MAT 5011 Real Analysis

Suggested Answer to Assignment 4

Page59, question22.

Proof Since there exists a point p such that $f(p) < \infty$ and for given $n, nd(x,p)$ is finite. Thus the definition of $g_n(x)$ is well-defined.

(1) Similarly as in the proof of question3 , we have

$\forall \epsilon > 0, |g_n(x) - g_n(y)| \leq nd(x,y) + \epsilon$. Let $\epsilon \to 0$ we get

$$|g_n(x) - g_n(y)| \leq nd(x,y).$$

This implies that $g_n(x)$ is continuous.

(2) Since for any n,

$$\inf_{p\in X}[f(p) + (n+1)d(x,p)] \geq \inf_{p\in X}[f(p) + nd(x,p)] + \inf_{p\in X} d(x,p) = \inf_{p\in X}[f(p) + nd(x,p)]$$

i.e. $g_n(x)$ is increasing.

Obviously, $g_n(x) \leq f(x) + nd(x,x) = f(x)$. This ends the proof.

(3) If there exists x_0 such that $g_n(x_0) \nrightarrow f(x_0)$. Since $f(x_0)$ is an upper bound of $\{g_n(x_0)\}$ and $\{g_n(x_0)\}$ is increasing. We have $g_n(x_0) \to a < f(x_0)$.

In the case of $f(x_0) < \infty$. By the definition, $\exists \{p_n\}$ such that $f(p_n) + nd(x_0, p_n) < \frac{a+f(x_0)}{2} < f(x_0)$. Then if $p_n \to x_0$, $\lim_{n\to\infty} f(p_n) \geq f(x_0)$ since $f(x)$ is lower semi-continuous. Contradiction. If $p_n \nrightarrow x_0$, there exists a constant $\delta > 0$, such that $d(x_0, p_n) \geq \delta, n$ large enough. Thus $nd(x_0, p_n) \to \infty$. Contradiction.

In the case of $f(x_0) = \infty$. By the definition, $\exists \{p_n\}$ such that $f(p_n) + nd(x_0, p_n) < 2a < f(x_0) = \infty$. Then similar discussion leads to contradiction. This finishes the proof.

Page60, question23.

Proof: Not necessary continuous nor upper semi-continuous. For instance $\mu(A) = 0$ if $0 \notin A$ and $\mu(A) = 1$ if $0 \in A$. Obviously such μ is a finite positive Borel measure.

Let $V=(0,1)$, then

$$\mu(V)=0, \quad \mu(V+\delta)\equiv 0, \quad \mu(V-\delta)\equiv 1, \quad \text{for} \quad 0<\delta<0.01.$$

But μ is lower semi-continuous. The proof is standard and here we omit it.

Page60, question24.

Proof: W.L.O.G., we may assume that f is a nonnegative real-value function. First we claim: $\forall\epsilon>0$, there exists a continuous function $g(x)\in C_c(R^1)$ such that $\int_{R^1}|f(x)-g(x)|<\frac{\epsilon}{2}$.

since $f\in L^1(R^1)$, there exists a simple function $s(x), \max_{x\in R^1}|s(x)|\le M$ such that $\int_{R^1} f(x)-s(x)<\frac{\epsilon}{4}$. By Lusin's Lemma there exists a continuous $g(x)\in C_c(R^1)$ such that

$$m(\{x:s(x)\neq g(x)\})<\frac{\epsilon}{4M}, \qquad \max_{x\in R^1}|g(x)|\le M.$$

It's easy to check that $\int_{R^1}|f(x)-g(x)|<\frac{\epsilon}{2}$.

Let $I=[-a,a], a>0$ be the support set of $g(x)$. By the uniform continuity of $g(x)$ there exists a step-function $\varphi(x)=\sum_{i=1}^N a_i\chi_{I_i}(x)$ such that $\int_{R^1}|g(x)-\varphi(x)|<\frac{\epsilon}{2}$. Thus

$$\int_{R^1}|f(x)-\varphi(x)|<\int_{R^1}|f(x)-g(x)|+\int_{R^1}|g(x)-\varphi(x)|<\epsilon.$$

where $\{I_i\}$ forms a partition of I.

Now we choose $\epsilon_n=\frac{1}{n}, n=1,2\ldots$, we get a sequence $\{\varphi_n\}$ such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}|f(x)-\varphi_n(x)|=0.$$

Page71, question3.

Proof: By M.I. we can get

$$\varphi(\frac{i}{2^k}x+(1-\frac{i}{2^k})y)\le\frac{i}{2^k}\varphi(x)+(1-\frac{i}{2^k})\varphi(y),$$

where $i,k\in\mathbb{N}$ and $1\le i<2^k$. And for any $\lambda\in[0,1]$, there exist $\{i_n,k_n\}, n\in\mathbb{N}$ such that $\frac{i_n}{2^{k_n}}\to\lambda$. By the continuity of $\varphi(x)$ we get that $\forall x,y\in(a,b)$,

$$\varphi(\lambda x+(1-\lambda)y)=\lim_{n\to\infty}\varphi(\frac{i_n}{2^{k_n}}x+(1-\frac{i_n}{2^{k_n}})\varphi(y)\le\lim_{n\to\infty}[\frac{i_n}{2^{k_n}}\varphi(x)+(1-\frac{i_n}{2^{k_n}})\varphi(y)]=\lambda\varphi(x)+(1-\lambda)\varphi(y).$$

Thus φ is convex.

Page71, question4.

Proof: (a) Let

$$E_1 = \{x \in X : |f(x)| \geq 1\}, \quad E_2 = \{x \in X : |f(x)| < 1\}.$$

Since $f \in L^r(\mu) \cap L^s(\mu)$ and $p \in (r, s)$, we have

$$\int_X |f|^p = \int_{E_1} |f|^p + \int_{E_2} |f|^p \leq \int_{E_1} |f|^s + \int_{E_2} |f|^r \leq \|f\|_s^s + \|f\|_r^r < \infty$$

Thus $f \in L^p(\mu)$.

(b) By Hölder's inequality, for any $\lambda \in (0, 1)$, we get

$$\int_X |f|^{\lambda p_1 + (1-\lambda)p_2} \leq (\int_X |f|^{\lambda p_1 \cdot \frac{1}{\lambda}})^\lambda \cdot (\int_X |f|^{(1-\lambda)p_2 \cdot \frac{1}{1-\lambda}})^{1-\lambda}.$$

i.e.

$$\int_X |f|^{\lambda p_1 + (1-\lambda)p_2} \leq (\int_X |f|_1^p)^\lambda \cdot (\int_X |f|_2^p)^{1-\lambda}.$$

Thus

$$\varphi(\lambda p_1 + (1-\lambda)p_2) \leq \varphi(p_1)^\lambda \cdot \varphi(p_2)^{1-\lambda}.$$

Since the function $y = \log x$ is increasing, then

$$\log \varphi(\lambda p_1 + (1-\lambda)p_2) \leq \lambda \log \varphi(p_1) + (1-\lambda) \log \varphi(p_2).$$

$\lambda = 0, 1$ is trivial. By (a) if $p_1 \in E, p_2 \in E$ then $(p_1, p_2) \subset E$. Thus $\log \varphi(p)$ is convex in the interior of E. For any $r < s, r, s \in E$. In $E_2, |f|^r - |f|^s \leq 2|f|^r, 2|f|^r \in L^1(E_2, \mu)$ and in $E_1, |f|^s - |f|^r \leq 2|f|^s, 2|f|^s \in L^1(E_1, \mu)$. By the L.D.C.T., $\forall \epsilon > 0, \exists \delta > 0$, if $|r - s| \leq \delta$, then

$$\int_{E_1} (|f|^s - |f|^r) < \frac{\epsilon}{2}, \int_{E_2} (|f|^r - |f|^s) < \frac{\epsilon}{2}.$$

Then

$$|\varphi(r) - \varphi(s)| \leq \int_{E_1} (|f|^s - |f|^r) + \int_{E_2} (|f|^r - |f|^s) < \epsilon.$$

Thus φ is continuous in E.

(c)By (a) E is path connected and then is connected.

E may be open. E.g., $X = [a, \infty], a > 0, f(x) = \frac{1}{x}$, then

$$\int_a^\infty \frac{1}{x^p} = \begin{cases} \infty, & p \in (0, 1] \\ \frac{a^{1-p}}{p-1}, & p > 1. \end{cases}$$

Thus $E = (1, \infty)$.

E may be closed. Let $X = (0, \infty), E_n = [n, n + \frac{1}{2^n n^2}], n \in \mathbb{N}, f(x) = \sum_{n=1}^\infty 2^n \chi_{E_n}$, then

$$\int_X f(x)^{1+\delta} = \sum_{n=1}^\infty 2^{n(1+\delta)} \frac{1}{2^n n^2} = \sum_{n=1}^\infty \frac{2^{n\delta}}{n^2} = \begin{cases} \infty, & \delta > 0 \\ \text{finite}, & \delta \le 0. \end{cases}$$

Thus $E = (0, 1]$ is relative closed.

E may be a point. Similar as the above, we define $F_n = [k_n, k_n + \frac{2^n}{n^2}], g(x) = \sum_{n=1}^\infty 2^{-n} \chi_{F_n}$ where we chose k_n such that $F_n \cap F_m = \emptyset, n \neq m$. Then

$$\int_X g(x)^{1+\delta} = \sum_{n=1}^\infty 2^{-n(1+\delta)} \frac{2^n}{n^2} = \sum_{n=1}^\infty \frac{1}{2^{n\delta} n^2} = \begin{cases} \infty, & \delta < 0 \\ \text{finite}, & \delta \ge 0. \end{cases}$$

Then $E_g = [1, \infty)$. Now let $f(x) = \sum_{n=1}^\infty 2^n \chi_{E_n} + \sum_{n=1}^\infty 2^{-n} \chi_{F_n}$, then $E = \{1\}$.

By the above example E can be any connected subset of $(0, \infty)$.

(d) If $r < p < s$, then there exists λ such that $p = \lambda r + (1 - \lambda)s$. By the convexity of $\log \varphi$, we have

$$\log \|f\|_p^p \le \lambda \log \|f\|_r^r + (1 - \lambda) \log \|f\|_s^s = \log[(\|f\|_r^r)^\lambda \cdot (\|f\|_s^s)^{(1-\lambda)}].$$

Thus

$$\|f\|_p^p \le (\|f\|_r^r)^\lambda \cdot (\|f\|_s^s)^{(1-\lambda)} \le (\max\{\|f\|_r, \|f\|_s\})^{\lambda r + (1-\lambda)s} = (\max\{\|f\|_r, \|f\|_s\})^p,$$

i.e.

$$\|f\|_p \le \max\{\|f\|_r, \|f\|_s\}.$$

Obviously, if $\|f\|_r < \infty$ and $\|f\|_s < \infty$, then $\|f\|_p < \infty$, i.e. $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) If $\|f\|_\infty < \infty$, then $\forall \epsilon > 0$, let

$$E_\epsilon = \{x \in X : 0 < \|f\|_\infty - \epsilon \le |f| \le \|f\|_\infty$$

then $\mu(E_\epsilon) < \infty$. If not,

$$\int_X |f|^r \geq \int_{E_\epsilon} (\|f\|_\infty - \epsilon)^r = \infty,$$

contradiction. Hence

$$\|f\|_p = (\int_X |f|^p)^{\frac{1}{p}} \geq (\int_{E_\epsilon} (\|f\|_\infty - \epsilon)^p)^{\frac{1}{p}} = (\|f\|_\infty - \epsilon)(\mu(E_\epsilon))^{\frac{1}{p}} \to \|f\|_\infty - \epsilon.$$

On the other hand for $p > r$,

$$\|f\|_p = (\int_X |f|^{p-r}|f|^r)^{\frac{1}{p}} \leq \|f\|_r^{\frac{r}{p}} \|f\|_\infty^{1-\frac{r}{p}}.$$

Thus

$$\lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

In conclusion we have $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Page71, question5.

Proof: (a) By Hölder's inequality, we have

$$\int_X |f|^r \leq (\int_X |f|^s)^{\frac{r}{s}} \cdot (\int_X 1)^{1-\frac{r}{s}} = (\int_X |f|^s)^{\frac{r}{s}} = \|f\|_s^r$$

since $\frac{r}{s}, 1 - \frac{r}{s} \in (0,1)$. Thus $\|f\|_r \leq \|f\|_s$.

(b) By (a) $\|f\|_r = \|f\|_s$ if and only if the equality holds in Hölder's inequality, i.e. $\frac{|f|^s}{1^{\frac{s}{s-r}}} =$ constant a.e.μ. Thus $|f| =$ constant a.e.μ.

(c) By (a), if $f \in L^s(\mu)$, then $\|f\|_r \leq \|f\|_s < \infty$. Hence $f \in L^r(\mu)$, i.e. $L^s(\mu) \subset L^r(\mu)$. On the other hand if $0 < r < s$, we should give the condition of $L^s(\mu) \supset L^r(\mu)$.

Claim: If $0 < r < s < \infty$. $L^r(\mu) \subset L^s(\mu)$ if and only if there exists $\epsilon_0 > 0$ such that for any Borel set $E \subset X$ with $0 < \mu(E) < \infty$ we have $\mu(E) > \epsilon_0$. Similarly, $L^r(\mu) \supset L^s(\mu)$ if and only if there exists $M > 0$ such that for any Borel set $E \subset X$ we have $\mu(E) < M$.

If there is a sequence of measurable sets $\{E_n\}$ with $0 < \mu(E_n) < \frac{1}{3^n}$. We may suppose that $E_n's$ are pointwise disjoint, $a_n = \mu(E_n) \to 0$ as $n \to \infty$ and $\sum_k a_{n_k}^{1-\frac{r}{s}} < \infty, \quad \sum_k a_{n_k}^{\frac{s}{r}-1} = \infty$ for an infinite subsequence $\{a_{n_k}\}$ since $1 - \frac{r}{s} < \frac{s}{r} - 1$. Define $f = \sum_k a_{n_k}^{\frac{1}{r}-\frac{1}{s}} \chi_{E_{n_k}}$, then $f \in L^r - L^s$. Contradiction.

Conversely, suppose $f \in L^r(\mu)$, then the support of f, $U =$ the closure of $\{x \in X : f(x) \neq 0\}$ is a $\sigma-$finite set. If $\mu(U) = 0$, it's clear that the claim is true. Now we assume $\mu(U) > 0$, i.e. $\mu(U) \geq \epsilon_0$. Then there exists a partition $\{E_n\}$ of U such that

$$\mu(E_n) \geq \epsilon_0, \forall A \subset E_n, \text{measurable, then} \mu(A) = 0 \quad \text{or} \quad \mu(E_n - A) = 0, n = 1, 2, \dots.$$

It suffices to prove the conclusion with $\mu(U)$ finite, and finite partition instead of $U\sigma$-finite and countable partition (E_n).

Let $\epsilon_0 \leq \mu(E_n) \leq n\epsilon_0$, then the proof is by induction on n.

$n = 1$, the partition of U is itself. $\forall A \subset U$ measurable, if $\mu(A) > 0$ then $\mu(A) \geq \epsilon_0$. So $\mu(U) = \mu(A) = \epsilon_0$ i.e.$\mu(U - A) = 0$.

Assume that the statement is true for all $n \leq k$. If $n = k + 1$, taking a measurable set A such that $\mu(A) \geq \epsilon_0, \mu(U - A) \geq \epsilon_0$. If not, $\forall B \subset U$, measurable, $\mu(B) = 0$ or $\mu(B) = U$, which is the case that we need. Thus $\mu(A) \leq k\epsilon_0$ and $\mu(U - A) \leq k\epsilon_0$. By the assumption there exists a partition $\{B_n\}$ of A and $\{C_n\}$ of $U - A$ such that $\{B_n\}, \{C_n\}$ have the above property. Thus $\{B_n, C_n\}$ forms a partition of U as we need.

Thus $f\ |_{E_n}$ is constant a.e. therefore f is bounded a.e. by a constant M since $f \in L^r(\mu)$. Thus $|f|^{s-r} \leq M^{s-r}$ a.e. Hence

$$\int |f|^s \leq M^{s-r} \int |f|^r < \infty.$$

If there are measurable sets $\{E_n\}$ with $b_n = \mu(E_n) \to \infty$ increasingly. We may suppose that $E_n's$ are pointwise disjoint and $\sigma-$ finite. We can find a subsequence $\{b_{n_k}\}$ such that $\sum_k b_{n_k}^{-\frac{s}{r}} < \infty, \quad \sum_k b_{n_k}^{-1} = \infty$.

Define $f = \sum_k b_{n_k}^{-\frac{1}{r}} \chi_{E_{n_k}}$, then $f \in L^s - L^r$. Contradiction.

Conversely, it is just the part (a).

(d) By Jensen's inequality we have

$$\log \|f\|_p = \frac{1}{p} \log \int_X |f|^p \geq \frac{1}{p} \int_X \log |f|^p = \int_X \log |f|.$$

On the other hand, $\forall x > 0, x - 1 - \log x \geq 0$. Thus $\|f\|_p^p - 1 - \log \|f\|_p^p \geq 0$, i.e.

$\frac{1}{p}(\|f\|_p^p - 1) \geq \log \|f\|_p$.

$$\int_X \log |f| \leq \log \|f\|_p \leq \int_X \frac{|f|^p - 1}{p}$$

since $\mu(X) = 1$.

Obviously $\frac{|f|^p-1}{p}$ is increasing with respect to p and $\lim_{p\to 0} \frac{|f|^p-1}{p} = \log |f|$. By monotone theorem we have

$$\lim_{p\to 0} \int_X \frac{|f|^p - 1}{p} = \int_X \log |f|.$$

Thus

$$\lim_{p\to 0} \|f\|_p = e^{\int_X \log |f| d\mu}.$$

Page72, question7.

Proof: First we consider $l^r(A)$ and $l^s(A)$ where $0 < r < s$. Then

$$\|f\|_\infty^r = \sup_{a\in A} |f(a)|^r \leq \sum_{a\in A} |f(a)|^r \leq \|f\|_r^r.$$

$$\|f\|_s^s = \int_X |f|^s = \int_X |f|^{s-r}|f|^r \leq \|f\|_\infty^{r-s}\|f\|_r^r \leq \|f\|_r^s.$$

i.e. $\|f\|_s \leq \|f\|_r$.

Part (a) of question 5 gives out an example which satisfies $L^s(\mu) \subset L^r(\mu)$.

Let $X = [0,\infty), m$ be the Lebesgue measure. Then $L^r(m) \neq L^s(m)$. E.g. $r = 1, s = 2, f_1 = |x|^{-\frac{1}{2}}\chi_{(0,1]}$, then $f_1 \in L^1$ and $\|f_1\|_2 = \infty$. On the other hand, $f_2 = |x|^{-1}\chi_{[1,\infty]}$, then $f_2 \in L^2$ and $\|f_2\|_1 = \infty$. The Claim of question5 has given the conditions on μ.

Page72, question11.

Proof: Since $fg \geq 1$ we have $\sqrt{fg} \geq 1$. By Hölder's inequality we have

$$1 \leq \int_\Omega \sqrt{f}\sqrt{g} d\mu \leq (\int_\Omega (\sqrt{f})^2)^{\frac{1}{2}} (\int_\Omega (\sqrt{g})^2)^{\frac{1}{2}} = (\int_\Omega f \cdot \int_\Omega g)^{\frac{1}{2}}.$$

Hence $\int_\Omega f \cdot \int_\Omega g \geq 1$.

Page72, question14.

Proof:(a) First assume that $f \geq 0, f \in C_c(R^+)$, then $\int_0^x f dt$ is differentiable. By $xF(x) = \int_0^x f dt$ we get $xF' + F = f$. Hence

$$\int_0^\infty F^p dx = xF^p \mid_0^\infty - \int_0^\infty pxF^{p-1}F' dx = p\int_0^\infty F^p dx - p\int_0^\infty fF^{p-1} dx.$$

That is

$$\frac{p-1}{p}\int_0^\infty F^p dx = \int_0^\infty fF^{p-1}dx \le (\int_0^\infty f^p dx)^{\frac{1}{p}}(\int_0^\infty F^{(p-1)q})^{\frac{1}{q}} = \|f\|_p \|F\|_p^{\frac{p}{q}} = \|f\|_p\|F\|_p^{p-1}$$

where $\frac{1}{p}+\frac{1}{q}=1$.Thus

$$\|F\|_p \le \frac{p}{p-1}\|f\|_p.$$

For general function $f \in C_c(R^+), f = f_+ - f_-$ where f_+, f_- are nonnegative, we have $\|F_\pm\|_p \le \frac{p}{p-1}\|f_\pm\|_p$ where $F_\pm$ is the image of $f_\pm$ by Hardy transform.

$$\begin{aligned}
\|F\|_p^p &= \int_0^\infty |\frac{1}{x}\int_0^x (f_+ - f_-)dt|^p dx = \int_0^\infty |F_+ - F_-|^p dx \\
&\le \int_0^\infty F_+^p dx + \int_0^\infty F_-^p dx \\
&\le (\frac{p}{p-1})^p(\int_0^\infty f_+^p dx + \int_0^\infty f_-^p dx) \\
&= (\frac{p}{p-1})^p \int_0^\infty (f_+ + f_-)^p dx
\end{aligned}$$

Thus

$$\|F\|_p \le \frac{p}{p-1}\|f\|_p.$$

If $f \in L^p(\mu)$, then there exists a sequence $f_n \in C_c(R^+)$ such that $f_n \to f$ a.e. and $\|f_n - f\|_p \to 0$ as $n \to \infty$. Thus

$$\int_0^x |f_n - f|dt \le x^{\frac{1}{q}}\|f_n - f\|_p \to 0, \quad n \to \infty,$$

thus $F_n \to F$ a.e.. By Fatou's Lemma we get

$$\int_0^\infty |F|^p dx \le \liminf_{n\to\infty}\int_0^\infty |F_n|^p dx \le (\frac{p}{p-1}\|f\|_p)^p.$$

Thus $\|F\|_p \le \frac{p}{p-1}\|f\|_p$.

(b): The Hölder's equality holds if and only if $f^p \equiv= \text{constant}F^{(p-1)q} = \text{constant}F^p$ a.e., i.e. $f = \text{constant}F$ a.e..Thus $f = \text{constant}$ a.e.. Since $f \in L^p(R)$, we get $f = 0$ a.e..

(c): Let

$$f(x) = \begin{cases} x^{-\frac{1}{p}}, & x \in [1, A] \\ 0, & \text{otherwise} \end{cases}$$

then $\|f\|_p^p = \log A$ and

$$F(x) = \begin{cases} 0, & x \in (0,1) \\ \frac{p}{p-1}(x^{-\frac{1}{p}} - x^{-1}), & x \in [1, A] \\ \frac{p}{p-1}(A^{1-\frac{1}{p}} - 1)x^{-1} & x \in (A, +\infty) \end{cases}$$

Then $\|F\|_p^p = I_1 + I_2$, where

$$I_2 = \int_A^\infty \{\frac{p}{p-1}(A^{1-\frac{1}{p}} - 1)x^{-1}\}^p = (\frac{p}{p-1})^p \frac{1}{p-1}(1 - A^{\frac{1}{p}-1})^p,$$

$$I_1 = \int_0^A \{\frac{p}{p-1}(x^{-\frac{1}{p}} - x^{-1})\}^p = (\frac{p}{p-1})^p \int_1^A (x^{-\frac{1}{p}} - x^{-1})^p.$$

Assume that the constant in the right hand side of part(a) can be replaced by $\alpha\frac{p}{p-1}$ for some $\alpha \in (0,1)$. Then there exists β such that $\alpha < \beta < 1$. Let A be large enough, there exists $A_0 < A$ such that for $x > A_0, x^{-\frac{1}{p}} - x^{-1} > \beta x^{-\frac{1}{p}}$. Then

$$I_1 > (\frac{p}{p-1})^p \beta^p \int_{A_0}^A x^{-1} = (\frac{p}{p-1})^p \beta^p (\log A - \log A_0)$$

and $\beta^p(\log A - \log A_0) > \alpha^p \log A$ since $\alpha < \beta$ and A large enough. This implies that

$$I_1 > (\alpha\frac{p}{p-1})^p \log A = (\alpha\frac{p}{p-1})^p \|f\|_p,$$

hence $\|F\|_p > (\alpha\frac{p}{p-1})^p \|f\|_p$ if A sufficiently large, contradiction.

(d): since $f > 0$, there exists x_0 such that $\int_0^{x_0} f dt \geq M$ where M is a positive number. Then

$$\int_{x_0}^\infty F dx = \int_{x_0}^\infty \frac{1}{x}\int_0^x f dt dx \geq \int_{x_0}^\infty \frac{1}{x}\int_0^{x_0} f dt dx \geq \int_{x_0}^\infty \frac{M}{x} = +\infty,$$

thus $F \notin L^1(R^+)$.

extra problem1.

Proof: For any sequence $\{t_n\}, t_n \to 0$ as $n \to \infty$, let $f_n(x) = f(x + t_n)$. Then $f_n(x) \in L^p$, $f_n(x) \to f(x)$ a.e. and $\|f_n\|_p = \|f\|_p$. It is well known that

$$|\alpha - \beta|^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p), \qquad p \geq 1.$$

Put $h_n = 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p \geq 0$ and $h_n \to 2^p|f|^p$, by Fatou's Lemma we have

$$\begin{aligned} 2^p \int_{R^k} |f|^p &\leq \liminf_{n\to\infty} \int_{R^k} [2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p] \\ &= 2^p \int_{R^k} |f|^p - \limsup_{n\to\infty} \int_{R^k} |f_n - f|^p. \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \int_{R^k} |f_n - f|^p = 0.$$

Since $\{t_n\}$ is arbitrary we get that

$$\lim_{t\to 0} \int_{R^k} |f(x+t) - f(x)|^p dx = 0.$$

Since $C_c(R^k)$ is dense in $L^p(R^k)$, then

$$\forall \varepsilon > 0, \exists g \in C_c, \quad \text{such that} \quad \|f(x) - g(x)\|_p \leq \frac{\varepsilon}{4}.$$

Let

$$f(x) = g(x) + h(x), \quad f_t = f(x-t) = g(x-t) + h(x-t) := g_t + h_t$$

where $h(x) \in L^p$ and $\|h(x)\|_p \leq \frac{\varepsilon}{4}$.

We need to show that $\lim_{|t|\to+\infty} \|f + f_t\|_p = 2^{\frac{1}{p}}\|f\|_p$.

$$\begin{aligned} \Big| \|f + f_t\|_p - 2^{\frac{1}{p}}\|f\|_p \Big| &\leq \Big| \|f + f_t\|_p - \|g + g_t\|_p \Big| + \Big| \|g + g_t\|_p - 2^{\frac{1}{p}}\|f\|_p \Big| \\ &\leq \|f + f_t - g - g_t\|_p + \Big| \|g + g_t\|_p - 2^{\frac{1}{p}}\|f\|_p \Big| \\ &= \|h + h_t\|_p + \Big| \|g + g_t\|_p - 2^{\frac{1}{p}}\|f\|_p \Big|. \end{aligned}$$

Obviously, $\|h + h_t\|_p \leq \frac{\varepsilon}{2}$. Since $g(x) \in C_c(R^k)$, then there exists M large enough such that if $|t| \geq M, \mathrm{supp} g \cap \mathrm{supp} g_t = \emptyset$. Then

$$\|g + g_t\|_p^p = \int_{R^k} |g + g_t|^p = \int_{\mathrm{supp} g} |g|^p + \int_{\mathrm{supp} g_t} |g_t|^p = 2\|g\|_p^p.$$

Hence if $|t| \geq M$,

$$\Big| \|g + g_t\|_p - 2^{\frac{1}{p}}\|f\|_p \Big| = 2^{\frac{1}{p}} \Big| \|g\|_p - \|f\|_p \Big| \leq 2^{\frac{1}{p}}\|g - f\|_p \leq \frac{\varepsilon}{2}.$$

That's to say

$$\left| \|f+f_t\|_p - 2^{\frac{1}{p}} \|f\|_p \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\lim_{|t|\to+\infty} \int_{R^k} |f(x)+f(x-t)|^p = 2\int_{R^k} |f|^p.$$

extra problem 2.

Proof: Use M.I.

By Hölder's inequality we know the validity of generalized Hölder's inequality for $n=2$. Assume that it's true for $n=k$. Then for $n=k+1$, let $g=f_2\cdots f_{k+1}$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$. Obviously

$$\frac{1}{p_2}+\cdots+\frac{1}{p_{k+1}}=\frac{1}{q_1},$$

that is

$$\frac{q_1}{p_2}+\cdots+\frac{q_1}{p_{k+1}}=1.$$

By Hölder's inequality, we have

$$\begin{aligned}
\int_X |f_1 f_2 \cdots f_{k+1}| &\leq \|f_1\|_{p_1}\|g\|_{q_1} = \|f_1\|_{p_1}\left(\int_X |f_2|^{q_1}\cdots|f_{k+1}|^{q_1}\right)^{\frac{1}{q_1}} \\
&\leq \|f_1\|_{p_1}\left(\||f_2|^{q_1}\|_{\frac{p_2}{q_1}}\cdots\||f_{k+1}|^{q_1}\|_{\frac{p_{k+1}}{q_1}}\right)^{\frac{1}{q_1}} \\
&\leq \|f_1\|_{p_1}\|f_2\|_{p_2}\cdots\|f_{k+1}\|_{p_{k+1}}.
\end{aligned}$$