MAT 5011 Real Analysis

Suggested Answer to Assignment 5

Page74, question18.

Proof: (a) For any $\epsilon > 0$, let $E_{n,\epsilon} = \{x : |f_n(x) - f(x)| > \epsilon\}$ and $S_\epsilon = \cap_{k=1}^{\infty} \cup_{n\geq k}^{\infty} E_{n,\epsilon}$.

Then for any $x \in S_\epsilon$, $f_n(x) \nrightarrow f(x)$, that is $\mu(S_\epsilon) = 0$.

Since $\mu(X) < \infty$ and $\cup_{n\geq k}^{\infty} E_{n,\epsilon}$ is decreasing, we get

$$\lim_{k\to\infty} \cup_{n\geq k}^{\infty} E_{n,\epsilon} = \mu(S_\epsilon).$$

Thus $\lim_{n\to\infty} E_{n,\epsilon} = 0$. Hence $f_n \to f$ in measure.

(b): Let $E_{n,\epsilon}$ be defined as part(a), then

$$\int_X |f_n - f|^p d\mu \geq \int_{E_{n,\epsilon}} |f_n - f|^p d\mu \geq \epsilon^p \mu(E_{n,\epsilon}).$$

Thus $\mu(E_{n,\epsilon}) \leq \epsilon^{-p} \int_X |f_n - f|^p d\mu \to 0$ as $n \to \infty$.

(c):Since $f_n \to f$ in measure, then $\{f_n\}$ is a Cauchy sequence in measure, that is , for any $\epsilon > 0, \exists N > 0$ such that for any $m, n > N$, we have

$$\mu(\{x : |f_m(x) - f_n(x)| \geq \epsilon\}) < \epsilon.$$

Thus there exists a subsequence $\{f_{n_k}\}$ such that

$$\mu(E_k) < \frac{1}{2^k}$$

where $E_k = \{x : |f_{n_{k+1}} - f_{n_k}| \geq \frac{1}{2^k}$.

Define $F_N = \cup_{k=N}^{\infty} E_k$. Then if $x \in F_N^c, i \geq j \geq N$,

$$|f_{n_i} - f_{n_j}| \leq |f_{n_i} - f_{n_{i-1}}| + \cdots + |f_{n_{j+1}} - f_{n_j}| \leq 2^{1-j}.$$

This implies that $\{f_{n_k}\}$ is a Cauchy sequence on F_N^c. Now let $F = \bigcap_{N=1}^{\infty} F_N$, then $\mu(F) = 0$ and for any $x \in F^c$, $f_{n_k}(x) \to f(x)$.

The converses of part(a) and (b) are false.

For part(a), the counterexample is the answer of question9 in chapter2 i.e. in assightment3.

For part(b), let $X = [0,1]$ and define $f_n = n^{\frac{1}{p}}\chi_{[0,\frac{1}{n}]}$. Obviously f_n converges in measure, but $\int_X f_n^p = 1 \neq 0$.

If $\mu(X) = \infty$, from the above proof, we know that part(b) and part(c) are still right. But part(a) is false. The counterexample is : let $X = \mathbb{R}, f_n = \chi_{[n,n+1]}$, then $f_n \to 0$ pointwisely. But f_n doesn't converge to 0 in measure.

Page74, question20.

Proof: For any $s, t \in \mathbb{R}, \lambda \in [0,1]$, define $f(x) = s\chi_{[0,\lambda]}(x) + t\chi_{[\lambda,1]}(x)$. Then

$$\varphi(s\lambda+t(1-\lambda)) = \varphi(\int_0^1 f dx) \leq \int_0^1 \varphi(f)dx = \int_0^\lambda \varphi(s)dx+\int_\lambda^1 \varphi(t)dx = \lambda\varphi(s)+(1-\lambda)\varphi(t).$$

Page74, question23.

Proof: Since $\|f\|_\infty > 0$ we get $\mu(x) > 0$. By Hölder's inequality, we have

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\int_X |f|^{n+1}d\mu}{\int_X |f|^n d\mu} \\ &\geq \frac{\int_X |f|^{n+1}d\mu}{(\int_X |f|^{n+1}d\mu)^{\frac{n}{n+1}}(\int_X 1^{n+1}d\mu)^{\frac{1}{n+1}}} \\ &= \|f\|_{n+1}(\mu(X))^{-\frac{1}{n+1}}\end{aligned}$$

According to question4 in chapter3, we have known $\lim_{n\to\infty}\|f\|_{n+1} = \|f\|_\infty$ since $\mu(X) < \infty, f \in L^\infty(\mu)$. Thus

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} \geq \|f\|_\infty.$$

On the other hand,

$$\frac{\int_X |f|^{n+1}d\mu}{\int_X |f|^n d\mu} \leq \|f\|_\infty \frac{\int_X |f|^n d\mu}{\int_X |f|^n d\mu} = \|f\|_\infty.$$

i.e. $\limsup_{n\to\infty}\frac{a_{n+1}}{a_n} \leq \|f\|_\infty$. Hence

$$\frac{a_{n+1}}{a_n} = \|f\|_\infty.$$

Page74, question24.

Proof: (a) By $0 < p < 1$, we get $|x^p - y^p| \leq |x-y|^p$ where $x, y \geq 0$. Thus

$$\int_X ||f|^p - |g|^p| d\mu \leq \int_X ||f| - |g||^p d\mu \leq \int_X |f-g|^p d\mu.$$

Next we verify that $\Delta(f,g) = \int_X |f-g|^p d\mu$ is a metric. Obviously we only need to verify the triangle inequality. For any f, g, h in $L^p(\mu)$,

$$\begin{aligned}
\Delta(f,g) &= \int_X |f-g|^p d\mu \\
&\leq \int_X (|f-h| + |h-g|)^p d\mu \\
&\leq \int_X |f-h|^p d\mu + \int_X |h-g|^p d\mu \\
&= \Delta(f,h) + \Delta(h,g)
\end{aligned}$$

Thus $L^p(\mu)$ is a complete space by Theorem3.11.

(b): Let $\varphi(x) = x^p, x \geq 0, p \geq 1$. Obviously, $\varphi(x)$ is convex. Thus if $x \geq y$, we have $\frac{\varphi(x)-\varphi(y)}{x-y} \leq \varphi'(x)$. Hence we have the following inequality

$$s^p - t^p \leq p(s-t)s^{p-1}, \qquad s \geq t \geq 0.$$

If $x, y \geq 0$, we get

$$|x^p - y^p| \leq p|x-y|(x^{p-1} + y^{p-1}).$$

By the above inequality and Minkowski inequality, we have

$$\begin{aligned}
\int_X ||f|^p - |g|^p| d\mu &\leq \int_X p||f| - |g||(|f|^{p-1} + |g|^{p-1}) d\mu \\
&\leq p(\int_X |f-g|^p)^{\frac{1}{p+1}} (\int_X (|f|^{p-1} + |g|^{p-1})^q d\mu)^{\frac{1}{q}} \\
&= p\|f-g\|_p \||f|^{p-1} + |g|^{p-1}\|_q \\
&\leq p\|f-g\|_p (\||f|^{p-1}\|_q + \||g|^{p-1}\|_q) \\
&= p\|f-g\|_p (\|f\|_p^{p-1} + \|g\|_p^{p-1}) \\
&\leq 2pR^{p-1}\|f-g\|_p
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Page92, question3.

Proof. By Lusin's Theorem, $C(T)$ is dense in $L^p(T)$ if $1 \leq p < \infty$. But the space $P(T)$ of all trigonometric polynomials is dense in $C(T)$. Hence P is dense in $L^p(T), 1 \leq p < \infty$. But for

$$u_n(t) = e^{int}, 0 \leq t < 2\pi,$$

$\{u_n\}$ is a basis for $P, n = 0, \pm 1, \pm 2, \ldots$

Therefore the set of all trigonometric polynomials

$$P = \sum_{n=-N}^{N} c_n u_n, \qquad c_n \in \mathbb{Q}$$

form a countable dense subset of P, hence of $L^p(T)$ in the L^p norm.

Therefore $L^p(T)$ is separable, $1 \leq p < \infty$.

Now let $K = \{\chi_{[0,x]} : 0 \leq x < 2\pi\} \subset L^\infty(T)$. We identify $L^\infty(T)$ to $L^\infty([0, 2\pi))$. If $x' < x$ then

$$\|\chi_{[0,x]} - \chi_{[0,x']}\| = \|\chi_{(x',x]}\| = 1.$$

Moreover K is uncountable. Let S be a dense subset of $L^\infty(T)$. Then for every $x \neq x'$ in $[0, 2\pi)$, then there are $s, s' \in S$ such that

$$\|\chi_{[0,x]} - s\|_\infty < \frac{1}{4}, \quad \|\chi_{[0,x']} - s'\|_\infty < \frac{1}{4}.$$

By the triangle inequality one has $\|s - s'\| > \frac{1}{2}$, hence $s \neq s'$. Thus we can define an injection from K to S. Hence S is uncountable. Thus L^∞ is not separable.

Page92, question6.

Proof. We use the following theorem:

A metric space K is compact iff every bounded sequence in K has a convergent subsequence.

Since $\|u_n - u_m\| = \sqrt{2}$ if $n \neq m$ and $\|u_n\| = 1, \forall n$, the set $\{u_1, u_2, \ldots\}$ is clearly closed and bounded, but contains no convergent subsequence, hence is not compact. Now $\bar{B}(0, 1)$ contains the sequence $\{u_n\}$ which has no convergent subsequence. This shows that $\bar{B}(0,1)$ is not compact, hence 0 has no compact neighborhood. Thus H is not locally compact.

Suppose $\sum \delta_n^2 < \infty$, and let $(x_k) = (\sum_{n=1}^{\infty} c_{n,k} u_n)$ be a sequence in S. We define by induction on m the integers $N_1, N_2, \dots$ and the sequences $(x_{k,1}), (x_{k,2}), \dots$ as follows

$$\text{(i)} \sum_{n=N_m}^{\infty} \delta_n^2 < \frac{1}{m^2}, \quad N_1 < N_2 < \dots (\sum \delta_n^2 < \infty);$$

(ii), $(x_{k,1})$ is a subsequence of (x_k) and $(x_{k,m+1})$ is a subsequence of $(x_{k,m}), m = 1, 2, \dots$ such that

$$\text{(iii)} \qquad \sum_{n=1}^{N_m} |c_{n,(k,m)} - c_{n,(k',m)}|^2 < \frac{1}{m^2} \qquad \forall k, k'.$$

By (i) and (iii), $\|x_{k,m} - x_{k',m}\|^2 < \frac{3}{m^2}$ for all k, k'. Hence if $y_m = x_{m,m}$, then (y_m) is a convergent sequence. By (ii) (y_m) is a subsequence of (x_k). If $y_m \to y = \sum c_n u_n$ as $m \to +\infty$, then $(y_m, u_n) = y_{m,n} \to y_n = (y, u_n)$, hence $|y_n| \leq \delta_n$. Thus $y \in S$. This shows that S is compact.

Now suppose $\sum_{n=1}^{\infty} \delta_n^2 = \infty$. Thus there are integers $N_1 = 1 < N_2 < \dots$ such that $\sum_{n=N_m+1}^{N_{m+1}} \delta_n^2 \geq 1$. If $x_m = c_1 u_1 + c_2 u_2 + \dots + c_{N_m} u_{N_m}$, then by Pythagoras' theorem ($\{u_n\}$ is an orthonormal system), for $m < m'$ we have

$$\|x_m - x_{m'}\|^2 = \sum_{n=N_m+1}^{N_{m+1}} \delta_n^2 \geq 1.$$

This shows that (x_m) contains no convergent subsequence. It is clear that $x_m \in S$, hence S is not compact.

We conclude that S is compact iff $\sum_1^{\infty} \delta_n^2 < \infty$.

Page93, question7.

Proof. Assume $\sum a_n^2 = \infty$, we shall find a sequence (b_n) such that $\sum b_n^2 = 1$ but $\sum a_n b_n$ divergence. Since $\sum a_n^2 = \infty$, we may find $0 = N_1 < N_2 < \dots$ such that

$$c_k := \sum_{n=N_k+1}^{N_{k+1}} a_n^2 \geq 2^k, \qquad k = 1, 2, \dots$$

We define $b_n = (2^k c_k)^{-\frac{1}{2}} a_n$ for $N_k + 1 \le n \le N_{k+1}, k = 1, 2, \ldots$ we can verify that

$$\sum_{n=N_k+1}^{N_{k+1}} b_n^2 = \frac{1}{2^k}, \qquad \sum_{n=N_k+1}^{N_{k+1}} a_n b_n = 2^{-\frac{k}{2}} c_k^{\frac{1}{2}} \ge 1.$$

Contradictions.

Page93, question11.

Proof. Let $E = \{u_n | u_n = (1 + \frac{1}{n}) e^{int}, n \in \mathbb{Z}\}$, then $E \subset L^2(T)$ and E is closed. Since $\|u_n\|_{L^2} = (1 + \frac{1}{n})$, which implies that E has no smallest element.

Page93, question13.

Proof. First consider the case $f(t) = e^{2\pi ikt}, k = 0, \pm 1, \pm 2, \ldots$. Obviously equality holds for $k = 0$. If $k \neq 0$, since α is irrational, $e^{2\pi ik\alpha} \neq 1$. So

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ikn\alpha} &= \frac{1}{N} \sum_{n=1}^{N} \left(e^{2\pi ik\alpha}\right)^n \\ &= \frac{e^{2\pi ik\alpha}}{N} \frac{e^{2\pi ikN\alpha - 1}}{e^{2\pi ik\alpha} - 1}. \end{aligned}$$

Hence

$$\left|\frac{1}{N} \sum_{n=1}^{N} e^{2\pi ikn\alpha}\right| = \frac{1}{N} \left|\frac{e^{2\pi ikN\alpha - 1}}{e^{2\pi ik\alpha} - 1}\right| \le \frac{2}{N|e^{2\pi ik\alpha} - 1|}.$$

This implies

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi ikn\alpha} = 0 = \int_0^1 e^{2\pi ikt} dt \quad (k = 0, \pm 1, \pm 2, \ldots)$$

It follows easily that if $P(t) = \sum_{k=-N}^{N} c_k e^{2\pi ikt}$, then equality holds with P instead of f.

Now if f is continuous with period 1, then $\forall \varepsilon > 0$, there is a $P(t) = \sum_{k=-N}^{N} c_k e^{2\pi ikt}$ such that $\|f - P\|_\infty < \varepsilon$. Thus with $g = f - P$, we have

$$\left|\frac{1}{N} \sum_{1}^{N} f(n\alpha) - \int_0^1 f(t) dt\right| \le \left|\frac{1}{N} \sum_{1}^{N} g(n\alpha) - \int_0^1 g(t) dt\right| + \left|\frac{1}{N} \sum_{1}^{N} P(n\alpha) - \int_0^1 P(t) dt\right|.$$

The first term at the right side of the inequality is $\le 2\varepsilon$ since $\|g\|_\infty \le \varepsilon$. Therefore

$$\lim_{N \to \infty} \sup \left|\frac{1}{N} \sum_{1}^{N} f(n\alpha) - \int_0^1 f(t) dt\right| \le 2\varepsilon.$$

Hence

$$\lim_{N\to\infty} |\frac{1}{N}\sum_{1}^{N} f(n\alpha) - \int_0^1 f(t)dt| = 0.$$