Suggested Answer to Assignment 1

- **Page 31, Problem 1.** Does there exist an infinite σ-algebra which has only countably many members?

Answer: There doesn't exist an infinite σ-algebra which has only countably many members. Suppose not, $\mathfrak{M}$ is a such σ-algebra on X. For $x \in X$, define

$$A_x \equiv \bigcap_{A\in\mathfrak{M}, x\in A} A.$$

Obviously $x \in A_x$. Since $\mathfrak{M}$ is countable, $A_x \in \mathfrak{M}$. We claim that if $y \in A_x$, then $A_y = A_x$. Suppose not, let $y \in A_x$, since $A_x \in \mathfrak{M}$, $A_y \subseteq A_x$. If also $x \in A_y$, then $A_x \subseteq A_y$ and thus $A_y = A_x$. This is a contradiction. Hence $x \notin A_y$. Since $x \in A_x \backslash A_y$ and $A_x \backslash A_y \in \mathfrak{M}$, $A_x \subseteq A_x \backslash A_y$. This is a contradiction since $y \in A_x$ and $y \notin A_x \backslash A_y$.

Set $\mathfrak{N} \equiv \{A_x : x \in X\}$, then $\mathfrak{N} \subset \mathfrak{M}$ and thus has only countable many members. Define $f : \mathfrak{M} \longrightarrow 2^{\mathfrak{N}}$ by

$$f(B) = \{A_x : x \in B\}, \quad \text{for } B \in \mathfrak{M}.$$

Obviously for $B \in \mathfrak{M}$ and $x \in B$, we have $A_x \subseteq B$.

We claim that f is bijective. First we show that f is surjective. Let E be a subset of $\mathfrak{N}$, then $E = \{A_x : x \in I\}$, where $I \subseteq X$ has only countable many members. Set $B = \bigcup_{x\in I} A_x$, then $I \subseteq B$ and $B \in \mathfrak{M}$. Furthermore, let $y \in B$, then $y \in A_x$ for some $x \in I$ and thus $A_y = A_x$. Hence $\{A_y : y \in B\} = \{A_x : x \in I\}$ and then

$$f(B) = \{A_x : x \in B\} = \{A_x : x \in I\} = E.$$

Second we show that f is injective. Let $B_1, B_2 \in \mathfrak{M}$ and $B_1 \neq B_2$. Then there exists $y \in B_1 \backslash B_2$ or $y \in B_2 \backslash B_1$. Without loss of generality, we assume $y \in B_1 \backslash B_2$ and then $A_y \in f(B_1)$. We claim that $A_y \notin f(B_2)$. Suppose not, there exists $x \in B_2$ such that $A_x = A_y$. Then $y \in A_y \subseteq A_x \subseteq B_2$, a contradiction. Therefore, $f(B_1) \neq f(B_2)$ if $B_1 \neq B_2$ and then f is injective.

Since f is bijective, the cardinality of $\mathfrak{M}$ equals that of $2^{\mathfrak{N}}$. Thus if $\mathfrak{N}$ has only n elements, then $\mathfrak{M}$ has 2^n elements. This contradicts with $\mathfrak{M}$ has infinite many numbers. On the other hand, if $\mathfrak{N}$ has countable infinity many numbers, then $\mathfrak{M}$ has uncountable many numbers. This is a contradiction and we complete the proof. $\square$

- **Page 31, Problem 3.** Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

Answer: By the argument in the proof of (c) in Theorem 1.12, we only need to prove $E_a = \{x : f(x) > a\}$ is measurable for all $a \in \mathbb{R}$. Let $a \in \mathbb{R}$, we can choose $\{r_n\} \subseteq \mathbb{Q}$, $r_n > a$ and $r_n \to a$ as $n \to \infty$. Since

$$E_a = f^{-1}((a, +\infty)) = f^{-1}(\bigcup_{n=1}^{\infty}[r_n, +\infty)) = \bigcup_{n=1}^{\infty} f^{-1}([r_n, +\infty)) = \bigcup_{n=1}^{\infty}\{x : f(x) \geq r_n\},$$

E_a is measurable. $\square$

- **Page 31, Problem 4.** Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(a)

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to} a_n.$$

(b)

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to} a_n + \limsup_{n\to} b_n$$

provide none of the sums is of the form $\infty - \infty$.

(c) If $a_n \le b_n$ for all n, then

$$\liminf_{n\to} a_n \le \liminf_{n\to} b_n.$$

Show by an example that strict inequality can hold in (b).

Answer:

(a) Since $\inf\limits_{k\ge n} a_k \le a_k$ for all $k \ge n$, $-\inf\limits_{k\ge n} a_k \ge -a_k$ for all $k \ge n$ and then $-\inf\limits_{k\ge n} a_k \ge \sup\limits_{k\ge n}(-a_k)$. Similarly, we have $\inf\limits_{k\ge n} a_k \le -\sup\limits_{k\ge n}(-a_k)$. Thus $-\inf\limits_{k\ge n} a_k = \sup\limits_{k\ge n}(-a_k)$. Hence

$$\begin{aligned}\limsup_{n\to\infty}(-a_n) = \lim_{n\to\infty}\big(\sup_{k\ge n}(-a_k)\big) &= \lim_{n\to\infty}\big(-\inf_{k\ge n} a_k\big)\\ &= -\lim_{n\to\infty}\big(\inf_{k\ge n} a_k\big)\\ &= -\liminf_{n\to} a_n.\end{aligned}$$

(b) Since $a_k \le \sup\limits_{k\ge n} a_k$ for all $k \ge n$ and $b_k \le \sup\limits_{k\ge n} b_k$ for all $k \ge n$, $a_k + b_k \le \sup\limits_{k\ge n} a_k + \sup\limits_{k\ge n} b_k$ for all $k \ge n$. Thus $\sup\limits_{k\ge n}(a_k+b_k) \le \sup\limits_{k\ge n} a_k + \sup\limits_{k\ge n} b_k$, which implies

$$\begin{aligned}\limsup_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty}\big(\sup_{k\ge n}(a_k+b_k)\big) &\le \lim_{n\to\infty}\big(\sup_{k\ge n} a_k + \sup_{k\ge n} b_k\big)\\ &\le \lim_{n\to\infty}\big(\sup_{k\ge n} a_k\big) + \lim_{n\to\infty}\big(\sup_{k\ge n} b_k\big)\\ &\le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.\end{aligned}$$

The strict inequality can holds, for example, let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

(c) If $a_n \le b_n$ for all n, then $\inf\limits_{k\ge n} a_k \le a_k \le b_k$ for all $k \ge n$ and thus $\inf\limits_{k\ge n} a_k \le \inf\limits_{k\ge n} b_k$. Hence

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\big(\inf_{k\ge n} a_k\big) \le \lim_{n\to\infty}\big(\inf_{k\ge n} b_k\big) \le \liminf_{n\to\infty} b_n. \quad \square$$

- **Page 32, Problem 5.** (a) Suppose $f : X \to [-\infty,\infty]$ and $g : X \to [-\infty,\infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$$

are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Answer: (a) Since

$$\{x : f(x) < g(x)\} = \bigcup_{r\in\mathbb{Q}}\{x : f(x) < r \le g(x)\} = \bigcup_{r\in\mathbb{Q}} \big(\{x : f(x) < r\} \cap \{x : g(x) \ge r\}\big),$$

and

$$\{x : f(x) = g(x)\} = \Big(\{x : f(x) < g(x)\} \bigcup \{x : g(x) < f(x)\}\Big)^c,$$

then both $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.

(b) Let f_n be a sequence of real measurable functions. Then $g = \limsup\limits_{n\to\infty} f_n$ and $h = \liminf\limits_{n\to\infty} f_n$ are measurable. Recall that a sequence of real numbers $\{a_n\}$ converges if and only if $\liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n$, hence

$$\big\{x : f_n(x) \text{ converges (to a finite limit)}\big\} = \bigcup_{n=1}^{\infty} \Big(\{x : g(x) = h(x)\} \bigcap \{x : -n < g(x) < n\}\Big).$$

By result in (a) we get the desired result. ☐

• **Page 32, Problem 6.** Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Answer: Set $\mathfrak{F} \equiv \{A \subset X : A \text{ is at most countable}\}$, $\mathfrak{G} \equiv \{A \subset X : A^c \text{ is at most countable}\}$. Since X is uncountable, $\mathfrak{F}$ and $\mathfrak{G}$ are disjoint, hence $\{\mathfrak{F}, \mathfrak{G}\}$ form a partition of $\mathfrak{M}$.

Now we show that $\mathfrak{M}$ is a σ-algebra. Obviously, $\emptyset \in \mathfrak{F}$ and $X \in \mathfrak{G}$.

(i) For $A \in \mathfrak{M}$, then $A \in \mathfrak{F}$ or $A \in \mathfrak{G}$, then $A^c \in \mathfrak{G}$ in the first case and $A^c \in \mathfrak{F}$ in the second. Hence $A^c \in \mathfrak{M}$.

(ii) Let $A_n \in \mathfrak{M}, n = 1, 2, \dots$ and $A = \bigcup_{n=1}^{\infty} A_n$. If $A_n \in \mathfrak{F}$ for all n, then A is at most countable, hence $A \in \mathfrak{F}$. If $A_{n_0} \in \mathfrak{G}$ for some n_0, then $A^c = \bigcap_{n=1}^{\infty} A_n^c \subset A_{n_0}^c$, with $A_{n_0}^c \in \mathfrak{F}$, hence $A^c \in \mathfrak{F}$ and then $A \in \mathfrak{G}$. In any case, $A \in \mathfrak{M}$. Hence $\mathfrak{M}$ is a σ-algebra.

To show μ is a measure, we only need to prove that μ is countably additive. First note that $\mu(B) = 0$ if $B \in \mathfrak{F}$ and $\mu(B) = 1$ is $B \in \mathfrak{G}$. Let $A_n \in \mathfrak{M}, n = 1, 2, \dots,$ satisfy $A_i \bigcap A_j = \emptyset$ if $i \neq j$. Set $A = \bigcup_{n=1}^{\infty} A_n$. If $A_n \in \mathfrak{F}$ for all n, then $A \in \mathfrak{F}$. Thus $\mu(A_n) = 0$ for all n and also $\mu(A) = 0$, hence $\mu(A) = 0 = \sum_{n=1}^{\infty} \mu(A_n)$. If $A_{n_0} \in \mathfrak{G}$ for some n_0, from the above discussion $A \in \mathfrak{G}$. Since for any $m \neq n_0$, $A_m \bigcap A_{n_0} = \emptyset$, which implies $A_m \subset A_{n_0}^c$ and then $A_m \in \mathfrak{F}$. Thus $\mu(A_{n_0}) = 1$, $\mu(A) = 1$ and $\mu(A_m) = 0$ for all $m \neq n_0$. Hence $\mu(A) = 1 = \sum_{n=1}^{\infty} \mu(A_n)$. In any case, countable additive property holds and then μ is a measure.

Claim: Let $f : \mathfrak{M} \to [-\infty, \infty]$ be a measurable function. Then there is exact one value α_0 such that $f^{-1}(\{\alpha_0\}) \in \mathfrak{G}$, that is $f = \alpha_0$ a.e., and then $\int_X f\, d\mu = \alpha_0$.

Proof of claim: First we prove the uniqueness. Suppose α_0 satisfies $f^{-1}(\{\alpha_0\}) \in \mathfrak{G}$, then by the definition of $\mathfrak{G}$, $\{x : f(x) \neq \alpha_0\}$ is at most countable. Hence there do not exist another such value.

Next we prove the existence. Set $A \equiv \{a \in \mathbb{R} : f^{-1}((a,\infty]) \in \mathfrak{F}\}$. If $A = \emptyset$, then we have $f^{-1}((n,\infty]) \in \mathfrak{G}$, for all $n \in \mathbb{Z}$. Thus $f^{-1}([-\infty,n]) \in \mathfrak{F}$, i.e., $f^{-1}([-\infty,n])$ is at most countable. Thus $f^{-1}([-\infty,\infty))$ is at most countable. Since X is uncountable, $f^{-1}(\{\infty\}) \in \mathfrak{G}$. If $A \neq \emptyset$, then $\alpha_0 \equiv \inf A \in [-\infty,\infty)$. If $A = -\infty$, then for all $n \in \mathbb{Z}$, $f^{-1}((-n,\infty]) \in \mathfrak{F}$. Thus $f^{-1}((-\infty,\infty])$ is at most countable. Since X is uncountable, $f^{-1}(\{-\infty\}) \in \mathfrak{G}$. If $A > -\infty$, then for all $n \in \mathbb{N}$, $f^{-1}((\alpha_0 - \frac{1}{n},\infty]) \in \mathfrak{G}$ and $f^{-1}((\alpha_0 + \frac{1}{n},\infty]) \in \mathfrak{F}$. Hence $f^{-1}([-\infty,\alpha_0)) \in \mathfrak{F}$ and $f^{-1}((\alpha_0,\infty]) \in \mathfrak{F}$. Since X is uncountable, $f^{-1}(\{\alpha_0\}) \in \mathfrak{G}$. □

• **Page 32, Problem 7.** Suppose $f_n : X \to [0,\infty]$ is measurable for $n = 1,2,3,\ldots, f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0, f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Answer: Set $g_n = f_1 - f_n, n = 1,2,\ldots$, since $f_n \leq f_1$ and $f_1 \in L^1(\mu)$, g_n's are well defined for almost all $x \in X$ and satisfy

(i) $0 \leq g_1 \leq g_2 \leq g_3 \leq \cdots \leq \infty$ on X,

(ii) $g_n(x) \to f_1(x) - f(x)$ as $n \to \infty$, for every $x \in X$.

Lebegue's Monotone Convergence Theorem implies that

$$\lim_{n\to\infty} \int_X (f_1 - f_n) d\,\mu = \int_X (f_1 - f)\, d\mu. \tag{1}$$

Since $f \in L^1(\mu)$, we have

$$\int_X (f_1 - f_n)\, d\mu = \int_X f_1\, d\mu - \int_X f_n\, d\mu, \quad \text{and} \quad \int_X (f_1 - f)\, d\mu = \int_X f_1\, d\mu - \int_X f\, d\mu.$$

Plug them into equation (1) we get the desired result.

Next we give an example to show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted. Let μ be the counting measure on the set $X \equiv \{1,2,3,\cdots\}$. Set $A_n \equiv \{n, n+1, n+2, \cdots\}$ and $f_n \equiv \chi_{A_n}$. Obviously, $f_1 \geq f_2 \geq \cdots \geq 0, f_n \to f \equiv 0$ as $n \to \infty$. But

$$\lim_{n\to\infty} \int_X f_n d\mu = \infty \neq 0 = \int_X f\, d\mu. \quad \square$$

• **Extra Problem 1.** Let $G \subset \mathbb{R}^n$ be an open set and $f : G \to \mathbb{R}$ be a real function. Show that the set $\{x \in G : f \text{ is not continuous at } x\}$ is a F_σ set.

Answer: Let

$$\varphi(x,\delta) \equiv \sup\{|f(s) - f(t)| : s,t \in B_\delta(x)\},$$

and

$$\varphi(x) \equiv \inf\{\varphi(x,\delta) : \delta > 0\}.$$

We claim that $\varphi(x)$ is upper semicontinuous, i.e., for every real a, $E_a \equiv \{x : \varphi(x) < a\}$ is open. Let $x_0 \in E_a$, then there exists $\delta_0 > 0$ such that

$$\sup\{|f(s) - f(t)| : s,t \in B_{\delta_0}(x)\} < a.$$

Thus for any $x \in B_{\delta_0/2}(x_0)$,

$$\varphi(x, \frac{\delta_0}{2}) = \sup\{|f(s) - f(t)| : s, t \in B_{\delta_0/2}(x)\} \leq \sup\{|f(s) - f(t)| : s, t \in B_{\delta_0}(x_0)\} < a.$$

Hence $\varphi(x) < a$ for all $x \in B_{\delta_0/2}(x_0)$. Then E_a is open and $\varphi(x)$ is upper semicontinuous.

By definition of continuity, we know that f is continuous at x if and only if $\varphi(x) = 0$. Hence

$$\{x : \varphi(x) > 0\} = \{x : f \text{ is not continuous at } \ x\}.$$

On the other hand

$$\{x : \varphi(x) > 0\} = \bigcup_{k=1}^{\infty} \{x : \varphi(x) \geq \frac{1}{k}\}.$$

Therefore $\{x \in G : f$ is not continuous at $x\}$ is a F_σ set since $\{x : \varphi(x) \geq \frac{1}{k}\}$ is closed. □

- **Extra Problem 2.** Suppose that f is a continuous function on $\mathbb{R}^1$. Show that the set $\{x | f$ is differentiable at $x\}$ is a $F_{\sigma\delta}$ set.

Answer: Recall the right upper Dini derivative of a continuous function f is denotes by D^+f, and defined by

$$D^+f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h};$$

the right lower Dini derivative D_+f, is defined by

$$D_+f(x) = \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h};$$

the left upper Dini derivative D^-f, is defined by

$$D^-f(x) = \limsup_{h \to 0^-} \frac{f(x+h) - f(x)}{h};$$

and the left lower Dini derivative D_-f, is defined by

$$D_-f(x) = \liminf_{h \to 0^-} \frac{f(x+h) - f(x)}{h}.$$

By the definition of differentiable,

$$\begin{aligned}
&\Big\{x | f \text{ is differentiable at } x\Big\} \\
=&\Big\{x | \infty > D_-f(x) = D^+f(x) = D_+f(x) = D^-f(x) > -\infty\Big\} \\
=&\bigcap_{m=1}^{\infty} \bigcup_{a \in \mathbb{Q}} \Big\{a + \frac{1}{m} > D^+f(x) \geq D_+f(x) > a, a + \frac{1}{m} > D^-f(x) \geq D_-f(x) > a\Big\} \\
=&\bigcap_{m=1}^{\infty} \bigcup_{a \in \mathbb{Q}} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \Big\{a + \frac{1}{m} - \frac{1}{n} \geq \frac{f(x + 1/k) - f(x)}{1/k} \geq a + \frac{1}{n}, \\
&\qquad a + \frac{1}{m} - \frac{1}{n} \geq \frac{f(x - 1/k) - f(x)}{-1/k} \geq a + \frac{1}{n}\Big\}.
\end{aligned}$$

Since f is continuous, the set

$$\left\{a+\frac{1}{m}-\frac{1}{n}\geq\frac{f(x+1/k)-f(x)}{1/k}\geq a+\frac{1}{n}, a+\frac{1}{m}-\frac{1}{n}\geq\frac{f(x-1/k)-f(x)}{-1/k}\geq a+\frac{1}{n}\right\}$$

is closed. Thus the set $\{x|f \text{ is differentiable at } x\}$ is a $F_{\sigma\delta}$ set. $\square$