential decay with respect to the spatial variable, one has to expand the solution to exponential order. There is, however, a major difference in the stability properties. Whereas for the Neumann boundary condition interior spikes are unstable, they are stable for the Robin boundary condition. The difference comes from the fact that the expression $\varphi_{\epsilon,P_\epsilon}(P_\epsilon)$ which plays a major role in the proof (see Section 3), has different signs for Neumann and Robin boundary condition, respectively.

When $N=1$ and $a<1$ the proof of Part (1) in Lemma A for the Robin boundary condition is similar to that in [33] for a boundary spike with the Neumann boundary condition: In both cases there are no small eigenvalues $\alpha_\epsilon = o(1)$.

The proof of (2) follows by a standard limiting process coupled with an argument of Dancer [2].

□

Notice that the eigenvalue problem in Part (ii) of Lemma A (near-boundary spike case) is a half-line nonlocal eigenvalue problem NLEP with a Robin boundary condition. This is a new type of NLEP which to the best of our knowledge has not been studied in the literature before. We will prove results on its spectral and stability properties in the next section.

From now on we assume that $\tau = 0$. By a regular perturbation argument the results also hold for the case of τ being sufficiently small.

2. Study of the NLEP: Proof of Theorems 1.2 and 1.3

In this section, we study the NLEP

$$\phi'' - \phi + pw^{p-1}\phi - \frac{qr}{s+1}\frac{\int_0^\infty w_{y_0}^{r-1}\phi\,dy}{\int_0^\infty w_{y_0}^r\,dy} w_{y_0}^p = \lambda\phi, \quad \phi \in H^1_{rob}(R^+), \tag{2.18}$$

where $w_{y_0}(y) = w(y-y_0)$ for some $y_0 > 0$. Let

$$L_0\phi := \phi'' - \phi + pw_{y_0}^{p-1}\phi, \quad \phi \in H^1_{rob}(R^+).$$

We set

$$L\phi := L_0\phi - \mu(p-1)\frac{\int_0^\infty w_{y_0}^{r-1}\phi\,dy}{\int_0^\infty w_{y_0}^r\,dy} w^p, \quad \phi \in H^1_{rob}(R^+),$$

where

$$\mu = \frac{qr}{(s+1)(p-1)} > 1.$$

We first prove

Lemma 2.1. *Let $\phi \in H^1_{rob}(R^+)$ satisfy*

$$\phi'' - \phi + pw_{y_0}^{p-1}\phi = 0, \quad \|\phi\|_{H^1(R^+)} = 1. \tag{2.19}$$

Then $\phi \equiv 0$.

Proof. Recall that the Robin boundary condition gives

$$a = \frac{w'_{y_0}(0)}{w_{y_0}(0)} \tag{2.20}$$

and by (1.15) w_{y_0} satisfies

$$w''_{y_0} = w_{y_0} - w^p_{y_0}, \quad \left(w'_{y_0}\right)^2 = w^2_{y_0} - \frac{2}{p+1}w^{p+1}_{y_0}. \tag{2.21}$$

We multiply (2.19) by $w_{y_0}^{'}$ and integrate. After integration by parts, using (2.21) and the Robin boundary condition for ϕ, we get

$$0=\phi^{'}(0)w_{y_0}^{'}(0)-\phi(0)w_{y_0}^{''}(0)=\phi(0)\left[aw_{y_0}^{'}(0)-w_{y_0}^{''}(0)\right]. \tag{2.22}$$

By (2.20) and (2.21) it follows that

$$aw_{y_0}^{'}(0)-w_{y_0}^{''}(0)=\frac{(w_{y_0}^{'}(0))^2-w_{y_0}(0)w_{y_0}^{''}(0)}{w_{y_0}(0)}=\frac{(w_{y_0}^{'}(0))^2-(w_{y_0}(0))^2+(w_{y_0}(0))^{p+1}}{w_{y_0}(0)}$$

$$=\frac{p-1}{p+1}\,w_{y_0}^{p}>0. \tag{2.23}$$

Thus from (2.22) we have

$$\phi(0)=0 \tag{2.24}$$

and finally we get $\phi^{'}(0)=0$ by the Robin boundary condition. By the uniqueness properties of ODEs, we conclude that $\phi(y)\equiv 0$ on R^+. The lemma is proved.

□

Lemma 2.1 implies, using the Fredholm Alternative, that the operator L_0, defined on $H^1_{rob}(R^+)$, is invertible.

Since

$$L_0 w_{y_0}=(p-1)w_{y_0}^p, \quad w_{y_0}^{'}(0)-aw_{y_0}(0)=0$$

we have

$$L_0^{-1}(w_{y_0}^p)=\frac{1}{p-1}w_{y_0}. \tag{2.25}$$

Another simple calculation shows that

$$L_0\left(\frac{1}{p-1}w_{y_0}+\frac{1}{2}yw_{y_0}^{'}\right)=w_{y_0}, \tag{2.26}$$

but note that $\frac{1}{p-1}w_{y_0}+\frac{1}{2}yw_{x_0}^{'}$ does not satisfy the Robin boundary condition. Thus, since $\frac{1}{p-1}w_{y_0}+\frac{1}{2}yw_{y_0}^{'}\notin H^1_{rob}(R^+)$, we do **not** have $L_0^{-1}(w_{y_0})=\frac{1}{p-1}w_{y_0}+\frac{1}{2}yw_{y_0}^{'}$. To overcome this difficulty and determine $L_0^{-1}(w_{y_0})$, we prove the following lemma.

Lemma 2.2. *We have*

$$L_0^{-1}(w_{y_0})=\frac{1}{p-1}w_{y_0}+\frac{1}{2}yw_{y_0}^{'}+Aw_{y_0}^{'},$$

where

$$A=\frac{a}{(p-1)(1-a^2)}.$$

Proof. We need to choose A such that

$$A(w_{y_0}^{''}(0)-aw_{y_0}(0))+\frac{1}{2}w_{y_0}^{'}(0)=0.$$

Using (2.23), we get

$$A = \frac{p+1}{2(p-1)} w_{y_0}^{'}(0) w_{y_0}^{-p}(0). \tag{2.27}$$

Now we use the explicit representation of the solution to the problem (1.15) which is given by

$$w(y) = \left(\frac{p+1}{2} \cosh^{-2} \frac{(p-1)y}{2} \right)^{1/(p-1)}.$$

We compute

$$w^{'}(y) = -\tanh \frac{(p-1)y}{2} w(y).$$

This gives

$$\frac{w^{'}(-y_0)}{w(-y_0)} = \tanh \frac{(p-1)y_0}{2} = a.$$

So y_0 can explicitly be expressed in terms of a as

$$y_0 = \frac{2}{p-1} \text{artanh } a$$

and we get

$$w(y_0) = \left(\frac{(p+1)(1-a^2)}{2} \right)^{1/(p-1)}.$$

Inserting this expression into (2.27), we get

$$A = \frac{p+1}{2(p-1)} \left(\frac{p+1}{2} \right)^{-1} \frac{a}{1-a^2} = \frac{a}{(p-1)(1-a^2)}$$

which proves the lemma.

□

Remarks. 1. The extra term $Aw_{y_0}^{'}$ in Lemma 2.2 only appears for Robin boundary conditions and is not present for the Neumann boundary condition. As we will see, the presence of this term under some extra conditions can lead to the destabilization of the near-boundary spike.

2. Note that $A \to \infty$ as $a \to 1$ and $A \to 0$ as $a \to 0$. The first limit will play a major role for the rest of the paper. The second limit is in agreement with intuition since in the limit $a \to 0$ the near-boundary spike for the Robin boundary condition approaches the boundary spike for the Neumann boundary condition, where this term does not occur.

We now compute the (sign of the) expression

$$\rho(y_0) := \int_0^\infty w_{y_0} L_0^{-1}(w_{y_0})\, dy$$

which will play the crucial in the stability analysis of the near-boundary spike.

From Lemma 2.2, we have

$$\begin{aligned} \rho(y_0) &= \int_0^\infty w_{y_0} L_0^{-1}(w_{y_0})\, dy \\ &= \frac{1}{p-1} \int_0^\infty w_{y_0}^2\, dy + \frac{1}{2} \int_0^\infty y w_{y_0} w_{y_0}^{'}\, dy + \frac{A}{p-1} \int_0^\infty w_{y_0} w_{y_0}^{'}\, dy \\ &= \left(\frac{1}{p-1} - \frac{1}{4} \right) \int_0^\infty w_{y_0}^2\, dy - \frac{A}{2(p-1)} w_{y_0}^2(0) \end{aligned}$$

$$= \left(\frac{1}{p-1} - \frac{1}{4}\right) \int_0^\infty w_{y_0}^2\, dy - \frac{a}{2(p-1)^2(1-a^2)} \left(\frac{(p+1)(1-a^2)}{2}\right)^{2/(p-1)}$$

$$= \left(\frac{1}{p-1} - \frac{1}{4}\right) \int_{-y_0}^\infty w^2\, dy - \frac{(p+1)^{2/(p-1)} a}{2^{(p+1)/(p-1)}(p-1)^2(1-a^2)^{(p-3)/(p-1)}}. \tag{2.28}$$

Let us differentiate $\rho(y_0)$ with respect to y_0:

$$\rho^{'}(y_0) = \frac{5-p}{4(p-1)} w_{y_0}^2(0) + \frac{p+1}{4(p-1)} \left[w_{y_0}^{2-p}(0) w_{y_0}^{''}(0) + (2-p) w_{y_0}^{1-p}(0)(w_{y_0}^{'}(0))^2\right]$$

$$= \frac{5-p}{4(p-1)} w_{y_0}^2(0) + \frac{p+1}{4(p-1)} \left[(w_{y_0} - w_{y_0}^p) w_{y_0}^{2-p} + (2-p) w_{y_0}^{1-p} \left(w_{y_0}^2 - \frac{2}{p+1} w_{y_0}^{p+1}\right)\right](0)$$

$$= \frac{(p-1)(3-p)}{4(p-1)} w_{y_0}^{3-p}(0) \tag{2.29}$$

by (2.21). We arrive at the following important proposition.

Proposition 2.3. *Suppose that* $1 < p \leq 3$. *Then*

$$\int_0^\infty w_{y_0} L_0^{-1}(w_{y_0})\, dy > 0. \tag{2.30}$$

Proof. For $1 < p \leq 3$, we get from (2.28) for $y_0 = 0$ (and so also $a = 0$) that

$$\rho(0) = \left(\frac{1}{p-1} - \frac{1}{4}\right) \int_0^\infty w^2\, dy > 0.$$

By (2.29) we compute $\rho^{'}(y_0) \geq 0$ for all $y_0 \in (0, \infty)$ and therefore $\rho(y_0) \geq 0$ for all $y_0 \in [0, \infty)$.

□

We now show that for $p > 3$, in contrast to Proposition 2.3, the integral $\int_0^\infty w_{y_0} L_0^{-1}(w_{y_0})\, dy$ may be negative.

Proposition 2.4. *Suppose that* $p > 3$. *Then for*

$$\frac{5-p}{p-1} \int_{-\frac{2}{p-1}\text{artanh } a}^\infty w^2\, dy < \frac{2^{(p-3)/(p-1)}(p+1)^{2/(p-1)} a}{(p-1)^2(1-a^2)^{(p-3)/(p-1)}} \tag{2.31}$$

it follows that $\int_0^\infty w_{y_0} L_0^{-1}(w_{y_0})\, dy$ *is negative. There exists some constant* $a_0(p) < 1$ *such for* $a_0(p) < a < 1$ *condition (2.31) holds.*

Proof. Condition (2.31) follows immediately from (2.28). The left hand side of (2.31) is positive and remains bounded for all $a \in (0, 1)$. The right hand side of (2.31) tends to 0 as $a \to 0^+$ and to $+\infty$ as $a \to 1^-$. By continuity, there exists some $a_0(p) \in (0, 1)$ such that (2.31) is true for $a_0(p) < a < 1$.

□

Next we need

Lemma 2.5. *The first eigenvalue of* L_0, *which we call* μ_1, *is positive. The second eigenvalue of* L_0 *is negative.*

Proof. Let

$$Q[u] = \frac{\int_0^\infty [(u')^2 + u^2]\, dy + au^2(0)}{(\int_0^\infty u^{p+1}\, dy)^{2/(p+1)}}.$$

Then w_{y_0} up to a scaling factor is the unique minimizer of $Q[u]$ in $H^2_{rob}(R^+)$.

Similar to the proof of Theorem 2.1 of [13], we see that the second eigenvalue of L_0 is non-positive, and hence is negative since by Lemma 2.1 the kernel is trivial.

Now, to study the case $r = 2$, we introduce a new operator

$$L_1\phi := L_0\phi - (p-1)\frac{\int_0^\infty w_{y_0}\phi\, dy}{\int_0^\infty w_{y_0}^2\, dy} w_{y_0}^p - (p-1)\frac{\int_0^\infty w_{y_0}^p\phi\, dy}{\int_0^\infty w_{y_0}^2\, dy} w_{y_0}$$
$$+ (p-1)\frac{\int_0^\infty w_{y_0}^{p+1}\, dy\, \int_0^\infty w_{y_0}\phi\, dy}{(\int_0^\infty w_{y_0}^2\, dy)^2} w_{y_0} \tag{2.32}$$

which is defined for all $\phi \in H^2_{rob}(R^+)$.

Then we have

Lemma 2.6. *(1) The operator L_1 is self-adjoint and the kernel of L_1 (denoted by X_1) is given by span $\{w_{y_0}\}$.*

(2) There exists a positive constant $c_0 > 0$ such that

$$L_1(\phi,\phi) := \int_0^\infty \left[(\phi')^2 + \phi^2 - pw_{y_0}^{p-1}\phi^2\right] dy$$
$$+\frac{2(p-1)\int_0^\infty w_{y_0}\phi\, dy\, \int_0^\infty w_{y_0}^p\phi\, dy}{\int_0^\infty w_{y_0}^2\, dy} - (p-1)\frac{\int_0^\infty w_{y_0}^{p+1}\, dy}{\left(\int_0^\infty w_{y_0}^2\, dy\right)^2}\left(\int_0^\infty w_{y_0}\phi\, dy\right)^2$$
$$\geq c_0 d^2_{L^2(R^+)}(\phi, X_1)$$

for all $\phi \in H^1_{rob}(0,\infty)$, where $d_{L^2(0,\infty)}$ denotes the distance in the L^2-norm.

Proof. By definition (2.32), it is an elementary calculation to show that $(L_1\phi,\psi)_{L^2(0,\infty)} = (L_1\psi,\phi)_{L^2(0,\infty)}$ for all $\phi,\psi \in H^1(0,\infty)$ which implies that the operator L_1 is self-adjoint.

Next we compute the kernel of L_1. It is easy to see that $w_{y_0} \in \text{kernel}(L_1)$. On the other hand, if $\phi \in \text{kernel}(L_1)$, then

$$L_0\phi = c_1(\phi)w_{y_0} + c_2(\phi)w_{y_0}^p = c_1(\phi)L_0\left(\frac{1}{p-1}w_{y_0} + \frac{1}{2}yw_{y_0}' + Aw_{y_0}'\right) + c_2(\phi)L_0\left(\frac{1}{p-1}w_{y_0}\right)$$

by Lemma 2.2, where

$$c_1(\phi) = (p-1)\frac{\int_0^\infty w_{y_0}^p\phi\, dy}{\int_0^\infty w_{y_0}^2\, dy} - (p-1)\frac{\int_0^\infty w_{y_0}^{p+1}\, dy \int_0^\infty w_{y_0}\phi\, dy}{(\int_0^\infty w_{y_0}^2\, dy)^2}, \quad c_2(\phi) = (p-1)\frac{\int_0^\infty w_{y_0}\phi\, dy}{\int_0^\infty w_{y_0}^2\, dy}.$$

Hence

$$\phi = c_1(\phi)L_0^{-1}(w_{y_0}) + c_2(\phi)L_0^{-1}(w_{y_0}^p) = c_1(\phi)L_0^{-1}(w_{y_0}) + \frac{1}{p-1}c_2(\phi)w_{y_0}. \tag{2.33}$$

Note that by (2.33)

$$c_1(\phi) = c_1(\phi)\left[(p-1)\frac{\int_0^\infty w_{y_0}^p L_0^{-1}(w_{y_0})\, dy}{\int_0^\infty w_{y_0}^2\, dy} - (p-1)\frac{\int_0^\infty w_{y_0}^{p+1}\, dy \int_0^\infty w_{y_0}L_0^{-1}(w_{y_0})\, dy}{\left(\int_0^\infty w_{y_0}^2\, dy\right)^2}\right]$$

$$= c_1(\phi)\left[1-(p-1)\frac{\int_0^\infty w_{y_0}^{p+1}\,dy\int_0^\infty w_{y_0}L_0^{-1}(w_{y_0})\,dy}{\left(\int_0^\infty w_{y_0}^2\,dy\right)^2}\right].$$

This implies that $c_1(\phi)=0$. By (2.33) and Lemma 2.1, this proves (1).

It remains to prove (2). Suppose (2) is not true, then by (1) there exists (α,ϕ) such that (i) $\alpha>0$, (ii) $\phi\perp w_{y_0}$, and (iii) $L_1\phi=\alpha\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0-\alpha)\phi=(p-1)\frac{\int_0^\infty w_{y_0}^p\phi\,dy}{\int_0^\infty w_{y_0}^2\,dy}w_{y_0}. \tag{2.34}$$

We first claim that $\int_0^\infty w_{y_0}^p\phi\,dy\neq 0$. In fact, if $\int_0^\infty w_{y_0}^p\phi\,dy=0$, then $\alpha>0$ is the first (principal) eigenvalue of L_0. By Proposition 2.5, $\alpha=\mu_1$ and ϕ has constant sign. This contradicts (ii).

Therefore we must have $\int_0^\infty w_{y_0}^p\phi\,dy\neq 0$ Hence $\alpha\neq\mu_1$ and $L_0-\alpha$ is invertible. So (2.34) implies

$$\phi=(p-1)\frac{\int_0^\infty w_{y_0}^p\phi\,dy}{\int_0^\infty w_{y_0}^2\,dy}(L_0-\alpha)^{-1}w_{y_0}.$$

Thus

$$\int_0^\infty w_{y_0}^p\phi\,dy=(p-1)\frac{\int_0^\infty w_{y_0}^p\phi\,dy}{\int_0^\infty w_{y_0}^2\,dy}\int_0^\infty((L_0-\alpha)^{-1}w_{y_0})w_{y_0}^p\,dy,$$

$$\int_0^\infty w_{y_0}^2\,dy=(p-1)\int_0^\infty((L_0-\alpha)^{-1}w_{y_0})w_{y_0}^p\,dy,$$

$$\int_0^\infty w_{y_0}^2\,dy=\int_0^\infty((L_0-\alpha)^{-1}w_{y_0})((L_0-\alpha)w_{y_0}+\alpha w_{y_0})\,dy,$$

$$0=\int_0^\infty((L_0-\alpha)^{-1}w_{y_0})w_{y_0}\,dy. \tag{2.35}$$

Let $h_1(\alpha)=\int_0^\infty((L_0-\alpha)^{-1}w_{y_0})w_{y_0}\,dy$. Then $h_1(0)=\int_0^\infty(L_0^{-1}w_{y_0})w_{y_0}\,dy=\rho_0(\alpha)>0$ by Proposition 2.3. Moreover $h_1'(\alpha)=\int_0^\infty((L_0-\alpha)^{-2}w_{y_0})w_{y_0}=\int_0^\infty((L_0-\alpha)^{-1}w_{y_0})^2\,dy>0$. This implies $h_1(\alpha)>0$ for all $\alpha\in(0,\mu_1)$. Clearly, since $\lim_{\alpha\to+\infty}h_1(\alpha)=0^-$, we also have $h_1(\alpha)<0$ for $\alpha\in(\mu_1,\infty)$.

This is a contradiction to (2.35), and completes the proof.

□

First we have the following theorem about (in)stability of a near-boundary spike in the case of Robin boundary condition including the exponents $r=2$, $1<p\le 3$, which is similar to results for an interior or a boundary spike in the case of Neumann boundary condition:

Theorem 2.7. *Suppose* $0<a<1$. *If*

$$r=2\quad\textit{and}\quad 1<p\le 3$$

or if

$$r=p+1\quad\textit{and}\quad 1<p<\infty,$$

then the following NLEP

$$\left\{\ \phi''-\phi+pw_{y_0}^{p-1}\phi-\mu(p-1)\frac{\int_0^\infty w_{y_0}^{r-1}\phi\,dy}{\int_0^\infty w_{y_0}^r\,dy}w_{y_0}^p=\lambda\phi,\quad \phi\in H^1_{rob}(R^+),\right. \tag{2.36}$$

is stable for $\mu>1$ *and unstable for* $\mu<1$.

In contrast, for the exponents $r = 2$, $p > 3$, we have the following instability result for the near-boundary spike in the case of Robin boundary condition:

Theorem 2.8. *If*

$$r = 2 \quad \text{and} \quad p > 3$$

then there exist some $a_0 \in (0,1)$ and $\mu_0(a) > 1$ such that for

$$a_0 < a < 1 \tag{2.37}$$

and

$$1 < \mu < \mu_0(a) \tag{2.38}$$

the NLEP (2.36) has a positive eigenvalue.

Remark. The number a_0 can be chosen according to (2.31).

To show the instability part in Theorems 2.7 and 2.8, we first prove the following result.

Theorem 2.9. *(1) If $\mu < 1$ and $r = 2$, $1 < p \le 3$ or $r = p+1$, $1 < p < \infty$, the NLEP (2.36) has a positive eigenvalue.*

(2) If $r = 2$ and

$$\int_0^\infty w_{y_0} L_0^{-1} w_{y_0}\, dy < 0,$$

then under the condition (2.38) the NLEP (2.36) has a positive eigenvalue.

Proof.

(1) Suppose $\mu < 1$. We look for a positive eigenvalue α to (2.36) which is equivalent to

$$\begin{cases} \phi = \mu(p-1)\frac{\int_0^\infty w_{y_0}^{r-1}\phi\, dy}{\int_0^\infty w_{y_0}^r\, dy}(L_0-\alpha)^{-1}w_{y_0}^p, \quad 0 < y < +\infty, \\ \phi'(0) - a\phi(0) = 0. \end{cases}$$

Multiplying by $w_{y_0}^{r-1}$ and integrating, we get

$$\int_0^\infty w_{y_0}^r\, dy = \mu(p-1)\int_0^\infty \left[(L_0-\alpha)^{-1}w_{y_0}^p\right] w_{y_0}^{r-1}\, dy.$$

Using the identity

$$(p-1)(L_0-\alpha)^{-1}w_{y_0}^{p-1} = w_{y_0} + \alpha(L_0-\alpha)^{-1}w_{y_0}$$

we get

$$\int_0^\infty w_{y_0}^r\, dy = \mu\left(\int_0^\infty w_{y_0}^r\, dy + \alpha\int_0^\infty \left[(L_0-\alpha)^{-1}w_{y_0}\right] w_{y_0}^{r-1}\, dy\right)$$

which is equivalent to

$$\frac{1}{\alpha}\left(\frac{1}{\mu}-1\right)\int_0^\infty w_{y_0}^r\, dy = \int_0^\infty \left[(L_0-\alpha)^{-1}w_{y_0}\right] w_{y_0}^{r-1}\, dy. \tag{2.39}$$

If $r = 2$ and $1 < p \le 3$ then by Proposition 2.3 the right hand side of (2.39) is positive for $\alpha = 0$.

If $r = p+1$ and $1 < p < \infty$ then the right hand side of (2.39) is positive for $\alpha = 0$ since

$$\int_0^\infty \left[L_0^{-1}w_{y_0}\right] w_{y_0}^p\, dy = \frac{1}{p-1}\int_0^\infty w_{y_0}^2\, dy > 0.$$

Therefore, as $\alpha \to 0^+$, the left hand side of (2.39) tends to $+\infty$ while the right hand side tends to some positive number. As $\alpha \to \mu_1^-$, the left hand side tends to some positive number while the right hand side tends to $+\infty$. By continuity, there exists a solution to (2.39).

(2) If $r = 2$, then (2.39) becomes

$$\frac{1}{\alpha}\left(\frac{1}{\mu}-1\right)\int_0^\infty w_{y_0}^2\,dy = \int_0^\infty \left[(L_0-\alpha)^{-1}w_{y_0}\right]w_{y_0}\,dy. \tag{2.40}$$

As $\alpha \to 0^+$, the left hand side of (2.40) tends to $-\infty$ while the right hand side tends to some negative number. As $\alpha \to \mu_1^-$, the left hand side tends to some negative number while the right hand side tends to $-\infty$.

By continuity, there exists a solution to (2.40).

□

Proof of Theorem 2.8:

The proof of Theorem 2.8 is completed by combining Proposition 2.4 and part (2) of Theorem 2.9.

□

Proof of Theorem 2.7:

The instability part of Theorem 2.7 is contained in part (1) of Theorem 2.9.

Now we prove the stability part of Theorem 2.7. We divide the proof into two cases:

Case 1. $r = 2, \quad 1 < p \le 3$.

Case 2. $r = p+1, \quad 1 < p < \infty$.

Let $\alpha_0 = \alpha_R + i\alpha_I$ be an eigenvalue and $\phi = \phi_R + i\phi_I$ an eigenfunction of (2.36). Since $\alpha_0 \neq 0$, we can choose $\phi \perp \text{kernel}(L_0)$. Then we obtain the two equations

$$L_0\phi_R - (p-1)\mu\frac{\int_0^\infty w_{y_0}\phi_R\,dy}{\int_0^\infty w_{y_0}^2\,dy}w_{y_0}^p = \alpha_R\phi_R - \alpha_I\phi_I, \tag{2.41}$$

$$L_0\phi_I - (p-1)\mu\frac{\int_0^\infty w_{y_0}\phi_I\,dy}{\int_0^\infty w_{y_0}^2\,dy}w_{y_0}^p = \alpha_R\phi_I + \alpha_I\phi_R. \tag{2.42}$$

Multiplying (2.41) by ϕ_R and (2.42) by ϕ_I and adding the two equations, we obtain

$$\begin{aligned} -\alpha_R\int_0^\infty(\phi_R^2+\phi_I^2)\,dy &= L_1(\phi_R,\phi_R) + L_1(\phi_I,\phi_I) \\ &+(p-1)(\mu-2)\frac{\int_0^\infty w_{y_0}\phi_R\,dy\int_0^\infty w_{y_0}^p\phi_R\,dy + \int_0^\infty w_{y_0}\phi_I\,dy\int_0^\infty w_{y_0}^p\phi_I\,dy}{\int_0^\infty w_{y_0}^2\,dy} \\ &+(p-1)\frac{\int_0^\infty w_{y_0}^{p+1}\,dy}{(\int_0^\infty w_{y_0}^2)^2\,dy}\left[\left(\int_0^\infty w_{y_0}\phi_R\,dy\right)^2 + \left(\int_0^\infty w_{y_0}\phi_I\,dy\right)^2\right]. \end{aligned}$$

Multiplying (2.41) by w_{y_0} and (2.42) by w_{y_0}, respectively, and integrating we obtain

$$(p-1)\int_0^\infty w_{y_0}^p\phi_R\,dy - (p-1)\mu\frac{\int_0^\infty w_{y_0}\phi_R\,dy}{\int_0^\infty w_{y_0}^2\,dy}\int_0^\infty w_{y_0}^{p+1}\,dy = \alpha_R\int_0^\infty w_{y_0}\phi_R\,dy - \alpha_I\int_0^\infty w_{y_0}\phi_I\,dy, \tag{2.43}$$

$$(p-1)\int_0^\infty w_{y_0}^p\phi_I\,dy - (p-1)\mu\frac{\int_0^\infty w_{y_0}\phi_I\,dy}{\int_0^\infty w_{y_0}^2\,dy}\int_0^\infty w_{y_0}^{p+1}\,dy = \alpha_R\int_0^\infty w_{y_0}\phi_I\,dy + \alpha_I\int_0^\infty w_{y_0}\phi_R\,dy. \tag{2.44}$$

Multiplying (2.43) by $\int_0^\infty w_{y_0}\phi_R\,dy$ and (2.44) by $\int_0^\infty w_{y_0}\phi_I\,dy$ and adding, we obtain

$$(p-1)\int_0^\infty w_{y_0}\phi_R\,dy\int_0^\infty w_{y_0}^p\phi_R\,dy+(p-1)\int_0^\infty w_{y_0}\phi_I\,dy\int_0^\infty w_{y_0}^p\phi_I\,dy$$
$$=\left(\alpha_R+(p-1)\mu\frac{\int_0^\infty w_{y_0}^{p+1}\,dy}{\int_0^\infty w_{y_0}^2\,dy}\right)\left(\left(\int_0^\infty w_{y_0}\phi_R\,dy\right)^2+\left[\int_0^\infty w_{y_0}\phi_I\,dy\right)^2\right].$$

Therefore we have

$$-\alpha_R\int_0^\infty(\phi_R^2+\phi_I^2)\,dy=L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)$$
$$+(p-1)(\mu-2)\left(\frac{1}{p-1}\alpha_R+\mu\frac{\int_0^\infty w_{y_0}^{p+1}\,dy}{\int_0^\infty w_{y_0}^2\,dy}\right)\frac{(\int_0^\infty w_{y_0}\phi_R\,dy)^2+(\int_0^\infty w_{y_0}\phi_I\,dy)^2}{\int_0^\infty w_{y_0}^2\,dy}$$
$$+(p-1)\frac{\int_0^\infty w_{y_0}^{p+1}\,dy}{\left(\int_0^\infty w_{y_0}^2\right)^2}\left[\left(\int_0^\infty w_{y_0}\phi_R\,dy\right)^2+\left(\int_0^\infty w_{y_0}\phi_I\,dy\right)^2\right].$$

Set

$$\phi_R=c_Rw_{y_0}+\phi_R^\perp,\quad \phi_R^\perp\perp X_1,\qquad \phi_I=c_Iw_{y_0}+\phi_I^\perp,\quad \phi_I^\perp\perp X_1,$$

where X_1 was defined in Lemma 2.6. Then

$$\int_0^\infty w_{y_0}\phi_R\,dy=c_R\int_0^\infty w_{y_0}^2\,dy,\quad \int_0^\infty w_{y_0}\phi_I\,dy=c_I\int_0^\infty w_{y_0}^2\,dy,$$
$$d^2_{L^2(R^+)}(\phi_R,X_1)=\|\phi_R^\perp\|^2_{L^2},\quad d^2_{L^2(R^+)}(\phi_I,X_1)=\|\phi_I^\perp\|^2_{L^2}.$$

By some straightforward computations, we have

$$L_1(\phi_R,\phi_R)+L_1(\phi_I,\phi_I)$$
$$+(\mu-1)\alpha_R(c_R^2+c_I^2)\int_0^\infty w_{y_0}^2\,dy+(p-1)(\mu-1)^2(c_R^2+c_I^2)\int_0^\infty w_{y_0}^{p+1}\,dy+\alpha_R(\|\phi_R^\perp\|^2_{L^2}+\|\phi_I^\perp\|^2_{L^2})=0.$$

By Lemma 2.6 (2), we get

$$(\mu-1)\alpha_R(c_R^2+c_I^2)\int_0^\infty w_{y_0}^2\,dy$$
$$+(p-1)(\mu-1)^2(c_R^2+c_I^2)\int_0^\infty w_{y_0}^{p+1}\,dy+(\alpha_R+a_1)(\|\phi_R^\perp\|^2_{L^2}+\|\phi_I^\perp\|^2_{L^2})\le 0.$$

Since $\mu>1$, we must have $\alpha_R<0$, which proves Theorem 2.7 in Case 1: $r=2$, $1<p\le 3$.

Now we consider **Case 2:** $r=p+1, 1<p<\infty$.

Then the nonlocal operator in (2.36) becomes

$$L\phi=L_0\phi-\mu(p-1)\frac{\int_0^\infty w_{y_0}^p\phi\,dy}{\int_0^\infty w_{y_0}^{p+1}\,dy}w_{y_0}^p.$$

We need to define yet another new operator:

$$L_2\phi:=L_0\phi-(p-1)\frac{\int_0^\infty w_{y_0}^p\phi\,dy}{\int_0^\infty w_{y_0}^{p+1}\,dy}w_{y_0}^p. \tag{2.45}$$

We have the following result.

Lemma 2.10. *(1) L_2 is self-adjoint and the kernel of L_2 (denoted by X_2) is spanned by w_{y_0}.*

(2) There exists a positive constant $c_3 > 0$ such that

$$L_2(\phi,\phi) := \int_0^\infty \left[(\phi')^2 + \phi^2 - p w_{y_0}^{p-1}\phi^2\right] dy + \frac{(p-1)\left(\int_0^\infty w_{y_0}^p \phi\, dy\right)^2}{\int_0^\infty w_{y_0}^{p+1}\, dy}$$
$$\geq c_3 d^2_{L^2(R^+)}(\phi, X_2), \ \forall \phi \in H^1_{rob}(R^+).$$

Proof:

The proof of (1) is similar to that of Lemma 2.6. We omit the details. It remains to prove (2). Suppose (2) is not true, then by (1) there exists (α,ϕ) such that (i) $\alpha > 0$, (ii) $\phi \perp w_{y_0}$, and (iii) $L_2\phi = \alpha\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha)\phi = \frac{(p-1)\int_0^\infty w_{y_0}^p \phi\, dy}{\int_0^\infty w_{y_0}^{p+1}\, dy} w_{y_0}^p. \tag{2.46}$$

Similar to the proof of Lemma 2.6, we have that $\int_0^\infty w_{y_0}^p \phi\, dy \neq 0$ and $\alpha \neq \mu_1$. Hence $L_0 - \alpha$ is invertible. So (2.46) implies

$$\phi = \frac{(p-1)\int_0^\infty w_{y_0}^p \phi\, dy}{\int_0^\infty w_{y_0}^{p+1}\, dy}(L_0-\alpha)^{-1} w_{y_0}^p.$$

Thus

$$\int_0^\infty w_{y_0}^p \phi\, dy = (p-1)\frac{\int_0^\infty w_{y_0}^p \phi\, dy}{\int_0^\infty w_{y_0}^{p+1}\, dy}\int_0^\infty ((L_0-\alpha)^{-1}w_{y_0}^p) w_{y_0}^p\, dy$$

and

$$\int_0^\infty w_{y_0}^{p+1}\, dy = (p-1)\int_0^\infty ((L_0-\alpha)^{-1}w_{y_0}^p) w_{y_0}^p\, dy. \tag{2.47}$$

Let

$$h_2(\alpha) = (p-1)\int_0^\infty ((L_0-\alpha)^{-1}w_{y_0}^p) w_{y_0}^p\, dy - \int_0^\infty w_{y_0}^{p+1}\, dy.$$

Then we have

$$h_2(0) = (p-1)\int_0^\infty (L_0^{-1}w_{y_0}^p) w_{y_0}^p\, dy - \int_0^\infty w_{y_0}^{p+1}\, dy = 0.$$

Moreover, we compute

$$h_2'(\alpha) = (p-1)\int_0^\infty ((L_0-\alpha)^{-2}w_{y_0}^p) w_{y_0}^p\, dy = (p-1)\int_0^\infty ((L_0-\alpha)^{-1}w_{y_0}^p)^2\, dy > 0.$$

This implies $h_2(\alpha) > 0$ for all $\alpha \in (0,\mu_1)$. Clearly, also $h_2(\alpha) < 0$ for $\alpha \in (\mu_1,\infty)$. This is a contradiction to (2.47) and the lemma is proved.

□

We now finish the proof of Theorem 2.7 in Case 2.

Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp \text{kernel}(L_0)$. Then, similarly to Case 1, we obtain the two equations

$$L_0\phi_R - (p-1)\mu\frac{\int_0^\infty w_{y_0}^p \phi_R\, dy}{\int_0^\infty w_{y_0}^{p+1}\, dy} w_{y_0}^p = \alpha_R\phi_R - \alpha_I\phi_I, \tag{2.48}$$

$$L_0\phi_I-(p-1)\mu\frac{\int_0^\infty w_{y_0}^p\phi_I\,dy}{\int_0^\infty w_{y_0}^{p+1}\,dy}w_{y_0}^p=\alpha_R\phi_I+\alpha_I\phi_R. \tag{2.49}$$

Multiplying (2.48) by ϕ_R and (2.49) by ϕ_I, integrating and adding, we obtain

$$-\alpha_R\int_0^\infty(\phi_R^2+\phi_I^2)\,dy=L_2(\phi_R,\phi_R)+L_2(\phi_I,\phi_I)$$
$$+(p-1)(\mu-1)\frac{(\int_0^\infty w_{y_0}^p\phi_R\,dy)^2+(\int_0^\infty w_{y_0}^p\phi_I\,dy)^2}{\int_0^\infty w_{y_0}^{p+1}\,dy}.$$

By Lemma 2.10 (2)

$$\alpha_R\int_0^\infty(\phi_R^2+\phi_I^2)+a_2d_{L^2}^2(\phi,X_1)+(p-1)(\mu-1)\frac{(\int_0^\infty w_{y_0}^p\phi_R)^2+(\int_0^\infty w_{y_0}^p\phi_I)^2}{\int_0^\infty w_{y_0}^{p+1}}\le 0$$

which implies $\alpha_R<0$ since $\mu>1$.

Theorem 2.7 is thus proved in Case 2: $r=p+1$, $1<p<\infty$.

□

Note that Theorem 2.7 implies Theorem 1.2, and Theorem 2.8 implies Theorem 1.3.

3. Eigenvalue Estimates: Proof of Theorem 1.1

In this section, we shall study eigenvalue estimates for $L_\epsilon:=\epsilon^2\Delta-1+p(u_\epsilon)^{p-1}$ and finish the proof of Theorem 1.1.

We will state a theorem for the small (i.e. $o(1)$) eigenvalues. But before we do this, let us first introduce some notation and give some important lemmas.

Let

$$d\mu_{P_0}(z)=\lim_{\epsilon\to 0}\frac{e^{-\frac{2|z-P_0|}{\epsilon}}dz}{\int_{\partial\Omega}e^{-\frac{2|z-P_0|}{\epsilon}}dz}. \tag{3.1}$$

It is easy to see that the support of $d\mu_{P_0}(z)$ is contained in $\bar{B}_{d(P_0,\partial\Omega)}(P_0)\cap\partial\Omega$.

A point P_0 is called a "nondegenerate peak point" if the following statements (H1) and (H2) hold: There exists $a\in R^N$ such that

$$\int_{\partial\Omega}e^{<z-P_0,a>}(z-P_0)d\mu_{P_0}(z)=0 \tag{H1}$$

and

$$\Big(\int_{\partial\Omega}e^{<z-P_0,a>}(z-P_0)_i(z-P_0)_jd\mu_{P_0}(z)\Big):=G(P_0)\text{ is nonsingular.} \tag{H2}$$

Such a vector a is unique. Moreover, $G(P_0)$ is a positive definite matrix. A geometric characterization of a nondegenerate peak point P_0 is the following:

$$P_0\in\text{ interior (convex hull of support}(d\mu_{P_0}(z)).$$

For a proof of the above, see Theorem 5.1 of [30].

Next, we introduce the following definition:

For each $P\in\Omega$, let $w_{\epsilon,P}$ be the unique solution of

$$\epsilon^2\Delta u-u+w^p\left(\frac{x-P}{\epsilon}\right)=0\text{ in }\Omega,\quad \epsilon\frac{\partial u}{\partial\nu}+au=0\text{ on }\partial\Omega. \tag{3.2}$$

Let $\varphi_{\epsilon,P}(x) = w\left(\frac{x-P}{\epsilon}\right) - w_{\epsilon,P}(x)$. Then $\varphi_{\epsilon,P}$ satisfies

$$\begin{cases} \epsilon^2\Delta\varphi_{\epsilon,P} - \varphi_{\epsilon,P} = 0 & \text{in } \Omega, \\ a\varphi_{\epsilon,P} + \epsilon\frac{\partial\varphi_{\epsilon,P}}{\partial\nu} = aw\left(\frac{x-P}{\epsilon}\right) + \epsilon\frac{\partial w\left(\frac{x-P}{\epsilon}\right)}{\partial\nu} & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

For $x \in \partial\Omega$, we have

$$aw\left(\frac{x-P}{\epsilon}\right) + \epsilon\frac{\partial w\left(\frac{x-P}{\epsilon}\right)}{\partial\nu} = aw\left(\frac{x-P}{\epsilon}\right) + w^{'}\left(\frac{x-P}{\epsilon}\right)\frac{< x-P,\nu >}{|x-P|}$$

$$= w\left(\frac{x-P}{\epsilon}\right)\left(a - \frac{< x-P,\nu >}{|x-P|} + O\left(\frac{\epsilon}{d(P,\partial\Omega)}\right)\right) \geq (a-1-\delta)w\left(\frac{x-P}{\epsilon}\right),$$

where $w^{'}(y) = \frac{dw(r)}{dr}$ for $r = |y|$ and $a - 1 - \delta > 0$. Therefore, there exist two positive constants C_1 and C_2 such that

$$C_1\varphi_{\epsilon,P,1} \leq \varphi_{\epsilon,P} \leq C_2\varphi_{\epsilon,P,1}, \tag{3.4}$$

where $\varphi_{\epsilon,P,1}$ satisfies

$$\begin{cases} \epsilon^2\Delta\varphi_{\epsilon,P,1} - \varphi_{\epsilon,P,1} = 0 & \text{in } \Omega, \\ \varphi_{\epsilon,P,1} + a^{-1}\epsilon\frac{\partial\varphi_{\epsilon,P,1}}{\partial\nu} = w\left(\frac{x-P}{\epsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

The study of $\varphi_{\epsilon,P,1}$ depends on the following lemma.

Lemma 3.1. *(Theorem 3.8 of* [32]*.) Suppose that $d(P,\partial\Omega) > d_0 > 0$. Let $\varphi^D_{\epsilon,P}$ be the unique solution of*

$$\begin{cases} \epsilon^2\Delta\varphi^D_{\epsilon,P} - \varphi^D_{\epsilon,P} = 0 & \text{in } \Omega, \\ \varphi^D_{\epsilon,P} = w\left(\frac{x-P}{\epsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Then for, any arbitrarily small $\delta > 0$, the following holds for ϵ sufficiently small:

$$\left|\epsilon\frac{\partial\varphi^D_{\epsilon,P}}{\partial\nu}\right| \leq (1+\delta)\varphi^D_{\epsilon,P}. \tag{3.7}$$

From Lemma 3.1 we conclude that on $\partial\Omega$

$$\varphi^D_{\epsilon,P} + a^{-1}\epsilon\frac{\varphi^D_{\epsilon,P}}{\partial\nu} \leq \varphi^D_{\epsilon,P}(1 + a^{-1}(1+\delta)) \leq (1 + a^{-1}(1+\delta))w\left(\frac{x-P}{\epsilon}\right)$$

and

$$\varphi^D_{\epsilon,P} + a^{-1}\epsilon\frac{\varphi^D_{\epsilon,P}}{\partial\nu} \geq \varphi^D_{\epsilon,P}(1 - a^{-1}(1-\delta)) \geq (1 - a^{-1}(1-\delta))w\left(\frac{x-P}{\epsilon}\right).$$

Using a comparison principle, it is straightforward to derive the following lemma:

Lemma 3.2. *There exist two positive constants C_1 and C_2 such that*

$$C_1\varphi^D_{\epsilon,P} \leq \varphi_{\epsilon,P,1} \leq C_2\varphi^D_{\epsilon,P}.$$

The convergence of (3.6) is well understood. It is studied in Section 4 of [20]. By Lemma 4.6 of [20] we have the following convergence results:

Lemma 3.3. *(i) Let* $V_\epsilon(y) := \varphi_{\epsilon,x_\epsilon}(x_\epsilon+\epsilon y)/\varphi_{\epsilon,x_\epsilon}(x_\epsilon)$. *Then* $V_\epsilon(y)\to V_0(y)$ *locally, where* $V_0(y)$ *is a solution of*

$$\Delta u - u = 0, \quad u(0)=1, \quad u>0 \ \ \text{in } R^N. \tag{3.8}$$

Moreover, for any $\sigma>0$,

$$\sup_{y\in\Omega_\epsilon} e^{-(1+\sigma)|y|}V_\epsilon(y)-V_0(y)\to 0. \tag{3.9}$$

(ii) As $\epsilon\to 0$,

$$-\epsilon\log(\varphi_{\epsilon,x_\epsilon}(x_\epsilon))\to 2d(x_0,\partial\Omega). \tag{3.10}$$

For $P\in\Omega$, let

$$\Omega_{\epsilon,P}=\{y|\epsilon y+P\in\Omega\},$$

$$S_\epsilon(u)=\Delta u-u+u^p \quad \text{for } u\in H^2_{rob}(\Omega_{\epsilon,P}), \qquad \partial_j=\frac{\partial}{\partial P_j},$$

$\mathcal{K}_{\epsilon,P}=\text{span }\{\partial_j w_{\epsilon,P}|j=1,...,N\}\subset H^1_{rob}(\Omega_{\epsilon,P})$, $\mathcal{K}^\perp_{\epsilon,P}=\left\{u\in H^1_{rob}(\Omega_{\epsilon,P})|\int_\Omega u\partial_j w_{\epsilon,P}=0, j=1,...,N\right\}$,

and

$$\mathcal{C}_{\epsilon,P}=\text{span }\{\partial_j w_{\epsilon,P}|j=1,...,N\}\subset L^2(\Omega_{\epsilon,P}),\ \mathcal{C}^\perp_{\epsilon,P}=\left\{u\in L^2(\Omega_{\epsilon,P})|\int_\Omega u\partial_j w_{\epsilon,P}=0, j=1,...,N\right\}.$$

Let $Q^0_\epsilon := P_0+\epsilon\frac{1}{2}d(P_0,\partial\Omega)a$, where P_0 is a nondegenerate peak-point (i.e. it satisfies ((H1)) and ((H2))) and $\Lambda := B_{\beta_0\epsilon}(Q^0_\epsilon)$, where β_0 is sufficiently small.

For each $P\in\Lambda$ we can find a solution $\varphi_{\epsilon,P}\in\mathcal{K}^\perp_{\epsilon,P}$ such that

$$S_\epsilon(w_{\epsilon,P}+\varphi_{\epsilon,P})\in\mathcal{C}_{\epsilon,P}$$

as was shown in [1].

Now we state our theorem on the small eigenvalues.

Theorem 3.4. *The eigenvalue problem*

$$\epsilon^2\Delta\phi-\phi+pu_\epsilon^{p-1}\phi=\tau^\epsilon\phi \ \ \text{in } \Omega,\ \frac{\partial\phi}{\partial\nu}+a\phi=0 \quad \text{on } \partial\Omega \tag{3.11}$$

admits the following set of $o(1)$) *eigenvalues:*

$$\tau^\epsilon_j=(c_0+o(1))\varphi_{\epsilon,P_0}(P_0)\lambda_j, \quad j=1,...,N,$$

where $\lambda_j, j=1,...,N$ *are the eigenvalues of the matrix* $G(P_0)$ *introduced in (H2) and*

$$c_0=2d^{-2}(P_0,\partial\Omega)\frac{\int_{R^N}pw^{p-1}w'V_0'(r)\,dy}{\int_{R^N}\left(\frac{\partial w}{\partial y_1}\right)^2 dy}<0, \tag{3.12}$$

where $V_r(r)$ *is the unique radial solution of the problem (3.8). Furthermore, the eigenfunction (suitably normalized) corresponding to* τ^ϵ_j, $j=1,...,N$ *is given by*

$$\phi^\epsilon_j=\sum_{l=1}^N(a_{j,l}+o(1))\epsilon\frac{\partial w_{\epsilon,P}}{\partial P_l}|_{P=P_\epsilon}, \tag{3.13}$$

where $\vec{a}_j = (a_{j,1}, ..., a_{j,N})^t$ *is the eigenvector of* $G(P_0)$ *corresponding to* λ_j*, namely*

$$G(P_0)\vec{a}_j = \lambda_j \vec{a}_j, \quad j = 1, ..., N.$$

Proof of Theorem 3.4: Let

$$u_\epsilon = w_{\epsilon,Q_\epsilon} + v_{\epsilon,Q_\epsilon}.$$

Let $(\tau^\epsilon, \phi_\epsilon)$ be a pair such that

$$L_\epsilon \phi_\epsilon = \tau^\epsilon \phi_\epsilon \text{ in } \Omega, \quad \frac{\partial \phi_\epsilon}{\partial \nu} + a\phi = 0 \text{ on } \partial\Omega. \tag{3.14}$$

We normalize ϕ_ϵ such that $\|\phi_\epsilon\|_\epsilon = 1$.

We now assume that $\tau_\epsilon \to 0$ as $\epsilon \to 0$. Then after a scaling and limiting process (see [17], [18] and [20]), we have $\tilde{\phi}_\epsilon(y) = \phi_\epsilon(Q_\epsilon + \epsilon y) \to \phi_0$, where ϕ_0 is a solution of

$$\Delta v - v + pw^{p-1} v = 0 \text{ in } R^N, \quad v \in H^1(R^N).$$

By Lemma 4.2 of [18], there exists s_j such that $\phi_0 = \sum_{j=1}^N s_j \frac{\partial w}{\partial y_j}$.

This suggests that we decompose ϕ_ϵ as $\phi_\epsilon = \sum_{j=1}^N s_j^\epsilon \epsilon \partial_j w_{\epsilon,Q_\epsilon} + \bar{\phi}_\epsilon$, where $\bar{\phi}_\epsilon \in \mathcal{K}^\perp_{\epsilon,Q_\epsilon}$ and $|s_j^\epsilon| \le C$. Since $\|\phi_\epsilon\|_\epsilon = 1$, we have $\|\bar{\phi}_\epsilon\|_\epsilon \le C$ and $\bar{\phi}_\epsilon$ satisfies

$$(L_\epsilon - \tau^\epsilon)\bar{\phi}_\epsilon + \sum_{j=1}^N s_j^\epsilon [p(u_\epsilon)^{p-1} \epsilon \partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1} \epsilon \partial_j w] = \tau^\epsilon \sum_{j=1}^N s_j^\epsilon \epsilon \partial_j w_{\epsilon,Q_\epsilon}. \tag{3.15}$$

Since $\tau^\epsilon \to 0$, then by the same argument as in Proposition 6.3 of [31] we have that $\pi_{\epsilon,Q_\epsilon} \circ (L_\epsilon - \tau^\epsilon) : \mathcal{K}^\perp_{\epsilon,Q_\epsilon} \to \mathcal{C}^\perp_{\epsilon,Q_\epsilon}$ is invertible. Since $\bar{\phi}_\epsilon \in \mathcal{K}^\perp_{\epsilon,Q_\epsilon}$, we have

$$\|\bar{\phi}_\epsilon\|_{H^1(\Omega_{\epsilon,Q_\epsilon})} = O\left(\left\| \sum_{j=1}^N s_j^\epsilon [p(u_\epsilon)^{p-1} \epsilon \partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1} \epsilon \partial_j w] \right\|_{L^2(\Omega_{\epsilon,Q_\epsilon})} \right)$$
$$= O\left((|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{(1+\sigma)/2}) \sum_{j=1}^N |s_j^\epsilon| \right).$$

Multiplying (3.15) by $\epsilon \partial_k (w_{\epsilon,Q_\epsilon})$ and integrating, we obtain

$$\sum_{j=1}^N s_j^\epsilon \int_{\Omega_{\epsilon,Q_\epsilon}} [p(u_\epsilon)^{p-1} \epsilon \partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1} \epsilon \partial_j w] \epsilon \partial_k w_{\epsilon,Q_\epsilon} \, dx$$
$$= \tau^\epsilon \sum_{j=1}^N \int_{\Omega_{\epsilon,Q_\epsilon}} s_j^\epsilon \epsilon \partial_j w_{\epsilon,Q_\epsilon} \epsilon \partial_k w_{\epsilon,Q_\epsilon} \, dx$$
$$+ \int_{\Omega_{\epsilon,Q_\epsilon}} [p(u_\epsilon)^{p-1} \bar{\phi}_\epsilon \epsilon \partial_k (w_{\epsilon,Q_\epsilon}) - pw^{p-1} \bar{\phi}_\epsilon \epsilon \partial_k w] \, dx + O(|\tau_\epsilon| \|\bar{\phi}_\epsilon\|_{H^1(\Omega_{\epsilon,Q_\epsilon})}) \tag{3.16}$$

We first estimate the left-hand side of (3.16). To begin with, we calculate,

$$-\int_{\Omega_{\epsilon,Q_\epsilon}} \left[pw^{p-1} \epsilon \frac{\partial w}{\partial P_j}|_{P=Q_\epsilon} - p(w_{\epsilon,Q_\epsilon} + v_{\epsilon,Q_\epsilon})^{p-1} \epsilon \frac{\partial w_{\epsilon,P}}{\partial P_k}|_{P=Q_\epsilon} \right] dy$$
$$= -\epsilon^2 \int_{\Omega_{\epsilon,Q_\epsilon}} \left[pw^{p-1} \frac{\partial w}{\partial P_j}|_{P=Q_\epsilon} - p(w_{\epsilon,Q_\epsilon})^{p-1} \frac{\partial w_{\epsilon,P}}{\partial P_j}|_{P=Q_\epsilon} \right] \frac{\partial w_{\epsilon,P}}{\partial P_k}|_{P=Q_\epsilon} \, dy + O(|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= -\epsilon^2 \int_{\Omega_{\epsilon,Q_\epsilon}} \frac{\partial}{\partial P_j}|_{P=Q_\epsilon} [w^p - (w_{\epsilon,Q_\epsilon})^p] \frac{\partial w_{\epsilon,P}}{\partial P_k}|_{P=Q_\epsilon} dy + O(|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= -\epsilon^2 \int_{\Omega_{\epsilon,Q_\epsilon}} \frac{\partial}{\partial P_j}|_{P=Q_\epsilon} \left[pw^{p-1}\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon+\epsilon y)\right] \frac{\partial w_{\epsilon,P}}{\partial P_k}|_{P=Q_\epsilon} + O(|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= 2\varphi_{\epsilon,P_0}(P_0)(1+o(1))$$

$$\times \int_{R^N} pw^{p-1} \int_{\partial\Omega} e^{<\frac{z-P_0}{|z-P_0|},y>} e^{<\frac{z-P_0}{|z-P_0|},\frac{2(Q_\epsilon-P_0)}{\epsilon}>} d\mu_{P_0}(z) \left(\frac{z-P_0}{|z-P_0|}\right)_j \frac{\partial w}{\partial y_k}\, dy + O(|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= 2\varphi_{\epsilon,P_0}(P_0)(1+o(1))$$

$$\times \int_{R^N} pw^{p-1} \int_{\partial\Omega} e^{<\frac{z-P_0}{|z-P_0|},y>} e^{<z-P_0,a>} \left(\frac{z-P_0}{|z-P_0|}\right)_j d\mu_{P_0}(z) \frac{\partial w}{\partial y_k}\, dy + O(|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma})$$

$$= \frac{2\gamma}{d^2(P_0,\partial\Omega)} \varphi_{\epsilon,P_0}(P_0) \left(\int_{\partial\Omega} e^{<z-P_0,a>}(z-P_0)_i(z-P_0)_k d\mu_{P_0}(z) + o(1)\right),$$

where

$$\gamma := \int_{R^N} pw^{p-1} w'(y) V_0'(y)\, dy.$$

For the left hand side of (3.16), we have

$$\text{l.h.s. of } (3.16)$$

$$= \sum_{j=1}^N s_j^\epsilon \left(\int_{\Omega_{\epsilon,Q_\epsilon}} [p(w_{\epsilon,Q_\epsilon})^{p-1}\epsilon\partial_j w_{\epsilon,Q_\epsilon} - pw^{p-1}\epsilon\partial_j w]\epsilon\partial_k w_{\epsilon,Q_\epsilon}\, dy\right) + O(|\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{1+\sigma}))$$

$$= \int_{\partial\Omega} e^{<z-P_0,a>} < \frac{z-P_0}{|z-P_0|}, s^\epsilon > \left(\frac{z-P_0}{|z-P_0|}\right)_k \varphi_{\epsilon,P_0}(P_0) d\mu_{P_0}(z)(-2\gamma + o(1))$$

where $s^\epsilon = (s_1^\epsilon, ..., s_N^\epsilon)$.

Similar but simpler computations for the right hand side of (3.16) give

$$\text{r.h.s. of } (3.16)$$

$$= \tau^\epsilon \sum_{j=1}^N s_j^\epsilon (B\delta_{jk} + o(1)) + O\left(\sum_{j=1}^N |s_j^\epsilon||\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{(1+\sigma)}\right) + O\left(|\tau^\epsilon|(\sum_{j=1}^N |s_j^\epsilon||\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)|^{(1+\sigma)/2})\right),$$

where $B = \int_{R^N} (\frac{\partial w}{\partial y_1})^2 dy$.

Hence we have

$$|\tau^\epsilon| = O(\varphi_{\epsilon,Q_\epsilon}(Q_\epsilon)) = O(\varphi_{\epsilon,P_0}(P_0))$$

and $\tau^\epsilon/\varphi_{\epsilon,P_0}(P_0) \to \tau_0$, $s^\epsilon \to s$, where (τ_0, s) satisfies

$$(-2\gamma)G(P_0)s = Bd^2(P_0,\partial\Omega)\tau_0 s.$$

Thus $\frac{Bd^2(P_0,\partial\Omega)}{-2\gamma}\tau_0$ is an eigenvalue of $G(P_0)$. Therefore $\tau^\epsilon/\varphi_{\epsilon,P_0}(P_0) \to \tau_j$, $s^\epsilon \to \vec{a}_j$ where

$$\tau_j = \frac{-2\gamma}{Bd^2(P_0,\partial\Omega)}\lambda_j, \quad G(P_0)\vec{a}_j = \lambda_j \vec{a}_j, \quad j = 1, ..., N.$$

By an argument of Dancer [2], we know that these are the only small eigenvalues.

This finishes the proof of Theorem 3.4.

□

Completion of the Proof of Theorem 1.1:

The small eigenvalues given by Theorem 3.4 all have negative real part. By a proof along the lines of the proof of Theorem 2.7 (replacing w_{y_0} by w and considering interior spikes instead of near-boundary spikes) the large eigenvalues all have negative real part. Finally, Theorem 1.1 follows by combining these two results.

□

4. Numerical Simulations

We show numerical simulations which display the various effects which have been analytically proved in this paper.

We consider the Gierer-Meinhardt system (1.1), (1.2) on $\Omega = (-1, 1)$ for the following parameters: diffusion constants $\epsilon^2 = 0.01$, $D = 10^9$, time relaxation constant $\tau = 10^{-9}$, Robin boundary condition parameters varying a_A, $a_H = 0$, reaction constants varying p and q, $r = 2$, $s = 0$.

First we consider the classical Gierer-Meinhardt system with $p = 2$, $q = 1$. We show stable near-boundary spikes for various a_A (Figure 1) and interior spikes for various a_A (Figure 2). We see that a change of a_A has strong influence on a near-boundary spike, but only a minor influence on interior spikes.

Then we numerically explore the instability of near-boundary spikes. We consider the Gierer-Meinhardt system for various p, $q = 2$, $r = 2$, $s = 0$. with Robin boundary condition for $a_A = 0.8$. We start with $p = 4.0$ and then increase p incrementally in steps of 0.01. The final steady state for the previous p is used as initial condition for the next one. The final steady state is displayed for $p = 4.5$, 4.8, 4.85 (Figure 3).

At $p = 4.86$ a rather dramatic change of stability is observed: The solution blows up in finite time (Figure 4). The simulations show a sharp peak, and after a finite time the simulation breaks down: The amplitudes of the solution become very large, and the finite element software is no longer able to resolve the solution since this peak occurs on a very small spatial scale. This is similar to phenomena which occur for supercritical systems. In some sense the Robin boundary condition is able to squeeze the threshold between sub- and supercritical to lower reaction rates.

Figure 1. Near-boundary spikes for variable constant a_A in the Robin boundary. We have chosen $a_A = 0.2$, 0.4, 0.6, 0.8. It is numerically stable (final state is shown for $t = 10,000$).

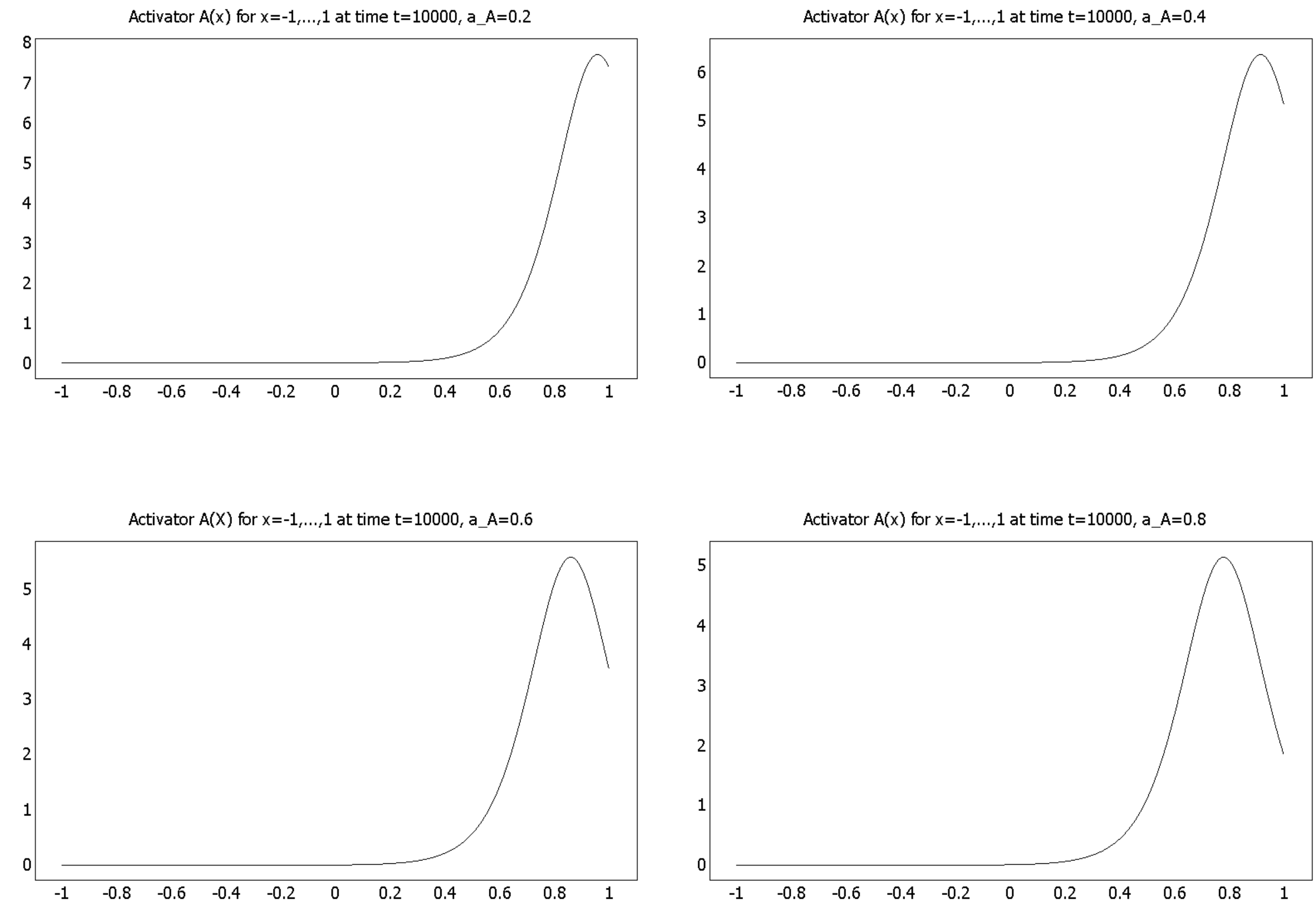


Figure 2. Interior-boundary spikes for variable constant a_A in the Robin boundary. We have chosen $a_A = 0.2$, 0.8. It is numerically stable (final state is shown for $t = 10,000$).

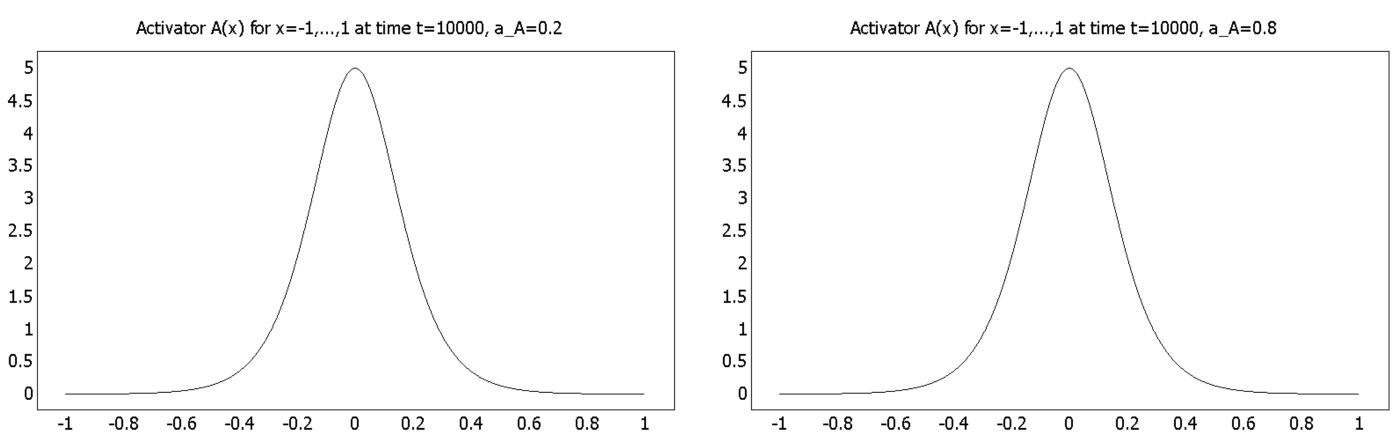


Figure 3. Stable near-boundary spikes. We choose constants $a_A = 0.8$, $q = 2$, $r = 2$, $s = 0$ and varying p. For $p = 4.0$, 4.5, 4.8, 4.85 the near-boundary spike is shown. It is numerically stable (final state is shown for $t = 10,000$).

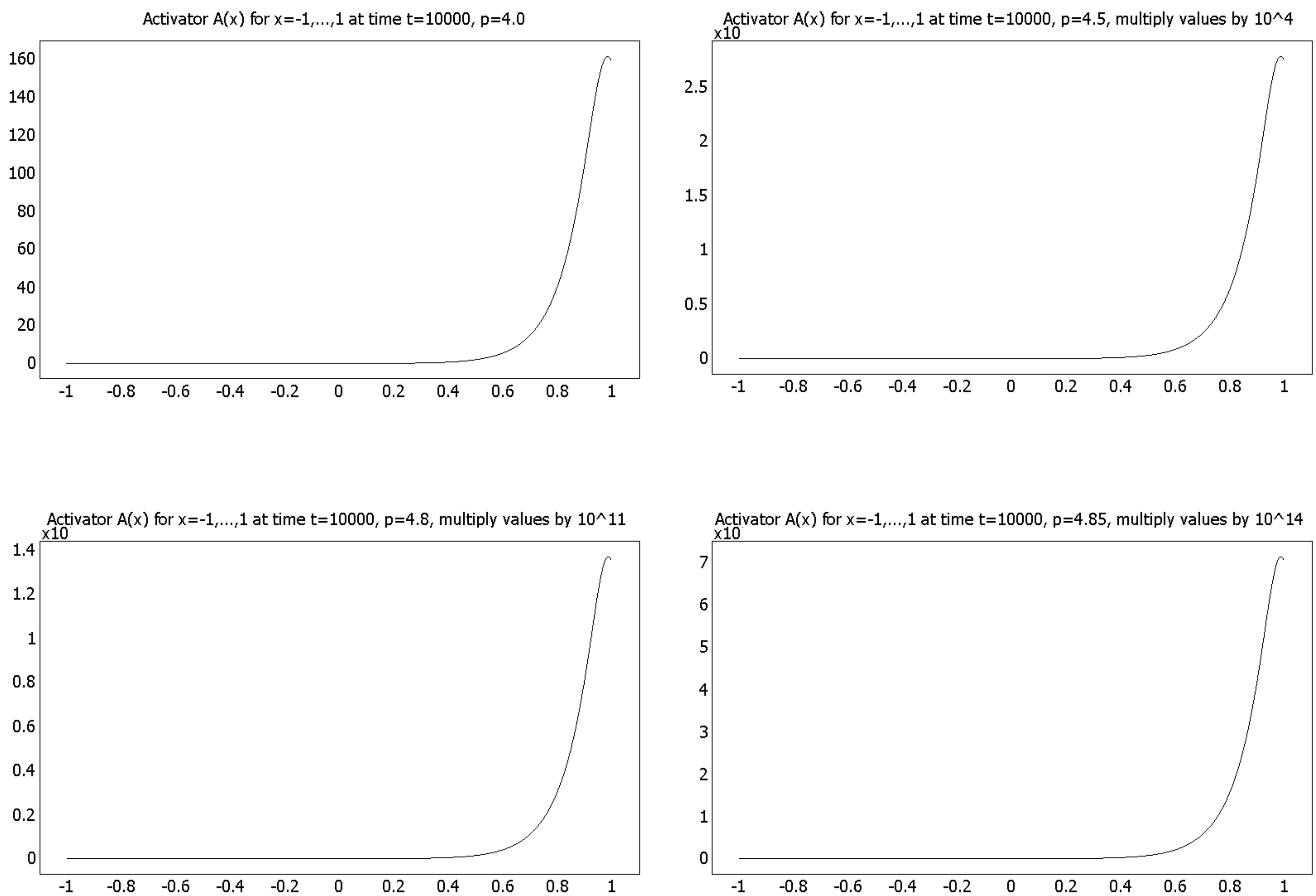
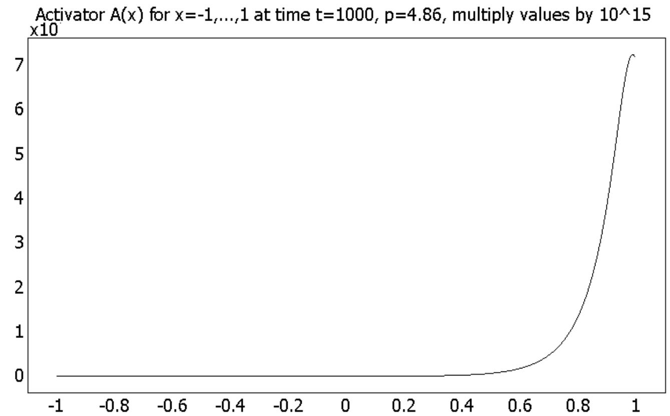
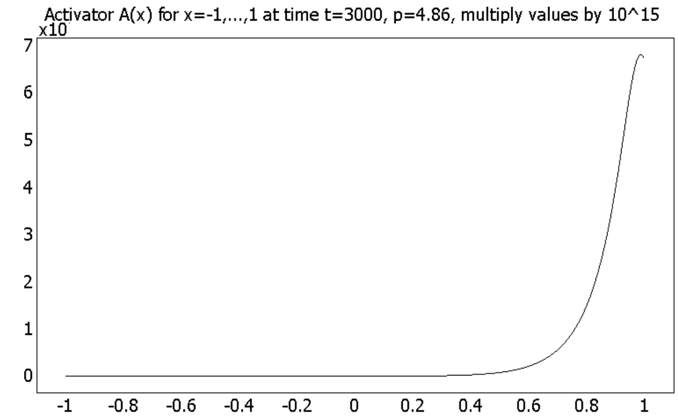
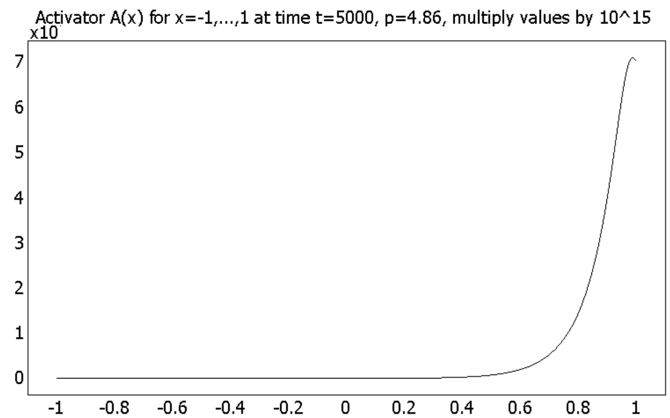
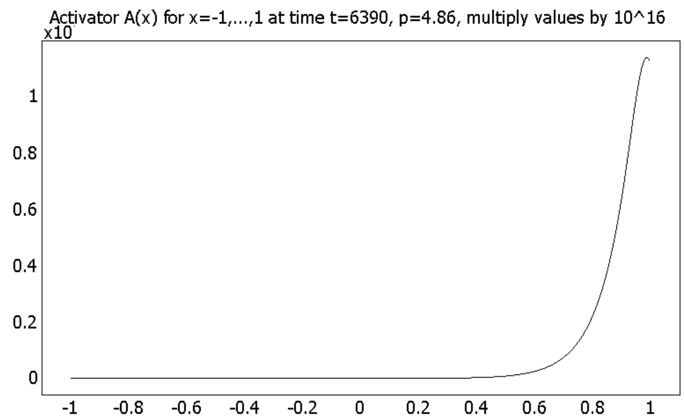


Figure 4. Unstable near-boundary spike. We choose constants $a_A = 0.8$, $q = 2$, $r = 2$, $s = 0$ and $p = 4.86$. The near-boundary spike is now numerically unstable. In the time evolution the amplitude increases (shown for $t = 1,000$, $3,000$, $5,000$, $6,390$). Then the simulation diverges.









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