MAT 5011 Real Analysis

Suggested Answer to Assignment 2

Page32, question 10.

Proof $\forall n \in \mathbb{N}, \exists M_n > 0$ such that $|f_n(x)| \leq M_n, \forall x \in X$.

Since $f_n \to f$ uniformly on X, $\exists N \in \mathbb{N}$, such that $n > N, |f_n(x) - f(x)| < 1, \forall x \in X$.

Thus for the above n, we have $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq M_n + 1$.

Furthermore $|f_n(x)| \leq |f(x) - f_n(x)| + |f(x)| \leq |f(x)| + 1$.

Since $\mu(X) < \infty$, $|f(x)| + 1 \in L^1(\mu)$. By L.D.C.T. we have

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Without $\mu(X) < \infty$, the conclusion doesn't hold. For example, let $X = [0, \infty)$, $f_n = \frac{1}{n}\chi_{[0,n]}$, then $f_n \to 0$ uniformly. But $\int_X f_n dx = 1 \neq 0 = \int_X 0dx$.

Page32, question12.

Proof Since $f \in L^1(\mu), \forall \epsilon > 0, \exists$ a simple function s such that $0 \leq s \leq |f|$ and $\int_X |f| d\mu - \int_X s d\mu < \frac{\epsilon}{2}$. According to the definition of simple function $\exists M > 0$ such that $\forall x \in X, s(x) \leq M$. Let $\delta = \frac{\epsilon}{2M+1}$. If $\mu(E) < \delta$ we have

$$\int_E |f| d\mu \leq \int_X |f| d\mu - \int_X s d\mu + \int_E s d\mu < \frac{\epsilon}{2} + M\frac{\epsilon}{2M+1} < \epsilon.$$

This ends the proof.

Page57, question1.

Proof (a) $\forall a \in R$, consider the set $E_\alpha = \{x : f_1(x) + f_2(x) < \alpha\}$. Using the definition we get the following relation

$$E_\alpha = \bigcup_{r,s\in\mathbb{Q}, r+s<\alpha} (f_1^{-1}(-\infty, r) \cap f_2^{-1}(-\infty, s))$$

Since f_1, f_2 are u.s.c., we get that E_α is open, i.e. $f_1 + f_2$ is u.s.c..

(b) $\forall a \in R$, consider the set $F_\alpha = \{x : f_1(x) + f_2(x) > \alpha\}$. Using the definition we get the following relation

$$F_\alpha = \bigcup_{r,s\in\mathbb{Q}, r+s>\alpha} (f_1^{-1}(r, \infty) \cap f_2^{-1}(s, \infty))$$

Since f_1, f_2 are l.s.c., we get that F_α is open, i.e. $f_1 + f_2$ is l.s.c..

(c) The conclusion doesn't hold. For example, let $\{r_i\}_{i\in\mathbb{N}}$ be the all rational numbers and $f_i = \chi_{r_i}$. Then $\{f_i\}_{i\in\mathbb{N}}$ are u.s.c. since the set $\{r_i\}$ is closed. But $\sum_{i=1}^{\infty} f_i = \chi_{\mathbb{Q}}$ is not u.s.c..

(d) Let $g_n = \sum_{i=1}^{n} f_i$. By part(b) and I.M. we know that g_n is l.s.c., nonnegative and then is increasing. Hence $\sum_{i=1}^{\infty} f_i = \sup_{n\in\mathbb{N}} g_n$ is l.s.c. by 2.8(c).

Without the condition "nonnegative", from the above proof we know that part(a) and part(b) remain true. As to part(d), the conclusion doesn't true. For example, similar as part(c), let $f_i = -\chi_{r_i}$. Then $\forall i \in \mathbb{N}, f_i$ is l.s.c.. But $\sum_{i=1}^{\infty} f_i = -\chi_{\mathbb{Q}}$ is not l.s.c..

From the proof of part(a), part(b), part(d) we know that a general topological space doesn't affect the truth of the statement.

Page57, question2.

Please refer to the proof of assignment1, extra problem1. We omit it here.

Page58, question3.

Proof $\forall \epsilon > 0$, by the definition of $\rho_E, \exists y_0 \in E$, such that $\rho_E(x) + \frac{\epsilon}{4} > \rho(x, y_0)$.

If $\rho(x, x') < \frac{\epsilon}{4}$,

$$\begin{aligned} |\rho_E(x) - \rho_E(x')| &= |\inf_{y\in E} \rho(x,y) - \inf_{y\in E} \rho(x',y)| \\ &\leq |\rho(x,y_0) - \rho(x',y_0)| + \frac{\epsilon}{2} \\ &\leq \rho(x,x') + \frac{\epsilon}{2} \leq \epsilon \end{aligned}$$

Thus $\rho_E(x)$ is uniformly continuous.

Let E be closed, then we know that $x \in E$ if and only if $\rho_E(x) = 0$. Obviously, $0 \leq f(x) \leq 1$. If B is a closed set and $A \cap B = \emptyset$, then $\rho_A(x) + \rho_B(x) > 0, \forall x \in X$. Thus $f(x)$ is continuous. The closure of suppt $f(x) \subset X - A$ since $f(x) \mid_A = 0$. Obviously $f(x) \mid_B = 1$. Hence $f(x)$ satisfies Urysohn's lemma if X is local cpt.

Otherwise let $X = C_{[0,1]}, K = \{0\}, V = X, \rho(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$.

Extra problem1: Suppose that $f_n \geq 0$, measurable and $\int_X f_n d\mu \to 0$ as $n \to \infty$. Does it follow that $f_n \to 0$, a.e. in μ?

Proof The conclusion doesn't hold.

Define

$$f_i(x) = \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n})}(x) + \frac{(-1)^n + 1}{2}\chi_1(x)$$

where

$$i = \Sigma_{j=1}^{n-1} 2^j + k + 1, k = 0, 1, ..., 2^n - 1, n = 1, 2, ...$$

then $\{f_i\}_{i \in \mathbb{N}}$ are measurable and

$$\int_0^1 f_i(x)dx = \frac{1}{2^n} \to 0 \quad \text{as} \quad n \to \infty.$$

But $f_i(x)$ doesn't converge at any $x \in [0, 1]$.

Extra problem2: Let $f \in L^1(\mu)$. Suppose that for any $g \in L^1(\mu)$, there holds $fg \in L^1(\mu)$. Show that $f(x)$ is bounded a.e.

Proof Since $f \in L^1(\mu)$, then $\mu(\{x : f(x) = \infty\}) = 0$, i.e. $f(x)$ is finite a.e. μ.

Now $\forall n \in \mathbb{N}$, let $A_n = \{x : n \leq |f(x)| < n + 1\}$. If there exists n_0 such that $\forall n > n_0$, $\mu(A_n) = 0$, then we get the desired result. If not, there exists a subsequence n_k such that $\mu(A_{n_k}) > 0$. Define

$$g(x) = \begin{cases} \frac{1}{\mu(A_{n_k}) n_k k}, & x \in A_{n_k} \\ 0, & \text{otherwise} \end{cases}$$

then

$$\int_X g d\mu = \sum_{k=1}^{\infty} \frac{1}{\mu(A_{n_k}) n_k k} \mu(A_{n_k}) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq C < \infty.$$

But

$$\int_X |fg| d\mu = \sum_{k=1}^{\infty} \int_{A_{n_k}} |f| \frac{1}{\mu(A_{n_k}) n_k k} d\mu \geq \sum_{k=1}^{\infty} n_k \frac{1}{\mu(A_{n_k}) n_k k} \mu(A_{n_k}) \geq \sum_{k=1}^{\infty} \frac{1}{k},$$

which is divergent, contradict to $fg \in L^1(\mu)$. This ends the proof.