MAT 5011 Real Analysis

Suggested Answer to Assignment 3

Page58, question4.

Proof Part(a). For any given open set $U \supset E_1 \cup E_2$, then $U \cap V_1$ and $U \cap V_2$ are both open sets which are disjoint and $E_i \subset U \cap V_i, i = 1, 2$. Thus

$$\mu(E_1) + \mu(E_2) \leq \mu(U \cap V_1) + \mu(U \cap V_2) \leq \mu(U).$$

Since U is arbitrary, we get $\mu(E_1) + \mu(E_2) \leq \mu(E_1 \cup E_2)$. By the subadditivity of μ we get $\mu(E_1) + \mu(E_2) = \mu(E_1 \cup E_2)$

Part(b) if $E \in \mathfrak{M}$, denoting E_1. By the inner regularity, $\exists$a compact set$K_1 \subset E_1$ such that $\mu(E_1 - K_1) < \frac{1}{2}$. Let $E_2 = E_1 - K_1$. By I.M. we get a sequence $\{E_n\}$ and a compact sequence $\{K_n\}$satisfying $K_n \subset E_n$ and $\mu(E_n - K_n) < \dfrac{1}{2^n}$ and $E_{n+1} = E_n - K_n$. By the construction E_n is decreasing. Let $\mathcal{N} = \bigcap_{n=1}^{\infty} E_n$, then $\mu(\mathcal{N}) = 0$. Of course, $E = \mathcal{N} \cup K_1 \cup K_2 \cup \cdots$

Page58, question5.

Proof By the construction of Cantor sets, we know the sum of the length of deleted open intervals in $[0, 1]$ is

$$\sum_{n=1}^{\infty} \frac{1}{3}(\frac{2}{3})^n = 1.$$

Thus $\mu(E) = 0$.

Page58, question8.

Proof We can construct in each $[n, n+1]$, a Borel set E_n of measure c_n with $0 < c_n < 1, \sum_n c_n = c < \infty$ such that for every open segment $I \subset [n, n+1], 0 < m(E \cap I) < m(I)$. For $n = 0$, put $c_0 = b$. we choose a sequence $\{\epsilon_m\}$ strictly decreasing to b with $b < \epsilon_m < 1$. For $m = 1, 2, \ldots, E_m$ is an open set constructed as follows:

(1): E_1 is the open dense subset of $[0, 1]$ of measure ϵ_1;

(2): Each E_m is the union of countably many disjoint open intervals $I_{m,k}$ of length $l_{m,k}, k = 1, 2, \ldots$ with $\sum_{k=1}^{\infty} l_{m,k} = \epsilon_m$.

(3): $E_{m+1} \subset E_m$ such that $E_{m+1} \cap I_{m,k}$ is open dense in $I_{m,k}$ has measure $\frac{\epsilon_{m+1}}{\epsilon_m} l_{m,k}$. In fact, $E_{m+1} = \frac{\epsilon_{m+1}}{\epsilon_m} m(E_m) = \epsilon_{m+1}$. Hence (2) is valid with $m+1$ instead of m.

Let $E = \cap_{m=1}^{\infty} E_m$ and $I = [a-\delta, a+\delta] \subset [0,1]$. Let n_0 satisfy $2^{-n_0} < \frac{\delta}{2}$ and E_{n_0} be dense in $[0,1]$. Then there exists a point $p \in E_{n_0}$ with $|p-a| < 2^{-n_0}, p \in I_{n_0,k_0}$ with $l_{n_0,k_0} \leq 2^{-n_0} < \frac{\delta}{2}$. So $I_{n_0,k_0} \subset I$.

$$m(I_{n_0,k_0} \cap E_n) = \frac{\epsilon_n}{\epsilon_{n_0}} l_{n_0,k_0}, \quad n \geq n_0$$

Thus

$$m(I_{n_0,k_0} \cap E) = \frac{c_0}{\epsilon_{n_0}} l_{n_0,k_0} > 0, \quad m(I_{n_0,k_0} \cap E^c) = (1 - \frac{c_0}{\epsilon_{n_0}}) l_{n_0,k_0} > 0.$$

By (1), $m(I \cap E) > 0$. By (2)$m(I \cap E^c) > 0$. Thus $0 < m(E \cap I) < m(I)$ and $m(E) < \infty$.

Page58, question9.

Proof: $\forall n$, there exist k and i such that $n = 2^k + i, 0 \leq i < 2^k$.

If $i = 0$, let

$$f_n(x) = \begin{cases} 1, & x \in [\frac{i}{2^k}, \frac{i+1}{2^k}] \\ 2 - 2^k x, & x \in (\frac{i+1}{2^k}, \frac{i+2}{2^k}) \\ 0, & \text{other points in } [0,1] \end{cases}$$

If $i = 2^k - 1$,let

$$f_n(x) = \begin{cases} 1, & x \in [\frac{i}{2^k}, \frac{i+1}{2^k}] \\ 2^k x - i + 1, & x \in (\frac{i-1}{2^k}, \frac{i}{2^k}) \\ 0, & \text{other points in } [0,1] \end{cases}$$

If $0 < i < 2^k - 1$, let

$$f_n(x) = \begin{cases} 2^k x - i + 1, & x \in (\frac{i-1}{2^k}, \frac{i}{2^k}) \\ 1, & x \in [\frac{i}{2^k}, \frac{i+1}{2^k}] \\ i + 2 - 2^k x, & x \in (\frac{i+1}{2^k}, \frac{i+2}{2^k}) \\ 0, & \text{other points in } [0,1] \end{cases}$$

Then $f_n(x)$ are continuous on $[0,1], 0 \le f_n(x) \le 1$ and

$$\int_0^1 f_n(x)dx = \begin{cases} \frac{3}{2^{k+1}}, & i = 0, 2^k - 1 \\ \frac{1}{2^{k-1}}, & 0 < i < 2^k - 1 \end{cases}$$

Thus $\int_0^1 f_n(x) \to 0$ as $n \to \infty$. However $f_n(x)$ doesn't converge for all $x \in [0,1]$.

Page58, question12.

Proof Let m be Lebesgue measure in R^1 and K be an arbitrary compact set of R^1. Obviously R^1 is locally compact Hausdorff space. Define

$$\mu(E) = m(E \cap K), \qquad \forall E \in \{\text{Borel sets}\}$$

Obviously, $\mu : \{\text{Borel sets}\} \to [0,\infty)$ is well defined since K is a compact set.
Let $\{E_n\}_{n=1}^{\infty}$ be Borel sets and $E = \bigcup_{n=1}^{\infty} E_n$ where $E_n \cap E_m = \emptyset, n \ne m$. Then

$$\mu(E) = m((\bigcup_{n=1}^{\infty} E_n) \cap K) = m(\bigcup_{n=1}^{\infty}(E_n \cap K)) = \sum_{n=1}^{\infty} m(E_n \cap K) = \sum_{n=1}^{\infty} \mu(E_n)$$

Thus μ is a Borel measure.
By the definition suppt$\mu \subset K$. Let suppt$\mu \cup A = K$, suppt$\mu \cap A = \emptyset$. Then

$$\mu(A) = 0 = m(A \cap K) = m(A)$$

Hence suppt$\mu = K$ a.e. m.

Another solution: The following is based on the definition of question 11 w.r.t. support of μ.
$\forall n \in \mathbb{N}, \cup_{x \in K} B(x, \frac{1}{n})$ is an open cover of K, thus there exists finite $x_{n,1}, \ldots, x_{n,t_n}$ where $t_n \in \mathbb{N}$. Let

$$A = \cup_{n=1}^{\infty} \cup_{i=1}^{t_n} \{x_{n,i}\},$$

then we can proof A is a countable basis of K. We may rearrange $x_{n,i}$ as $\{y_1, y_2, \ldots, \}$. Define

$$\mu(y_i) = \frac{1}{2^i} \quad \text{otherwise} \quad \mu(x) = 0.$$

Obviously μ is Borel measurable and $\mu(K) = 1$. For any proper cpt subset H of K, there exists at least y_{n_0} such that $y_{n_0} \in K - H$. Thus $\mu(H) < \mu(K)$.

Page59, question14.

Proof: Since $f = u + iv, u = u^+ - u^-, v = v^+ - v^-$, W.L.O.G., we may assume that f is a positive real-value function.

Let $\{U_i\}_{i=1}^{\infty}$ be the countable basis of $\mathbb{R}$. Then $V_i = f^{-1}(U_i)$ is Lebesgue measurable. Hence there exists a F_σ set F_i such that $m(V_i - F_i) = 0$.

Define

$$g(x) = \begin{cases} f(x), & x \in \bigcup F_i \\ 0, & \text{otherwise} \end{cases}$$

$$h(x) = \begin{cases} f(x), & x \in \bigcup F_i \\ +\infty, & \text{otherwise} \end{cases}$$

Obviously $g(x)$ and $h(x)$ are Borel measurable. Then $g(x) = h(x)$ a.e. $[m]$ and $g(x) \leq f(x) \leq h(x)$ for every $x \in R^k$.

Page59, question17.

Proof Denote

$$\rho(P_1, P_2) = \begin{cases} |y_1 - y_2|, & x_1 = x_2 \\ 1 + |y_1 - y_2|, & x_1 \neq x_2 \end{cases}$$

where $P_i = (x_i, y_i) \in X, i = 1, 2$. First prove this is a metric.

Obviously $\rho(P_1, P_2) \geq 0, \forall P_1, P_2 \in X, \rho(P_1, P_2) = \rho(P_2, P_1)$ and $\rho(P_1, P_2) = 0$ if and only if $x_1 = x_2, y_1 = y_2$.

$\forall P_i = (x_i, y_i) \in X, i = 1, 2, 3$. Since $|y_1 - y_2| + |y_2 - y_3| \geq |y_1 - y_3|$, it's easy to verify ρ satisfy triangle inequality. Here we omit it.

Let τ_1 be the discrete topology on R^1(that is, every singleton $\{x\}$ is an open set in τ_1). Then every point $x \in R^1$ has the compact set $\{x\}$ as a neighborhood, so that (R^1, τ_1) is a locally compact Hausdorff space. It is clear by the statement in the exercise that the topology space (X, τ) is exactly the topological product of (R^1, τ_1) and (R^1, τ_2) where τ_2 being the usual topology of R^1. Therefore it is clear that (X, τ) is locally compact Hausdorff space.

If K is compact in X, the first projection $pr_1(K)$ is compact in (R^1, τ_1). Hence it is a finite set. Therefore K is a finite union

$$\{x_1\} \times K_1 \cup \cdots \cup \{x_n\} \times K_n,$$

K_i is a compact set in R^1 for $i = 1, 2, \ldots, n$. If $f : X \to \mathbb{C}$ has compact support, then the support of f is contained in $\{x_1, \ldots, x_n\} \times R^1$.
thus for $f \in C_c(X)(f$ continuous with compact support),

$$\bigwedge f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y) dy$$

is a linear functional on $C_c(X)$ and $\bigwedge f \geq 0$ if $f \geq 0$.
By the proof of Riesz's theorem. the measure μ defined by the following two equalities:

$$(1) \qquad \mu(V) = \sup\{\bigwedge f : f \prec V\} = \sup\{\mu(K) : K \subset V, K \quad \text{compact}\}$$

$$(2) \qquad \mu(E) = \inf\{\mu(V) : E \subset V, V \quad \text{open}\}$$

is a representing measure for $\bigwedge$. We observe that if $K = \{x\} \times K'$, where K' is compact in R^1, then by (1) for μ and (2) for m instead of μ, $\mu(K) = m(K')(m$ stands for Lebesgue measure on R^1).
Thus μ is characterized by the identity

$$\mu(\{x\} \times [a, b]) = b - a, x \in R^1.$$

Let V be an open set containing $R^1 \times \{0\}$. Then for $x \in R^1, (x, 0) \in V$, so that there exists an $\epsilon_x > 0$ with

$$\{x\} \times [-\epsilon_x, \epsilon_x] \subset V.$$

This implies that there must be an n with uncountably many $\epsilon_x \geq \frac{1}{n}$(If this is not the case, then ϵ_x for at most countably many x, which contradicts the fact that R^1 is uncountable).
Let $K_x = \{x\} \times [-\frac{\epsilon_x}{2}, \frac{\epsilon_x}{2}]$, for $\epsilon_x \geq \frac{1}{n}$. For

$$K = K_{x_1} \cup \cdots \cup K_{x_m}, \mu(K) \geq \frac{m}{n},$$

hence $\mu(V) \geq \sup\{\frac{m}{n} : m = 1, 2, \ldots\} = \infty$.

Thus we have proved that if V is an open set containing $R^1 \times \{0\}$, then $\mu(V) = \infty$, this implies $\mu(R^1 \times \{0\}) = \infty$.

Now if K is a compact subset of $R^1 \times \{0\}$(K compact in R^1), then

$$K = \{x_1, \cdots, x_n\} \times \{0\}$$

hence evidently $\mu(K) = 0$.

Therefore , for $E = R^1 \times \{0\}$, we find that $\mu(E) = \infty$ and $\sup\{\mu(K) : K \quad \text{compact}, K \subset E\} = 0$, that is μ is not inner regular.

Page59, question20.

Proof Let

$$f_n(x) = \begin{cases} \frac{1}{x}, & x \in [\frac{1}{n+2}, \frac{1}{n+1}] \\ (n+2)^2[(n+3)x-1], & x \in [\frac{1}{n+3}, \frac{1}{n+2}] \\ (n+1)^2(1-nx), & x \in [\frac{1}{n+1}, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}$$

then $f_n(x), n = 1, 2, \cdots$ is continuous, $f_n(x) \to 0, \forall x \in [0,1]$.

$$\int_0^1 f_n(x)dx \leq \int_{\frac{1}{n+3}}^{\frac{1}{n}} \frac{1}{x} = \ln \frac{n+3}{n} \to 0.$$

But $\sup f_n(x) = \frac{1}{x}, \forall x \in [0, \frac{1}{2}], \sup f_n(x) \geq 0, \forall x \in [\frac{1}{2}, 1]$.

Hence $\sup f_n(x)$ is not contained in $L^1([0,1], m)$ where m is Lebesgue measure.