

# Non-compactness of the Prescribed $Q$ – curvature Problem in Large Dimensions

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## Abstract

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $N \geq 5$  and  $Q_g$  be its  $Q$  curvature. The prescribed  $Q$  curvature problem is concerned with finding metric of constant  $Q$  curvature in the conformal class of  $g$ . This amounts to finding a positive solution to

$$P_g(u) = cu^{\frac{N+4}{N-4}}, u > 0 \text{ on } M$$

where  $P_g$  is the Paneitz operator. We show that for dimensions  $N \geq 25$ , the set of all positive solutions to the prescribed  $Q$  curvature problem is *non-compact*.

## 1 Introduction

Let  $(M, g)$  be a Riemannian manifold of dimension  $N$ . A basic question in conformal geometry is the following: can one change the original metric  $g$  conformally into a new metric  $g'$  with prescribed properties? This means that one searches for some positive function  $\psi$  (conformal factor) such that  $g' = \psi g$  and the new metric  $g'$  has prescribed properties.

A best known example is the so-called Yamabe problem. For  $N > 2$ , let  $L_g := -\frac{4(N-1)}{N-2}\Delta_g + S_g$  be the conformal Laplacian, where  $\Delta_g$  is the Laplace-Beltrami operator and  $S_g$  is the scalar curvature. If one sets the conformal factor  $\psi = u^{\frac{4}{N-2}}$  ( $u > 0$ ), then it is well known that  $L_g$  has the following conformal covariance property:

$$L_g(u\varphi) = u^{\frac{N+2}{N-2}}L_{g'}(\varphi) \quad \forall \varphi \in C^\infty(M).$$

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If one prescribes the scalar curvature  $S_{g'}$  for the metric  $g'$  then  $u$  has to satisfy the second-order equation

$$L_g(u) = u^{\frac{N+2}{N-2}} L_{g'}(1) = S_{g'} u^{\frac{N+2}{N-2}}. \quad (1)$$

In the case when  $S_{g'}$  is a constant, this is the Yamabe problem. In the case when  $S_{g'}$  is a prescribed function, it is called the Nirenberg problem.

Precisely, the Yamabe equation is

$$\frac{4(N-1)}{N-2} \Delta_g u - S_g u + c u^{\frac{N+2}{N-2}} = 0. \quad (2)$$

The question that whether the set of all solutions to the Yamabe PDE is compact in the  $C^\infty$  – topology has been widely studied. It has been conjectured that this should be true unless  $(M, g)$  is conformally equivalent to the round sphere (see [27, 28, 29]). The case of the round sphere  $\mathbb{S}^N$  is special in that (2) is invariant under the action of the conformal group on  $\mathbb{S}^N$ , which is non-compact. The Compactness Conjecture has been verified in low dimensions and locally conformally flat by R. Schoen [28, 29]. He also proposed a strategy to proving the conjecture in the non-locally conformally flat case. Developing further this strategy, the conjecture is proved in low dimensions:  $N = 3$  by Li-Zhu [22],  $N = 4, 5$  by Druet [10],  $N = 6, 7$  by Li-Zhang [20, 21] and Marques [23]. Recently this conjecture is shown to be true by Khuri-Marques-Schoen [16] for dimensions  $N \leq 24$  under the Positive Mass condition. On the other hand, the Compactness Conjecture is not true for  $N \geq 25$  in the recent papers by Brendle [3] and Brendle-Marques [4]. More precisely, given any integer  $N \geq 25$ , there exists a smooth Riemannian metric  $g$  on  $\mathbb{S}^N$  such that set of constant scalar curvature metrics in the conformal class of  $g$  is non-compact. Moreover, the blowing-up sequences obtained in [3, 4] form exactly one bubble. The construction relies on a gluing procedure based on some local standard metric. The non-compactness of Yamabe problem in the  $C^k$  – topology is studied by Ambrosetti-Malchiodi [1] and Berti-Malchiodi [2]. A complete description of blow-up behavior of Yamabe type problems can be found in the book by Druet-Hebey-Robert [11].

Besides the conformal Laplacian  $L_g$ , there are many other operators which enjoy a conformal covariance property. A particularly interesting one is the fourth order operator  $P_g$  which was discovered by Paneitz in 1983 ([24]), which can be written for  $N \geq 5$  as follows:

$$P_g = \Delta_g^2 - \operatorname{div}_g(a_N S_g g + b_N \mathcal{R}_g) d + \frac{N-4}{2} Q_g, \quad (3)$$

where  $\operatorname{div}_g$  is the divergence of a vector-field,  $d$  is the differential operator,

$$a_N = \frac{(N-2)^2 + 4}{2(N-1)(N-2)}, \quad b_N = -\frac{4}{N-2},$$

and

$$Q_g = -\frac{1}{2(N-1)} \Delta_g S_g + \frac{N^3 - 4N^2 + 16N - 16}{8(N-1)^2(N-2)^2} S_g^2 - \frac{2}{(N-2)^2} |\mathcal{R}_g|^2. \quad (4)$$

Here  $\mathcal{R}_g$  the Ricci tensor,  $|\mathcal{R}_g|^2 = \sum_{i,j} \mathcal{R}^{ij} \mathcal{R}_{ij}$  where  $\mathcal{R}^{ij} = \sum_{s,t} g^{is} \mathcal{R}_{st} g^{tj}$ .  $Q_g$  is the so called  $Q$ -curvature. In the case  $N > 4$ , the conformal factor is usually chosen in the form  $\psi = u^{\frac{4}{N-4}}$  ( $u > 0$ ) and the conformal covariance property of the Paneitz operator reads as follows:

$$P_g(u\varphi) = u^{\frac{N+4}{N-4}} P_{g'}(\varphi) \quad \forall \varphi \in C^\infty(M).$$

If one prescribes the  $Q$ -curvature for the metric  $g'$  by a function  $Q_{g'}$  this leads to the equation

$$P_g(u) = u^{\frac{N+4}{N-4}} P_{g'}(1) = \frac{N-4}{2} Q_{g'} u^{\frac{N+4}{N-4}},$$

which is a fourth-order analogue of (1). We refer to the survey articles of Chang [5] and Chang-Yang [8] and the lecture notes [6, 7] for more background information on the Paneitz operator. Recently, there are more and more interests in using higher order partial differential equations in the study of conformal geometry. See [9, 13, 30] and references therein.

There is an analogue problem to the Yamabe problem, that is, to find metrics of constant  $Q$ -curvature in the conformal class of  $g$ . The problem can be transformed to solve the following  $Q$ -curvature equation, for  $N \geq 5$ ,

$$P_g u = \frac{N-4}{2} u^{\frac{N+4}{N-4}}, \quad u > 0 \quad \text{on } M. \quad (5)$$

Clearly, every solution of (5) is a critical point of the functional

$$\mathcal{E}_g(u) = \frac{\int_M (\Delta_g u)^2 + \sum_{i,j} (a_N S_g g^{ij} + b_N \mathcal{R}^{ij}) \partial_i u \partial_j u + \frac{N-4}{2} Q_g u^2 \, d\text{vol}_g}{\left( \int_M u^{\frac{2N}{N-4}} \, d\text{vol}_g \right)^{\frac{N-4}{N}}}. \quad (6)$$

Consider

$$P(g) = \inf \left\{ \mathcal{E}_g(u) : u \in H^2(M), u > 0, u \not\equiv 0 \right\}.$$

We refer to  $P(g)$  as the Paneitz energy. Clearly it is a conformal invariant.

A similar question is whether or not the set of all positive solutions to the  $Q$ -curvature equation is compact. As far as we know, the only results in this direction are by Hebey-Frederic [15] and Qing-Raske [25]. Both papers give positive answers provided  $M$  is locally conformally flat and satisfies some other assumptions. Compactness is also studied for non geometric potentials of Paneitz operator in Hebey-Robert-Wen [14].

In this paper, we prove the non-compactness of the set of solutions to the  $Q$ -curvature problem in large dimensions. We construct a blowing-up sequence consisting of exactly one bubble. More precisely we prove the following theorem.

**Theorem 1.1.** *Assume  $N \geq 25$ . Then there exists a  $C^\infty$  Riemannian metric  $g$  on  $\mathbb{S}^N$  and a sequence of positive function  $u_n$  ( $n \in \mathbb{N}$ ) with the following properties:*

- i)  $g$  is not conformally flat,
- ii)  $u_n$  is a positive solution of the  $Q$  – curvature equation (5) for all  $n$ ,
- iii)  $\mathcal{E}_g(u_n) < P(\mathbb{S}^N)$  for all  $n$  and  $\mathcal{E}_g(u_n) \rightarrow P(\mathbb{S}^N)$  as  $n \rightarrow \infty$ ,
- iv)  $\sup_{\mathbb{S}^N} u_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

Here  $P(\mathbb{S}^N)$  denotes the Paneitz energy of the round metric on  $\mathbb{S}^N$ .

For convenience (and by stereo-graphic projection), we will work on  $\mathbb{R}^N$  instead of  $\mathbb{S}^N$ . Let  $g$  be a smooth metric on  $\mathbb{R}^N$  which agrees with the Euclidean metric outside a ball of radius 1. We will assume throughout the paper that  $\det g(x) = 1$  for all  $x \in \mathbb{R}^N$ , so that the volume form associated with  $g$  agrees with the Euclidean volume form. Precisely, we will consider

$$P_g u = \frac{N-4}{2} u^{\frac{N+4}{N-4}} \quad \text{in } \mathbb{R}^N. \quad (7)$$

Our goal is to construct solutions to the  $Q$  – curvature equation (7) on  $(\mathbb{R}^N, g)$ . Though we shall follow the main ideas in [3] and [4], there are some major difficulties in fourth order equations. The main difficulty is that we need to ensure that  $u$  is strictly positive on  $\mathbb{R}^N$ . In the Yamabe problem case, one constructs solutions of the form

$$u = u_0 + \phi$$

where  $u_0$  is the standard bubble and  $\phi$  is the error. As long as  $\|\phi\|_{H^1}$  is small, it can be shown easily that  $u > 0$ . In the prescribed  $Q$  curvature problem, even if we can show that  $\|\phi\|_{H^2(\mathbb{R}^N)}$  is small, it is not guaranteed that  $u$  is positive. To overcome this difficulty, we need to use a weighted  $L^\infty$  norm

$$\|\phi\|_* = \|u_0^{-1} \phi\|_{L^\infty(\mathbb{R}^N)}$$

and we have to show that  $\|\phi\|_*$  is small which then implies that  $u$  is positive. We use a technical framework which is more closely related to the finite dimensional Liapunov-Schmidt reduction procedure, as in [12], [26], and [31]. Another difficulty is the choice of the auxiliary function  $f(s)$ . In the second order case, a quadratic polynomial is chosen ([4]).

The organization of the paper is as follows: In Section 2, we introduce the special metric  $g$  in this paper. In Section 3, the first approximation of the solutions is given. In Section 4, we calculate the corresponding  $\Delta_g$ ,  $S_g$ ,  $R_g$ ,  $Q_g$  under this special metric  $g$ , and then acquire the estimate of the energy functional. In Section 5, the invertibility of the linearized operator is settled. In Section 6, we solve the nonlinear problem. In Section 7, a variational reduction procedure is processed. In Section 8, we show that the energy can be approximated by an auxiliary function. In Section 9, we prove that this auxiliary function has a critical point, which is a strict local minimum. In Section 10, we prove the main theorem by a gluing method. In Section 11, some inequalities used in the paper is proven.

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## 2 The Special Metric $g$

In this section we introduce the metric which will be used in this paper.

In what follows, we fix a multi-linear form  $W: \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ . We assume that  $W_{ikj\ell}$  satisfy all the algebraic properties of the Weyl tensor. Moreover, we assume that some components of  $W$  are non-zero, so that

$$\sum_{i,j,k,\ell=1}^n (W_{ikj\ell} + W_{i\ell jk})^2 > 0.$$

For simplicity, we put

$$H_{ij}(y) = \sum_{p,q=1}^n W_{ipjq} y_p y_q$$

and

$$\bar{H}_{ij}(y) = f(|y|^2) H_{ij}(y),$$

where  $f(|y|^2)$  will be chosen later. (Specifically, we shall choose  $f(s) = \tau - 12000s + 2411s^2 - 135s^3 + s^4$ . The number  $\tau$  depends only on the dimension  $N$  and will be chosen later.) It is easy to see that  $H_{ij}(y)$  is trace-free,  $\sum_{i=1}^n y_i H_{ij}(y) = 0$  and  $\sum_{i=1}^n \partial_i H_{ij}(y) = 0$  for all  $y \in \mathbb{R}^N$ .

We consider a Riemannian metric of the form  $g(x) = e^{h(x)}$ , where  $h(x)$  is a trace-free symmetric two-tensor on  $\mathbb{R}^N$  satisfying  $h(0) = 0$ ,  $h(x) = 0$  for  $|x| \geq 1$ ,

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| + |\partial^3 h(x)| + |\partial^4 h(x)| \leq \alpha$$

for all  $x \in \mathbb{R}^N$ , where  $\alpha > 0$  is a fixed small number, and

$$h_{ij}(x) = \mu \varepsilon^8 f(\varepsilon^{-2}|x|^2) H_{ij}(x)$$

for  $|x| \leq \rho$ . We assume that the parameter  $\varepsilon$ ,  $\mu$  and  $\rho$  are chosen such that  $\mu \leq 1$  and  $\varepsilon \leq \rho \leq 1$ . Note that  $\sum_{i=1}^N x_i h_{ij}(x) = 0$  and  $\sum_{i=1}^N \partial_i h_{ij}(x) = 0$  for  $|x| \leq \rho$ .

For later purpose, we need to understand the Green's function of  $\Delta_g^2$ . Denote  $\tilde{G}(x, y)$  be the Green's function of  $\Delta_g$  in  $\mathbb{R}^N$ . Then the Green's function of  $\Delta_g^2$  is

$$G(x, y) = \int_{\mathbb{R}^N} \tilde{G}(x, z) \tilde{G}(y, z) dz.$$

Since  $|\tilde{G}(x, y)| \leq C[d(x, y)]^{2-N}$ ,  $|\nabla \tilde{G}(x, y)| \leq C[d(x, y)]^{1-N}$  and  $|\nabla^2 \tilde{G}(x, y)| \leq C[d(x, y)]^{-N}$ , we know that

$$\begin{aligned} |G(x, y)| &\leq C[d(x, y)]^{4-N}, \\ |\nabla G(x, y)| &\leq C[d(x, y)]^{3-N}, \end{aligned}$$

$$|\nabla^2 G(x, y)| \leq C[d(x, y)]^{2-N}.$$

Here  $d(x, y)$  is the distance between  $x$  and  $y$  under the metric  $g$ . It is easy to see that

$$d(x, y) = [1 + O(\alpha)]|x - y|$$

since  $g = e^h$  and  $\alpha$  is sufficiently small.

*Notations* In what follows, we use  $C$  to denote the variable constant which is independent of  $\alpha$  and  $\varepsilon$ .  $|O(A)| \leq CA$  and  $o(A)/A \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

### 3 First Approximation of the Solutions

In this section we will provide an ansatz for solutions of Problem (7).

Denote

$$u_0(x) = \gamma_N \left( \frac{\lambda \varepsilon}{\lambda^2 \varepsilon^2 + |x - \xi|^2} \right)^{\frac{N-4}{2}},$$

where  $\gamma_N = [N(N-4)^2(N-2)(N+2)/2]^{-\frac{N-4}{8}}$ ,  $\lambda > 0$  and  $\varepsilon > 0$ . It is well known [17] that  $u_0$  is the only positive solution of

$$\Delta^2 u_0 = \frac{N-4}{2} u_0^{\frac{N+4}{N-4}} \quad \text{in } \mathbb{R}^N.$$

Observe that  $u(x)$  satisfies (7) if and only if  $v(y) = \varepsilon^{\frac{N-4}{2}} u(\varepsilon y)$  satisfies

$$P_{\tilde{g}} v = \frac{N-4}{2} v^{\frac{N+4}{N-4}} \quad (8)$$

where  $\tilde{g}(y) = g(\varepsilon y)$ . Denote

$$\tilde{u}_0(y) = \varepsilon^{\frac{N-4}{2}} u_0(\varepsilon y) = \gamma_N \left( \frac{\lambda'}{\lambda'^2 + |y - \xi'|^2} \right)^{\frac{N-4}{2}},$$

where  $\xi' = \xi/\varepsilon$ ,  $\lambda' = \lambda/\varepsilon$ . The configuration set for  $(\xi', \lambda')$  is

$$\Lambda = \left\{ (\xi', \lambda') \in \mathbb{R}^N \times \mathbb{R} : |\xi'| \leq 1, \frac{1}{2} < \lambda' < \frac{3}{2} \right\}.$$

We will look for a solution to (7) with the form  $\tilde{u}_0(y) + \phi(y)$ . It is easy to check that  $\phi$  must be a solution of

$$P_{\tilde{g}} \phi - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \phi = -R + N(\phi) \quad \text{in } \mathbb{R}^N, \quad (9)$$

where

$$R(y) = P_{\tilde{g}} \tilde{u}_0 - \frac{N-4}{2} \tilde{u}_0^{\frac{N+4}{N-4}}, \quad (10)$$

$$N(\phi) = \frac{N-4}{2}(\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} - \frac{N-4}{2}\tilde{u}_0^{\frac{N+4}{N-4}} - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\phi. \quad (11)$$

A main step in solving (9) for small  $\phi$  is that of a solvability theory for the linearized operator  $P_{\tilde{g}} - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}$ .

## 4 Preliminary estimates

In this section we will mainly estimate the energy functional. From now on, denote  $\tilde{h}(y) = h(\varepsilon y)$  and  $\tilde{\mathcal{R}}, S_{\tilde{g}}, Q_{\tilde{g}}$  are the corresponding Ricci tensor, scalar curvature,  $Q$ -curvature under the metric  $\tilde{g}$ .

**Lemma 4.1.** *For  $|y| \leq \rho/\varepsilon$ , it holds*

$$\begin{aligned} \tilde{\mathcal{R}}_{ij} = & -\sum_m \frac{1}{2}\partial_{mm}\tilde{h}_{ij} + \sum_{m,s} \frac{1}{2}[\tilde{h}_{ms}(\partial_{ms}\tilde{h}_{ij}) - (\partial_s\tilde{h}_{mj})(\partial_m\tilde{h}_{si})] \\ & + \sum_{m,s} \frac{1}{4}\left[(\partial_i\tilde{h}_{ms})(\partial_m\tilde{h}_{sj}) + (\partial_j\tilde{h}_{ms})(\partial_m\tilde{h}_{si}) - (\partial_j\tilde{h}_{ms})(\partial_i\tilde{h}_{sm}) \right. \\ & \quad \left. - \tilde{h}_{ms}(\partial_{mi}\tilde{h}_{sj}) - \tilde{h}_{ms}(\partial_{mj}\tilde{h}_{si}) - (\partial_{mm}\tilde{h}_{js})\tilde{h}_{si} - \tilde{h}_{js}(\partial_{mm}\tilde{h}_{si})\right] \\ & + O(|\tilde{h}|^2|\partial^2\tilde{h}| + |\tilde{h}||\partial\tilde{h}|^2). \end{aligned}$$

For  $|y| \leq \rho/\varepsilon$ , we have

$$\tilde{\mathcal{R}}_{ij} = O(\alpha\varepsilon^2).$$

*Proof.* Recall that  $\tilde{\Gamma}_{kl}^i = \sum_m \frac{1}{2}\tilde{g}^{im}(\partial_l\tilde{g}_{mk} + \partial_k\tilde{g}_{ml} - \partial_m\tilde{g}_{kl})$ . Since  $\tilde{h}$  is trace-free, we have  $\det \tilde{g} = 1$  for all  $y \in \mathbb{R}^N$ . This implies  $\sum_i \tilde{\Gamma}_{ik}^i = \sum_{i,\ell} \frac{1}{2}\tilde{g}^{i\ell}\partial_k\tilde{g}_{i\ell} = \frac{1}{2}\partial_k \log |\tilde{g}| = 0$ . Therefore, we obtain

$$\begin{aligned} \tilde{\mathcal{R}}_{ij} &= \sum_m \partial_m \tilde{\Gamma}_{ji}^m - \sum_m \partial_j \tilde{\Gamma}_{mi}^m + \sum_{\ell,m} \tilde{\Gamma}_{m\ell}^m \tilde{\Gamma}_{ji}^\ell - \sum_{\ell,m} \tilde{\Gamma}_{j\ell}^m \tilde{\Gamma}_{mi}^\ell \\ &= \sum_m \partial_m \tilde{\Gamma}_{ji}^m - \sum_{\ell,m} \tilde{\Gamma}_{j\ell}^m \tilde{\Gamma}_{mi}^\ell. \end{aligned} \quad (12)$$

Direct calculation shows

$$\begin{aligned} \sum_m \partial_m \tilde{\Gamma}_{ji}^m &= \sum_m \frac{1}{2}\left[\partial_{mi}\tilde{h}_{mj} + \partial_{mj}\tilde{h}_{mi} - \partial_{mm}\tilde{h}_{ij}\right] \\ &+ \sum_{m,s} \frac{1}{2}\left[(\partial_m\tilde{h}_{ms})(\partial_s\tilde{h}_{ij}) + \tilde{h}_{ms}(\partial_{ms}\tilde{h}_{ij})\right] \\ &+ \sum_{m,s} \frac{1}{4}\left[(\partial_i\tilde{h}_{ms})(\partial_m\tilde{h}_{sj}) - (\partial_m\tilde{h}_{ms})(\partial_i\tilde{h}_{sj}) + (\partial_j\tilde{h}_{ms})(\partial_m\tilde{h}_{si})\right] \end{aligned}$$

$$\begin{aligned}
& - (\partial_m \tilde{h}_{ms})(\partial_j \tilde{h}_{si}) - 2(\partial_m \tilde{h}_{js})(\partial_m \tilde{h}_{si}) + (\partial_{mi} \tilde{h}_{ms}) \tilde{h}_{sj} \\
& - (\partial_{mi} \tilde{h}_{sj}) \tilde{h}_{ms} + (\partial_{mj} \tilde{h}_{ms}) \tilde{h}_{si} - (\partial_{mj} \tilde{h}_{si}) \tilde{h}_{ms} - (\partial_{mm} \tilde{h}_{js}) \tilde{h}_{si} \\
& - (\partial_{mm} \tilde{h}_{si}) \tilde{h}_{js} \Big] + O(|\tilde{h}|^2 |\partial^2 \tilde{h}| + |\tilde{h}| |\partial \tilde{h}|^2),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\ell, m} \tilde{\Gamma}_{j\ell}^m \tilde{\Gamma}_{mi}^\ell &= \sum_{\ell, m} \frac{1}{2} \left[ (\partial_\ell \tilde{h}_{mj})(\partial_m \tilde{h}_{\ell i}) - (\partial_\ell \tilde{h}_{mj})(\partial_\ell \tilde{h}_{mi}) \right] + \frac{1}{4} (\partial_j \tilde{h}_{m\ell})(\partial_i \tilde{h}_{\ell m}) \\
&+ O(|\tilde{h}| |\partial \tilde{h}|^2).
\end{aligned}$$

Since  $\sum_m \partial_m \tilde{h}_{mk} = 0$  for  $|y| \leq \rho/\varepsilon$ , the lemma concludes from (12).  $\square$

Thus we have the following conclusion for  $\tilde{\mathcal{R}}^{ij} = \sum_{s,t} \tilde{g}^{is} \tilde{\mathcal{R}}_{st} \tilde{g}^{tj}$ .

**Corollary 4.2.** *For  $|y| \leq \rho/\varepsilon$ , we have*

$$\begin{aligned}
\tilde{\mathcal{R}}^{ij} &= - \sum_m \frac{1}{2} \partial_{mm} \tilde{h}_{ij} + \sum_{m,s} \frac{1}{2} [\tilde{h}_{ms} (\partial_{ms} \tilde{h}_{ij}) - (\partial_s \tilde{h}_{mj})(\partial_m \tilde{h}_{si})] \\
&+ \sum_{m,s} \frac{1}{4} \left[ (\partial_i \tilde{h}_{ms})(\partial_m \tilde{h}_{sj}) + (\partial_j \tilde{h}_{ms})(\partial_m \tilde{h}_{si}) - (\partial_j \tilde{h}_{ms})(\partial_i \tilde{h}_{sm}) \right. \\
&\quad \left. - \tilde{h}_{ms} (\partial_{mi} \tilde{h}_{sj}) - \tilde{h}_{ms} (\partial_{mj} \tilde{h}_{si}) + (\partial_{mm} \tilde{h}_{js}) \tilde{h}_{si} + h_{js} (\partial_{mm} \tilde{h}_{si}) \right] \\
&+ O(|\tilde{h}|^2 |\partial^2 \tilde{h}| + |\tilde{h}| |\partial \tilde{h}|^2).
\end{aligned}$$

For  $|y| \leq \rho/\varepsilon$ , it holds

$$\tilde{\mathcal{R}}^{ij} = O(\alpha \varepsilon^2).$$

**Lemma 4.3.** *For  $|y| \leq \rho/\varepsilon$ , there holds*

$$S_{\tilde{g}} = -\frac{1}{4} \sum_{k,\ell,m} (\partial_\ell \tilde{h}_{mk})^2 + O(|\tilde{h}|^2 |\partial^2 \tilde{h}| + |\tilde{h}| |\partial \tilde{h}|^2).$$

For  $|y| \geq \rho/\varepsilon$ , we have

$$S_{\tilde{g}} = O(\alpha \varepsilon^2).$$

*Proof.* The detailed proof can be found in [3, Prop. 26], noting that  $\sum_m \partial_m \tilde{h}_{mk} = 0$  in  $|y| \leq \rho/\varepsilon$ .  $\square$

The above lemma and a direct computation show the following conclusion.

**Corollary 4.4.** *For  $|y| \leq \rho/\varepsilon$ , we have*

$$\Delta_{\tilde{g}} S_{\tilde{g}} = -\frac{1}{2} \sum_{i,k,\ell,m} (\partial_{i\ell} \tilde{h}_{mk})^2 - \frac{1}{2} \sum_{i,k,\ell,m} (\partial_\ell \tilde{h}_{mk})(\partial_{i\ell} \tilde{h}_{mk})$$

$$+ O(|\partial\tilde{h}|^2|\partial^2\tilde{h}| + |\tilde{h}||\partial^2\tilde{h}|^2 + |\tilde{h}||\partial\tilde{h}||\partial^3\tilde{h}| + |\tilde{h}|^2|\partial^4\tilde{h}|).$$

For  $|y| \geq \rho/\varepsilon$ , it holds

$$\Delta_{\tilde{g}}S_{\tilde{g}} = O(\alpha\varepsilon^4).$$

Now it is ready to estimate  $Q_{\tilde{g}}$ .

**Lemma 4.5.** For  $|y| \leq \rho/\varepsilon$ , we have

$$\begin{aligned} Q_{\tilde{g}} &= \frac{1}{4(N-1)} \sum_{i,k,\ell,m} \left[ (\partial_{i\ell}\tilde{h}_{mk})^2 + (\partial_{\ell}\tilde{h}_{mk})(\partial_{i\ell}\tilde{h}_{mk}) \right] \\ &\quad - \frac{1}{2(N-2)^2} \sum_{i,j,m,s} (\partial_{mm}\tilde{h}_{ij})(\partial_{ss}\tilde{h}_{ij}) \\ &\quad + O(|\partial\tilde{h}|^2|\partial^2\tilde{h}| + |\tilde{h}||\partial^2\tilde{h}|^2 + |\tilde{h}||\partial\tilde{h}||\partial^3\tilde{h}| + |\tilde{h}|^2|\partial^4\tilde{h}|). \end{aligned}$$

For  $|y| \geq \rho/\varepsilon$ , there holds

$$Q_{\tilde{g}} = O(\alpha\varepsilon^4).$$

*Proof.* This is a direct result of (4), Lemma 4.1, Corollary 4.2 and 4.4.  $\square$

Our next goal is to estimate  $R(y)$  defined in (10).

**Lemma 4.6.** For  $|y| \leq \frac{\rho}{\varepsilon}$ ,

$$\begin{aligned} \Delta_{\tilde{g}}^2\tilde{u}_0 - \Delta^2\tilde{u}_0 &= -(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) - 2(\partial_s\tilde{h}_{ij})(\partial_{sij}\tilde{u}_0) - 2\tilde{h}_{ij}(\partial_{ssij}\tilde{u}_0) \\ &\quad + O(|\tilde{h}||\partial^2\tilde{h}||\partial^2\tilde{u}_0| + O(|\tilde{h}||\partial\tilde{h}||\partial^3\tilde{u}_0| + O(|\tilde{h}|^2)|\partial^4\tilde{u}_0|). \end{aligned}$$

For  $|y| \geq \frac{\rho}{\varepsilon}$ ,

$$\Delta_{\tilde{g}}^2\tilde{u}_0 - \Delta^2\tilde{u}_0 = O\left(\frac{\alpha\varepsilon}{(1+|y-\xi'|)^{N-1}}\right).$$

*Proof.* The computations follow easily from the definition of  $\Delta_{\tilde{g}}$  and the properties of  $h$ .  $\square$

By Lemma 4.3, Corollary 4.2 and the properties of  $h$ , it is also not difficult to verify the following two lemmas.

**Lemma 4.7.** For  $|y| \leq \frac{\rho}{\varepsilon}$ ,

$$\sum_{i,j} \partial_j(S_{\tilde{g}}\tilde{g}^{ij}\partial_i\tilde{u}_0) = O\left(\frac{\mu^2\varepsilon^{20}}{(1+|y-\xi'|)^{N-20}}\right).$$

For  $|y| \geq \frac{\rho}{\varepsilon}$ ,

$$\sum_{i,j} \partial_j(S_{\tilde{g}}\tilde{g}^{ij}\partial_i\tilde{u}_0) = O\left(\frac{\alpha\varepsilon^2}{(1+|y-\xi'|)^{N-2}}\right).$$

**Lemma 4.8.** For  $|y| \leq \frac{\rho}{\varepsilon}$ ,

$$\begin{aligned} \sum_{i,j} \partial_j (\tilde{\mathcal{R}}^{ij} \partial_i \tilde{u}_0) &= -\frac{1}{2} (\partial_{jmm} \tilde{h}_{ij}) (\partial_i \tilde{u}_0) - \frac{1}{2} (\partial_{mm} \tilde{h}_{ij}) (\partial_{ij} \tilde{u}_0) \\ &\quad + O\left(\frac{\mu^2 \varepsilon^{20}}{(1+|y-\xi'|)^{N-20}}\right). \end{aligned}$$

For  $|y| \geq \frac{\rho}{\varepsilon}$ ,

$$\sum_{i,j} \partial_j (\tilde{\mathcal{R}}^{ij} \partial_i \tilde{u}_0) = O\left(\frac{\alpha \varepsilon^2}{(1+|y-\xi'|)^{N-2}}\right).$$

Combining the above results, we have the following estimate for  $R(y)$ .

**Proposition 4.9.** It holds

$$R(y) \leq \begin{cases} C \frac{\mu \varepsilon^{10}}{(1+|y-\xi'|)^{N-10}} & \text{for } |y| \leq \frac{\rho}{\varepsilon}, \\ C \frac{\alpha \varepsilon}{(1+|y-\xi'|)^{N-1}} & \text{for } \frac{\rho}{\varepsilon} \leq |y| \leq \frac{1}{\varepsilon}, \\ 0 & \text{for } |y| \geq \frac{1}{\varepsilon}. \end{cases}$$

Let us consider the energy functional  $E_{\tilde{g}}(v)$  associated to Problem (8), namely

$$\begin{aligned} E_{\tilde{g}}(v) &= \frac{1}{2} \int_{\mathbb{R}^N} (\Delta_{\tilde{g}} v)^2 + \sum_{i,j} (a_N S_{\tilde{g}} \tilde{g}^{ij} + b_N \tilde{\mathcal{R}}_{\tilde{g}}^{ij}) \partial_i v \partial_j v + \frac{N-4}{2} Q_{\tilde{g}} v^2 dy \\ &\quad - \frac{(N-4)^2}{4N} \int_{\mathbb{R}^N} v^{\frac{2N}{N-4}} dy. \end{aligned}$$

In what follows, we will calculate the energy  $E_{\tilde{g}}(\tilde{u}_0)$ , which is a important step for the existence of the solutions of our equation. First we have

**Lemma 4.10.** For  $|y| \leq \rho/\varepsilon$ ,

$$\begin{aligned} &(\Delta_{\tilde{g}} \tilde{u}_0)^2 - (\Delta \tilde{u}_0)^2 \\ &= \sum_{i,j,k,\ell} \tilde{h}_{i\ell} (\partial_i \tilde{h}_{j\ell}) (\partial_{kk} \tilde{u}_0) (\partial_j \tilde{u}_0) - \sum_{i,j,k} 2 \tilde{h}_{ij} (\partial_{kk} \tilde{u}_0) (\partial_{ij} \tilde{u}_0) \\ &\quad + \sum_{i,j,k,\ell} \tilde{h}_{i\ell} \tilde{h}_{j\ell} (\partial_{kk} \tilde{u}_0) (\partial_{ij} \tilde{u}_0) + \left( \sum_{i,j} \tilde{h}_{ij} (\partial_{ij} \tilde{u}_0) \right)^2 \\ &\quad + O(|\tilde{h}| |\partial \tilde{h}|^2) |\partial \tilde{u}_0|^2 + O(|\tilde{h}|^2 |\partial \tilde{h}|) |\partial \tilde{u}_0| |\partial^2 \tilde{u}_0| + O(|\tilde{h}|^3) |\partial^2 \tilde{u}_0|^2. \end{aligned} \quad (13)$$

While for  $|y| \geq \rho/\varepsilon$ , we have

$$(\Delta_{\tilde{g}} \tilde{u}_0)^2 - (\Delta \tilde{u}_0)^2 = O(\alpha (1+|y-\xi'|)^{4-2N}). \quad (14)$$

*Proof.* It is easy to check that

$$\begin{aligned} (\Delta_{\tilde{g}}\tilde{u}_0)^2 - (\Delta\tilde{u}_0)^2 &= \left( \sum_{i,j} (\partial_i \tilde{g}^{ij}) (\partial_j \tilde{u}_0) \right)^2 + \sum_{i,j,k,\ell} 2(\partial_k \tilde{g}^{k\ell}) \tilde{g}^{ij} (\partial_\ell \tilde{u}_0) (\partial_{ij} \tilde{u}_0) \\ &\quad + \sum_{i,j,k,\ell} (\tilde{g}^{ij} - \delta_{ij}) (\tilde{g}^{k\ell} + \delta_{k\ell}) (\partial_{ij} \tilde{u}_0) (\partial_{k\ell} \tilde{u}_0). \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} &\left( \sum_{i,j} \partial_i \tilde{g}^{ij} \partial_j \tilde{u}_0 \right)^2 \\ &= \sum_{i,j,k,\ell} (\partial_k \tilde{h}_{ik}) (\partial_\ell \tilde{h}_{j\ell}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 + O(|\tilde{h}| |\partial \tilde{h}|^2) |\partial \tilde{u}_0|^2, \\ &\sum_{i,j,k,\ell} 2\tilde{g}^{ij} (\partial_k \tilde{g}^{k\ell}) \partial_\ell \tilde{u}_0 \partial_{ij} \tilde{u}_0 \\ &= - \sum_{i,k,\ell} 2(\partial_k \tilde{h}_{k\ell}) \partial_\ell \tilde{u}_0 \partial_{ii} \tilde{u}_0 + \sum_{i,k,\ell,m} \left[ (\partial_k \tilde{h}_{km}) \tilde{h}_{k\ell} + (\partial_k \tilde{h}_{m\ell}) \tilde{h}_{km} \right] \partial_\ell \tilde{u}_0 \partial_{ii} \tilde{u}_0 \\ &\quad + \sum_{i,j,k,\ell} 2\tilde{h}_{ij} (\partial_k \tilde{h}_{k\ell}) \partial_\ell \tilde{u}_0 \partial_{ij} \tilde{u}_0 + O(|\tilde{h}|^2 |\partial \tilde{h}|) |\partial \tilde{u}_0| |\partial^2 \tilde{u}_0| \end{aligned}$$

and

$$\begin{aligned} &\sum_{i,j,k,\ell} (\tilde{g}^{ij} - \delta_{ij}) (\tilde{g}^{k\ell} + \delta_{k\ell}) (\partial_{ij} \tilde{u}_0) (\partial_{k\ell} \tilde{u}_0) \\ &= - \sum_{i,j,k} 2\tilde{h}_{ij} \partial_{ij} \tilde{u}_0 \partial_{kk} \tilde{u}_0 + \sum_{i,j,k,m} \tilde{h}_{im} \tilde{h}_{mj} \partial_{ij} \tilde{u}_0 \partial_{kk} \tilde{u}_0 \\ &\quad + \left( \sum_{i,j} \tilde{h}_{ij} \partial_{ij} \tilde{u}_0 \right)^2 + O(|\tilde{h}|^3) |\partial^2 \tilde{u}_0|^2. \end{aligned}$$

Therefore, for  $|y| \leq \rho/\varepsilon$ ,  $\sum_i \partial_i \tilde{h}_{ij} = 0$  yields (13). Since  $\tilde{h} = 0$  for  $|y| \geq 1/\varepsilon$ , (14) can be easily gotten. This concludes the proof.  $\square$

**Lemma 4.11.** *It holds*

$$\begin{aligned} &\int_{\mathbb{R}^N} (\Delta_{\tilde{g}}\tilde{u}_0)^2 - (\Delta\tilde{u}_0)^2 \\ &= - \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,k,\ell} \tilde{h}_{i\ell} \tilde{h}_{j\ell} (\partial_{ik} \tilde{u}_0) (\partial_j \tilde{u}_0) + \int_{B_{\frac{\rho}{\varepsilon}}} \left( \sum_{i,j} \tilde{h}_{ij} (\partial_{ij} \tilde{u}_0) \right)^2 \\ &\quad + O(\mu^3 \varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right). \end{aligned}$$

*Proof.* Since for  $|y| \leq \rho/\varepsilon$ ,

$$\begin{aligned} & |\tilde{h}||\partial\tilde{h}|^2|\partial\tilde{u}_0|^2 + |\tilde{h}|^2|\partial\tilde{h}||\partial\tilde{u}_0||\partial^2\tilde{u}_0| + |\tilde{h}|^3|\partial^2\tilde{u}_0|^2 \\ & \leq C\mu^3\varepsilon^{30}(1+|y-\xi'|)^{34-2N} \\ & \leq C\mu^3\varepsilon^{\frac{20N}{N-1}}(1+|y-\xi'|)^{\frac{20N}{N-1}+4-2N}, \end{aligned}$$

from Lemma 4.10 we have

$$\begin{aligned} & \int_{\mathbb{R}^N} (\Delta_{\tilde{g}}\tilde{u}_0)^2 - (\Delta\tilde{u}_0)^2 \\ & = \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,k,\ell} \tilde{h}_{i\ell}(\partial_i\tilde{h}_{j\ell})(\partial_{kk}\tilde{u}_0)(\partial_j\tilde{u}_0) - \sum_{i,j,k} 2\tilde{h}_{ij}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) \\ & \quad + \sum_{i,j,k,\ell} \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) + \left( \sum_{i,j} \tilde{h}_{ij}(\partial_{ij}\tilde{u}_0) \right)^2 dy \\ & \quad + O(\mu^3\varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right). \end{aligned} \tag{15}$$

On the other hand, integrating by parts and using  $\sum_i \partial_i \tilde{h}_{ij} = 0$  for  $|y| \leq \rho/\varepsilon$  and  $\tilde{h}(y) = 0$  for  $|y| \geq 1/\varepsilon$ , we know

$$\begin{aligned} & \sum_{i,j,k,\ell} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{i\ell}(\partial_i\tilde{h}_{j\ell})(\partial_{kk}\tilde{u}_0)(\partial_j\tilde{u}_0) + \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) \\ & = \sum_{i,j,k,\ell} \int_{\mathbb{R}^N} \partial_i(\tilde{h}_{i\ell}\tilde{h}_{j\ell})(\partial_{kk}\tilde{u}_0)(\partial_j\tilde{u}_0) + \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right) \\ & = - \sum_{i,j,k,\ell} \int_{\mathbb{R}^N} \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{ikk}\tilde{u}_0)(\partial_j\tilde{u}_0) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right) \\ & = - \sum_{i,j,k,\ell} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{ikk}\tilde{u}_0)(\partial_j\tilde{u}_0) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right). \end{aligned} \tag{16}$$

Next, direct computation shows

$$\begin{aligned} \tilde{u}_0(\partial_{ijkk}\tilde{u}_0) & = \frac{N}{(N-3)(N^2-4N+8)}(\partial_{ijkk}\tilde{u}_0^2) + \frac{N^2+4N}{N^2-4N+8}(\partial_{ij}\tilde{u}_0)(\partial_{kk}\tilde{u}_0) \\ & \quad + \frac{4(N-4)^2(N-2)N}{N^2-4N+8}\tilde{u}_0^2\frac{|y-\xi'|^2\delta_{ij}}{(\varepsilon^2+|y-\xi'|^2)^3} \\ & \quad - \frac{4(N-4)^2(N^2-2)}{N^2-4N+8}\tilde{u}_0^2\frac{\delta_{ij}}{(\varepsilon^2+|y-\xi'|^2)^2}. \end{aligned}$$

Recalling  $\tilde{h}$  is trace-free, we get

$$\sum_{i,j,k} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ij}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0)$$

$$\begin{aligned}
&= \sum_{i,j,k} \int_{\mathbb{R}^N} \partial_{ij}(\tilde{h}_{ij}\tilde{u}_0)(\partial_{kk}\tilde{u}_0) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right) \\
&= \sum_{i,j,k} \int_{\mathbb{R}^N} \tilde{h}_{ij}\tilde{u}_0(\partial_{ijkk}\tilde{u}_0) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}$$

Since  $\sum_{i=1}^N \partial_i \tilde{h}_{ij} = 0$  for  $|y| \leq \frac{\rho}{\varepsilon}$ , it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \tilde{h}_{ij}(\partial_{ijkk}\tilde{u}_0^2) = \int_{\mathbb{R}^N} (\partial_{ijkk}\tilde{h}_{ij})\tilde{u}_0^2 \\
&= \int_{\mathbb{R}^N \setminus B_{\frac{\rho}{\varepsilon}}} (\partial_{ijkk}\tilde{h}_{ij})\tilde{u}_0^2 = O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}$$

Thus

$$\sum_{i,j,k} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ij}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) = O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right). \quad (17)$$

The proof of the lemma is completed by (15), (16) and (17).  $\square$

**Lemma 4.12.** For  $|y| \leq \rho/\varepsilon$ ,

$$\sum_{i,j} S_{\tilde{g}} \tilde{g}^{ij} \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 = -\frac{1}{4} \sum_{i,k,\ell,m} (\partial_\ell \tilde{h}_{mk})^2 (\partial_i \tilde{u}_0)^2 + O(|\tilde{h}| |\partial \tilde{h}|^2) |\partial \tilde{u}_0|^2.$$

For  $|y| \geq \rho/\varepsilon$ ,

$$\sum_{i,j} S_{\tilde{g}} \tilde{g}^{ij} \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 = O(\alpha \varepsilon^2 (1 + |y - \xi'|)^{6-2N}).$$

*Proof.* Recalling that  $g = e^h$ , this lemma is an easy consequence of Lemma 4.3.  $\square$

**Lemma 4.13.** We have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_{i,j} a_N S_{\tilde{g}} \tilde{g}^{ij} \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \\
&= -\frac{a_N}{4} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,k,\ell,m} (\partial_\ell \tilde{h}_{mk})^2 (\partial_i \tilde{u}_0)^2 + O(\mu^3 \varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}$$

*Proof.* This is a result of Lemma 4.12 by direct calculation.  $\square$

**Lemma 4.14.** It holds

$$\int_{\mathbb{R}^N} \sum_{i,j} b_N \tilde{\mathcal{R}}^{ij} \partial_i \tilde{u}_0 \partial_j \tilde{u}_0$$

$$\begin{aligned}
&= -\frac{b_N}{4} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_j \tilde{h}_{ms})(\partial_i \tilde{h}_{sm})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&\quad - \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \left[ \tilde{h}_{ms}(\partial_s \tilde{h}_{ij}) - \tilde{h}_{si}(\partial_s \tilde{h}_{mj}) + \tilde{h}_{sj}(\partial_i \tilde{h}_{ms}) \right. \\
&\qquad \qquad \qquad \left. - \tilde{h}_{ms}(\partial_i \tilde{h}_{sj}) \right] \partial_m(\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) \\
&\quad + \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \tilde{h}_{is}(\partial_{mm} \tilde{h}_{js})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&\quad + O(\mu^3 \varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}$$

*Proof.* From Corollary 4.2, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_{i,j} b_N \tilde{\mathcal{R}}^{ij} \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \\
&= -\frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m} (\partial_{mm} \tilde{h}_{ij}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \\
&\quad - \frac{b_N}{4} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_j \tilde{h}_{ms})(\partial_i \tilde{h}_{sm})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&\quad + \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \left[ \tilde{h}_{ms}(\partial_{ms} \tilde{h}_{ij}) - (\partial_s \tilde{h}_{mj})(\partial_m \tilde{h}_{si}) + (\partial_i \tilde{h}_{ms})(\partial_m \tilde{h}_{sj}) \right. \\
&\qquad \qquad \qquad \left. - \tilde{h}_{ms}(\partial_{mi} \tilde{h}_{sj}) \right] (\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&\quad + \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \tilde{h}_{is}(\partial_{mm} \tilde{h}_{js})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&\quad + O(\mu^3 \varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}$$

Since

$$\partial_i \tilde{u}_0 \partial_j \tilde{u}_0 - \frac{(N-4)^2}{4(N-3)} \partial_{ij} \tilde{u}_0^2 = \frac{(N-4)^2}{2(N-3)} \tilde{u}_0^2 \frac{\delta_{ij}}{\lambda'^2 + |y - \xi'|^2},$$

it is easy to check, since  $\sum_{i=1}^N \partial_i \tilde{h}_{ij} = 0$ ,

$$\begin{aligned}
\left| \sum_{i,j,m} \int_{B_{\frac{\rho}{\varepsilon}}} (\partial_{mm} \tilde{h}_{ij}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right| &\leq C \int_{\mathbb{R}^n \setminus B_{\frac{\rho}{\varepsilon}}} |\partial^4 \tilde{h}| \tilde{u}_0^2 + O\left(\alpha \rho^2 \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right) \\
&= O\left(\alpha \rho^2 \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}$$

Integrating by parts, we know

$$\sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ms}(\partial_{ms} \tilde{h}_{ij})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0)$$

$$\begin{aligned}
&= \sum_{i,j,m,s} \int_{\mathbb{R}^n} \partial_m [\tilde{h}_{ms}(\partial_s \tilde{h}_{ij})] \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 + O(\alpha^2 \rho^2 \left(\frac{\varepsilon}{\rho}\right)^{N-4}) \\
&= - \sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ms}(\partial_s \tilde{h}_{ij}) \partial_m (\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) + O(\alpha^2 \rho \left(\frac{\varepsilon}{\rho}\right)^{N-4}), \\
&\quad \sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} (\partial_s \tilde{h}_{jm})(\partial_m \tilde{h}_{is})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&= \sum_{i,j,m,s} \int_{\mathbb{R}^n} \partial_m [\tilde{h}_{is}(\partial_s \tilde{h}_{jm})] \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 + O(\alpha^2 \rho^2 \left(\frac{\varepsilon}{\rho}\right)^{N-4}) \\
&= - \sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{is}(\partial_s \tilde{h}_{jm}) \partial_m (\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) + O(\alpha \rho \left(\frac{\varepsilon}{\rho}\right)^{N-4}), \\
&\quad \sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} (\partial_i \tilde{h}_{ms})(\partial_m \tilde{h}_{sj})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&= - \sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{sj}(\partial_i \tilde{h}_{ms})(\partial_m (\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0)) + O(\alpha \rho \left(\frac{\varepsilon}{\rho}\right)^{N-4})
\end{aligned}$$

and

$$\begin{aligned}
&\quad \sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ms}(\partial_m \tilde{h}_{sj})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&= - \sum_{i,j,m,s} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ms}(\partial_i \tilde{h}_{js})(\partial_m (\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0)) + O(\alpha \rho \left(\frac{\varepsilon}{\rho}\right)^{N-4}).
\end{aligned}$$

The proof is complete.  $\square$

**Lemma 4.15.**

$$\begin{aligned}
&\frac{N-4}{2} \int_{\mathbb{R}^N} Q_{\tilde{g}} \tilde{u}_0^2 \\
&= \frac{N-4}{8(N-1)} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,k,\ell,m} \left[ (\partial_{i\ell} \tilde{h}_{mk})^2 + (\partial_{\ell} \tilde{h}_{mk})(\partial_{i\ell} \tilde{h}_{mk}) \right] \tilde{u}_0^2 \\
&\quad - \frac{N-4}{4(N-2)^2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_{mm} \tilde{h}_{ij})(\partial_{ss} \tilde{h}_{ij}) \tilde{u}_0^2 \\
&\quad + O\left(\mu^3 \varepsilon^{\frac{20N}{N-1}}\right) + O\left(\alpha \rho^4 \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}$$

*Proof.* The lemma can be easily deduced from Lemma 4.5.  $\square$

Hence, we have the following estimate of  $E_{\tilde{g}}(\tilde{u}_0)$ .

**Proposition 4.16.**

$$\begin{aligned}
2E_{\tilde{g}}(\tilde{u}_0) &= 2E - \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,k,\ell} \tilde{h}_{i\ell} \tilde{h}_{j\ell} (\partial_{ikk} \tilde{u}_0) (\partial_j \tilde{u}_0) + \int_{B_{\frac{\rho}{\varepsilon}}} \left( \sum_{i,j} \tilde{h}_{ij} (\partial_{ij} \tilde{u}_0) \right)^2 \\
&\quad - \frac{a_N}{4} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,k,\ell,m} (\partial_\ell \tilde{h}_{mk})^2 (\partial_i \tilde{u}_0)^2 \\
&\quad - \frac{b_N}{4} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_j \tilde{h}_{ms}) (\partial_i \tilde{h}_{sm}) (\partial_i \tilde{u}_0) (\partial_j \tilde{u}_0) \\
&\quad - \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \left[ \tilde{h}_{ms} (\partial_s \tilde{h}_{ij}) - \tilde{h}_{si} (\partial_s \tilde{h}_{mj}) + \tilde{h}_{sj} (\partial_i \tilde{h}_{ms}) \right. \\
&\quad \quad \quad \left. - \tilde{h}_{ms} (\partial_i \tilde{h}_{sj}) \right] \partial_m (\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) \\
&\quad + \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \tilde{h}_{is} (\partial_{mm} \tilde{h}_{js}) (\partial_i \tilde{u}_0) (\partial_j \tilde{u}_0) \\
&\quad + \frac{N-4}{8(N-1)} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,k,\ell,m} \left[ (\partial_{i\ell} \tilde{h}_{mk})^2 + (\partial_\ell \tilde{h}_{mk}) (\partial_{i\ell} \tilde{h}_{mk}) \right] \tilde{u}_0^2 \\
&\quad - \frac{N-4}{4(N-2)^2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_{mm} \tilde{h}_{ij}) (\partial_{ss} \tilde{h}_{ij}) \tilde{u}_0^2 \\
&\quad + O\left(\mu^3 \varepsilon^{\frac{20N}{N-1}}\right) + O\left(\alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right),
\end{aligned}$$

where  $E$  is the constant such that

$$E = \frac{N-4}{N} \int_{\mathbb{R}^N} \left( \frac{1}{1+|y|^2} \right)^N dy.$$

**Remark:** Note that

$$E = P(\mathbb{S}^N). \quad (18)$$

## 5 Linearized Operator

In this section we develop the invertibility theory for the linearized operator  $P_{\tilde{g}} - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}}$  in suitable weighted  $L^\infty$  spaces.

We define two norms

$$\begin{aligned}
\|\phi\|_* &= \sup_{y \in \mathbb{R}^N} \sum_{i=0}^2 \left[ \frac{1}{\frac{\mu \varepsilon^{10}}{(1+|y-\xi'|)^{N-14+i}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4+i}} + \frac{(1+|y-\xi'|)^{N-4+i}}{\alpha} \right] |\partial^i \phi(y)|, \\
\|\zeta\|_{**} &= \sup_{y \in \mathbb{R}^N} \left[ \frac{\chi_{\{|y-\xi'| \leq \frac{\rho}{\varepsilon}\}} (1+|y-\xi'|)^{N-10}}{\mu \varepsilon^{10}} + \frac{\chi_{\{\frac{\rho}{\varepsilon} \leq |y-\xi'| \leq \frac{1}{\varepsilon}\}} (1+|y-\xi'|)^{N-1}}{\alpha \varepsilon} \right. \\
&\quad \left. + \frac{\chi_{\{|y-\xi'| \geq \frac{1}{\varepsilon}\}} (1+|y-\xi'|)^{N+\sigma}}{\alpha} \right] |\zeta(y)|,
\end{aligned}$$

where  $\chi_S$  is the characteristic function on the set  $S$ , and  $0 < \sigma < 1$  is a small constant.

Denote

$$Z_0 = \frac{\partial \tilde{u}_0}{\partial \lambda'}, \quad Z_j = \frac{\partial \tilde{u}_0}{\partial \xi_j'} \quad j = 1, \dots, N.$$

First, we consider the following problem. Given  $\zeta \in C^\alpha(\mathbb{R}^N)$ , find a function  $\phi$  such that for certain constants  $c_i$ ,  $i = 0, 1, \dots, N$ ,

$$\begin{cases} P_{\tilde{g}} \phi - \frac{N+4}{2} \tilde{u}_0^{\frac{s}{N-4}} \phi = \zeta + \sum_{i=0}^N c_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \phi \chi Z_i = 0, \end{cases} \quad (19)$$

where  $\chi(y)$  is a cut-off function satisfying  $\chi(y) = 1$  for  $|y - \xi'| \leq r_0$ ,  $\chi(y) = 0$  for  $|y - \xi'| \geq r_0 + 1$ . Here  $r_0 > 0$  is large but fixed.

**Proposition 5.1.** *Assume  $N \geq 25$ ,  $(\xi', \lambda') \in \Lambda$  and  $\alpha$  is small and fixed. Then for small  $\varepsilon$ , there is a unique solution  $\phi$  to (19). Moreover*

$$\|\phi\|_* \leq C \|\zeta\|_{**}$$

where  $C$  is independent of  $\alpha$  and  $\varepsilon$ .

To prove the above proposition, we need the following priori estimate.

**Lemma 5.2.** *Under the assumptions of Proposition 5.1, for any solution  $\phi$  to (19), there exists a constant  $C$  such that*

$$\|\phi\|_* \leq C \|\zeta\|_{**}.$$

*Proof.* We use the contradiction argument as in [12] and [26]. Assume there are sequences  $\varepsilon_n \rightarrow 0$  and the corresponding  $\zeta_n, \phi_n$  such that  $\|\zeta_n\|_{**} \rightarrow 0$  but  $\|\phi_n\|_* = 1$ . For abbreviation, we omit the subscript  $n$  in the following proof. Testing the equation against  $\bar{\chi} Z_j$  and integrating by parts four times, where  $\bar{\chi}(y)$  is a smooth cut-off function satisfying  $\bar{\chi}(y) = 1$  for  $|y - \xi'| \leq \frac{\rho}{4\varepsilon}$ ,  $\bar{\chi}(y) = 0$  for  $|y - \xi'| \geq \frac{\rho}{2\varepsilon}$  and  $|\nabla^i \bar{\chi}| \leq C(\frac{\varepsilon}{\rho})^i$ ,  $1 \leq i \leq 4$ . we get

$$\sum_i c_i \int_{\mathbb{R}^N} \chi Z_i Z_j = \int_{\mathbb{R}^N} \left( P_{\tilde{g}_n}(\bar{\chi} Z_j) - \frac{N+4}{2} \tilde{u}_0^{\frac{s}{N-4}} \bar{\chi} Z_j \right) \phi - \int_{\mathbb{R}^N} \zeta \bar{\chi} Z_j.$$

This defines a linear system in the  $c_i$  which is “almost diagonal” as  $\varepsilon$  approaches zero, since we have

$$\int_{\mathbb{R}^N} \chi Z_0 Z_j = \delta_{0j} \int \chi \left( \frac{\partial \tilde{u}_0}{\partial \lambda'} \right)^2;$$

$$\int_{\mathbb{R}^N} \chi Z_i Z_j = \delta_{ij} \int \chi \left( \frac{\partial \tilde{u}_0}{\partial y_j} \right)^2 \quad \forall i = 1, \dots, N.$$

On the other hand, using  $\Delta^2 Z_j - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} Z_j = 0$  and the estimates of  $S_{\tilde{g}}, \tilde{\mathcal{R}}^{st}, Q_{\tilde{g}}$  in the previous section, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \left( P_{\tilde{g}}(\bar{\chi} Z_j) - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \bar{\chi} Z_j \right) \phi \\ &= \int_{\mathbb{R}^N} \left\{ \Delta_{\tilde{g}}^2(\bar{\chi} Z_j) - \bar{\chi} \Delta^2(Z_j) - \sum_{s,t} \partial_s \left[ (a_N S_{\tilde{g}} \tilde{g}^{st} + b_N \tilde{\mathcal{R}}^{st}) \partial_t(\bar{\chi} Z_j) \right] \right. \\ & \quad \left. + \frac{N-4}{2} Q_{\tilde{g}} \bar{\chi} Z_j \right\} \phi \\ &= o(\mu \varepsilon^{10}) \|\phi\|_*. \end{aligned}$$

It is also easy to get

$$\int_{\mathbb{R}^N} \zeta \bar{\chi} Z_j \leq C \mu \varepsilon^{10} \|\zeta\|_{**}.$$

Thus we conclude

$$|c_i| \leq o(\mu \varepsilon^{10}) \|\phi\|_* + C \mu \varepsilon^{10} \|\zeta\|_{**} \quad \forall i = 0, \dots, N,$$

so  $c_i = o(\mu \varepsilon^{10})$ .

Next we claim that, for any fixed  $R > 0$ ,

$$\|\phi\|_{L^\infty(B_R(\xi'))} = o(\mu \varepsilon^{10}).$$

Indeed, by elliptic regularity we can get a  $\hat{\phi}$  such that  $\frac{\phi}{\mu \varepsilon^{10}} \rightarrow \hat{\phi}$  in  $C_{\text{loc}}^4(\mathbb{R}^N)$  and

$$\Delta^2 \hat{\phi} - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \hat{\phi} = 0 \quad \text{in } \mathbb{R}^N.$$

This implies  $\hat{\phi}$  is a linear combination of the functions  $Z_j$ ,  $j = 0, 1, \dots, N$ , see [19]. On the other hand, the assumed orthogonality conditions on  $\phi$  yields  $\int_{\mathbb{R}^N} \hat{\phi} \chi Z_j = 0$  for all  $j$ . Hence  $\hat{\phi} \equiv 0$ , which concludes the claim.

Now rewrite the equation in the following form

$$\begin{aligned} \phi(y) &= \int_{\mathbb{R}^N} G(y, z) \sum_{i,j} \partial_j \left[ (a_N S_{\tilde{g}} \tilde{g}^{ij} + b_N \tilde{\mathcal{R}}^{ij}) \partial_i \phi \right] (z) dz \\ & \quad - \int_{\mathbb{R}^N} \frac{N-4}{2} G(y, z) Q_{\tilde{g}}(z) \phi(z) dz + \int_{\mathbb{R}^N} G(y, z) \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \phi(z) dz \\ & \quad + \int_{\mathbb{R}^N} G(y, z) \zeta(z) dz + \sum_i c_i \int_{\mathbb{R}^N} G(y, z) \chi Z_i(z) dz. \end{aligned} \quad (20)$$

We make now the following observations:

Owing to  $S_{\tilde{g}}(z) = O(\alpha\varepsilon^2)$ ,  $\partial S_{\tilde{g}} = O(\alpha\varepsilon^3)$  and  $\tilde{\mathcal{R}}^{ij}(z) = O(\alpha\varepsilon^2)$ ,  $\partial\tilde{\mathcal{R}}^{ij}(z) = O(\alpha\varepsilon^3)$ , we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} G(y, z) \sum_{i,j} \partial_j \left[ (a_N S_{\tilde{g}} \tilde{g}^{ij} + b_N \tilde{\mathcal{R}}^{ij}) \partial_i \phi \right] (z) dz \right| \\ &= O(\alpha\varepsilon^3) \int_{B_{1/\varepsilon}} G(y, z) |\partial\phi(z)| dz + O(\alpha\varepsilon^2) \int_{B_{1/\varepsilon}} G(y, z) |\partial^2\phi(z)| dz \\ &:= I + II. \end{aligned} \quad (21)$$

For  $|y - \xi'| \leq \frac{\rho}{2\varepsilon}$ ,

$$\begin{aligned} |I| &\leq C\alpha\varepsilon^3 \|\phi\|_* \left\{ \int_{|z| \leq \frac{\rho}{\varepsilon}} \frac{1}{|y-z|^{N-4}} \left[ \frac{\mu\varepsilon^{10}}{(1+|z-\xi'|)^{N-13}} + \alpha\left(\frac{\varepsilon}{\rho}\right)^{N-3} \right] dz \right. \\ &\quad \left. + \int_{\frac{\rho}{\varepsilon} \leq |z| \leq \frac{1}{\varepsilon}} \frac{1}{|y-z|^{N-4}} \frac{\alpha}{(1+|z-\xi'|)^{N-3}} dz \right\} \\ &\leq C\alpha \|\phi\|_* \left[ \frac{\mu\varepsilon^{10}\varepsilon^3(1+|y-\xi'|)^3}{(1+|y-\xi'|)^{N-14}} + \alpha\rho^3\left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] \\ &\leq C\alpha\rho^3 \|\phi\|_* \left[ \frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}} + \alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4} \right], \end{aligned} \quad (22)$$

where we use, for any  $0 < s, k < N$  such that  $s + k < N$ ,

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z-\xi'|)^{N-k}} dz \leq C(1+|y-\xi'|)^{k+s-N}. \quad (23)$$

The proof of (23) is standard and is given in the appendix. Similarly, for  $|y-\xi'| \leq \frac{\rho}{2\varepsilon}$ ,

$$|II| \leq C\alpha\rho^2 \|\phi\|_* \left[ \frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}} + \alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4} \right]. \quad (24)$$

On the other hand, for  $|y - \xi'| \geq \frac{\rho}{2\varepsilon}$ ,

$$\begin{aligned} |I| &\leq C\alpha\varepsilon^3 \|\phi\|_* \left\{ \int_{|z| \leq \frac{\rho}{\varepsilon}} \frac{1}{|y-z|^{N-4}} \left[ \frac{\mu\varepsilon^{10}}{(1+|z-\xi'|)^{N-13}} + \alpha\left(\frac{\varepsilon}{\rho}\right)^{N-3} \right] dz \right. \\ &\quad \left. + \int_{\frac{\rho}{\varepsilon} \leq |z| \leq \frac{1}{\varepsilon}} \frac{1}{|y-z|^{N-4}} \frac{\alpha}{(1+|z-\xi'|)^{N-3}} dz \right\} \\ &\leq C\rho^3 \|\phi\|_* \frac{\alpha}{(1+|y-\xi'|)^{N-4}}, \end{aligned} \quad (25)$$

where we use, for any  $0 < s, k < N$  such that  $s + k < N$ ,

$$\int_{B_r} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z-\xi'|)^{N-k}} dz \leq Cr^k(1+|y-\xi'|)^{s-N}, \quad (26)$$

which is a direct result of (23) and the proof is also given in the appendix. A similar proof also gives

$$|II| \leq C\rho^2 \|\phi\|_* \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} \quad \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \quad (27)$$

Since  $Q_{\tilde{g}}(z) = O(\alpha\varepsilon^4)$ , we similarly have

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \frac{N-4}{2} G(y, z) Q_{\tilde{g}}(z) \phi(z) dz \right| \\ & \leq C\alpha\rho^4 \|\phi\|_* \left[ \frac{\mu\varepsilon^{10}}{(1 + |y - \xi'|)^{N-14}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] \quad \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \end{aligned} \quad (28)$$

and for  $|y - \xi'| \geq \frac{\rho}{2\varepsilon}$ ,

$$\left| \int_{\mathbb{R}^N} \frac{N-4}{2} G(y, z) Q_{\tilde{g}}(z) \phi(z) dz \right| \leq C\rho^4 \|\phi\|_* \frac{\alpha}{(1 + |y - \xi'|)^{N-4}}. \quad (29)$$

Therefore, combining (21), (22), (24), (25), (27)-(29), we finally have

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} G(y, z) \sum_{i,j} \partial_i \left[ (a_N S_{\tilde{g}} \tilde{g}^{ij} + b_N \tilde{\mathcal{R}}^{ij}) \partial_j \phi \right] (z) dz \right. \\ & \quad \left. - \int_{\mathbb{R}^N} \frac{N-4}{2} G(y, z) Q_{\tilde{g}}(z) \phi(z) dz \right| \\ & \leq \begin{cases} C\alpha\rho^2 \|\phi\|_* \left[ \frac{\mu\varepsilon^{10}}{(1 + |y - \xi'|)^{N-14}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C\rho^2 \|\phi\|_* \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases} \end{aligned} \quad (30)$$

Next note that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} G(y, z) \frac{N+4}{2} \tilde{u}_0(z)^{\frac{8}{N-4}} \phi(z) dz \right| \\ & \leq C \left\{ \int_{B_R(\xi')} + \int_{R < |z - \xi'| < \frac{\rho}{\varepsilon}} + \int_{|z - \xi'| > \frac{\rho}{\varepsilon}} \right\} \frac{1}{|y - z|^{N-4}} \frac{\phi(z)}{(1 + |z - \xi'|)^8}. \end{aligned}$$

Since  $\|\phi\|_{L^\infty(B_R(\xi'))} = o(\mu\varepsilon^{10})$ , it is easy to check that

$$\begin{aligned} & \int_{B_R(\xi')} \frac{1}{|y - z|^{N-4}} \frac{\phi(z)}{(1 + |z - \xi'|)^8} dz \\ & = \begin{cases} o(1) \left[ \frac{\mu\varepsilon^{10}}{(1 + |y - \xi'|)^{N-14}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ o(\mu\varepsilon^{10}) \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases} \end{aligned}$$

$$\begin{aligned} & \int_{R < |z - \xi'| < \frac{\rho}{\varepsilon}} \frac{1}{|y - z|^{N-4}} \frac{\phi(z)}{(1 + |z - \xi'|)^8} dz \\ & \leq \begin{cases} \frac{C}{R^3} \|\phi\|_* \left[ \frac{\mu \varepsilon^{10}}{(1 + |y - \xi'|)^{N-14}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C \alpha \left(\frac{\varepsilon}{\rho}\right)^4 \|\phi\|_* \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases} \end{aligned}$$

$$\begin{aligned} & \int_{|z - \xi'| \geq \frac{\rho}{\varepsilon}} \frac{1}{|y - z|^{N-4}} \frac{\phi(z)}{(1 + |z - \xi'|)^8} dz \\ & \leq \begin{cases} C \left(\frac{\varepsilon}{\rho}\right)^4 \|\phi\|_* \left[ \frac{\mu \varepsilon^{10}}{(1 + |y - \xi'|)^{N-14}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C \left(\frac{\varepsilon}{\rho}\right)^3 \|\phi\|_* \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} G(y, z) \frac{N+4}{2} \tilde{u}_0(z)^{\frac{8}{N-4}} \phi(z) dz \right| \\ & \leq \begin{cases} \left( \frac{C}{R^3} \|\phi\|_* + o(1) \right) \left[ \frac{\mu \varepsilon^{10}}{(1 + |y - \xi'|)^{N-14}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C \left(\frac{\varepsilon}{\rho}\right)^3 \|\phi\|_* \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases} \end{aligned} \tag{31}$$

Similarly,

$$\int_{\mathbb{R}^N} G(y, z) \zeta(z) dz = \left\{ \int_{|z - \xi'| < \frac{\rho}{\varepsilon}} + \int_{\frac{\rho}{\varepsilon} < |z - \xi'| < \frac{1}{\varepsilon}} + \int_{|z - \xi'| > \frac{1}{\varepsilon}} \right\} G(y, z) \zeta(z).$$

Using (23) and (26), we have

$$\left| \int_{|z - \xi'| < \frac{\rho}{\varepsilon}} G(y, z) \zeta(z) \right| \leq \begin{cases} C \|\zeta\|_{**} \frac{\mu \varepsilon^{10}}{(1 + |y - \xi'|)^{N-14}} & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C \|\zeta\|_{**} \frac{\mu \rho^{10}}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}, \end{cases}$$

$$\left| \int_{\frac{\rho}{\varepsilon} < |z - \xi'| < \frac{1}{\varepsilon}} G(y, z) \zeta(z) \right| \leq \begin{cases} C \|\zeta\|_{**} \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C \|\zeta\|_{**} \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon} \end{cases}$$

and

$$\left| \int_{|z-\xi'| \geq \frac{1}{\varepsilon}} G(y, z) \zeta(z) \right| \leq \begin{cases} C \|\zeta\|_{**} \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C \|\zeta\|_{**} \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases}$$

So

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} G(y, z) \zeta(z) dz \right| \\ & \leq \begin{cases} C \|\zeta\|_{**} \left[ \frac{\mu \varepsilon^{10}}{(1 + |y - \xi'|)^{N-4}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ C \|\zeta\|_{**} \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases} \end{aligned} \quad (32)$$

Since we have know  $c_i = o(\mu \varepsilon^{10})$ , it holds

$$\begin{aligned} & \sum_i c_i \int_{\mathbb{R}^N} G(y, z) \chi(z) Z_i(z) dz \\ & \leq C |c_i| \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-4}} \chi(z) \frac{1}{(1 + |z - \xi'|)^{N-4}} dz \\ & = \begin{cases} o(1) \left[ \frac{\mu \varepsilon^{10}}{(1 + |y - \xi'|)^{N-4}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4} \right] & \text{for } |y - \xi'| \leq \frac{\rho}{2\varepsilon}, \\ o(\mu \varepsilon^{10}) \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} & \text{for } |y - \xi'| \geq \frac{\rho}{2\varepsilon}. \end{cases} \end{aligned} \quad (33)$$

Now we obtain that, by combining (30)-(33) and choosing  $R$  large enough,

$$\left[ \frac{1}{\frac{\mu \varepsilon^{10}}{(1 + |y - \xi'|)^{N-4}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4}} + \frac{(1 + |y - \xi'|)^{N-4}}{\alpha} \right] |\phi| \leq C \|\zeta\|_{**} + o(1).$$

Taking the derivative in (20), we have

$$\begin{aligned} \partial_{y_i} \phi(y) &= \int_{\mathbb{R}^N} \partial_{y_i} G(y, z) \sum_{s,t} \partial_s \left[ (a_N S_{\tilde{g}} \tilde{g}^{st} + b_N \tilde{\mathcal{R}}^{st}) \partial_t \phi \right] (z) dz \\ &\quad - \int_{\mathbb{R}^N} \frac{N-4}{2} \partial_{y_i} G(y, z) Q_{\tilde{g}}(z) \phi(z) dz \\ &\quad + \int_{\mathbb{R}^N} \partial_{y_i} G(y, z) \frac{N+4}{2} \tilde{u}_0^{\frac{s}{N-4}} \phi(z) dz \\ &\quad + \int_{\mathbb{R}^N} \partial_{y_i} G(y, z) \zeta(z) dz \\ &\quad + \sum_i c_i \int_{\mathbb{R}^N} \partial_{y_i} G(y, z) Z_i(z) dz. \end{aligned}$$

Since  $|\partial_{y_i} G(y, z)| \leq C[1 + O(\alpha)]|x - y|^{3-N}$ , similarly we can prove that

$$\left[ \frac{1}{\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-13}} + \alpha(\frac{\varepsilon}{\rho})^{N-3}} + \frac{(1+|y-\xi'|)^{N-3}}{\alpha} \right] |\partial\phi| \leq C\|\zeta\|_{**} + o(1).$$

It is also similar to get that

$$\left[ \frac{1}{\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-12}} + \alpha(\frac{\varepsilon}{\rho})^{N-2}} + \frac{(1+|y-\xi'|)^{N-2}}{\alpha} \right] |\partial^2\phi| \leq C\|\zeta\|_{**} + o(1).$$

So we finally have

$$\|\phi\|_* \leq C\|\zeta\|_{**} + o(1),$$

which is a contradiction.  $\square$

*Proof of Proposition 5.1.* Consider the space

$$\mathcal{H} = \left\{ \phi \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \chi Z_j \phi = 0 \quad \forall j = 0, 1, \dots, N \right\}$$

endowed with the inner product  $(\phi, \psi) = \int_{\mathbb{R}^N} \Delta_g \phi \Delta_g \psi$ . Problem (19) expressed in weak form is equivalent to that of finding a  $\phi \in \mathcal{H}$  such that, for any  $\psi \in \mathcal{H}$ ,

$$\begin{aligned} (\phi, \psi) = & - \int_{\mathbb{R}^N} \sum_{i,j} (a_N S_{\bar{g}} g^{ij} + b_N \tilde{\mathcal{R}}^{ij}) \partial_i \phi \partial_j \psi + \frac{N-4}{2} Q_{\bar{g}} \phi \psi \\ & + \int_{\mathbb{R}^N} \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \phi \psi + \int_{\mathbb{R}^N} \zeta \psi. \end{aligned}$$

With the aid of Riesz's representation theorem, this equation can be rewritten in  $\mathcal{H}$  in the operator form

$$\phi = K(\phi) + \tilde{\zeta}$$

with certain  $\tilde{\zeta} \in \mathcal{H}$  which depends linearly on  $\zeta$  and  $K$  is a compact operator in  $\mathcal{H}$ . Fredholm's alternative guarantees unique solvability of this problem for any  $\tilde{\zeta}$  provided that the homogeneous equation  $\phi = K(\phi)$  has only the zero solution in  $\mathcal{H}$ , which is equivalent to (19) with  $\zeta = 0$ . Thus existence of a unique solution follows from Lemma 5.2. This finishes the proof.  $\square$

**Remark 5.3.** *The result of Proposition 5.1 implies that the unique solution  $\phi = T(\zeta)$  of (19) defines a continuous linear map from the weighted  $L^\infty$  space  $L_{**}^\infty$ , equipped with norm  $\|\cdot\|_{**}$ , into the weighted  $L^\infty$  space  $L_*^\infty$ , equipped with  $\|\cdot\|_*$ .*

It is important for later purposes to understand the differentiability of the operator  $T$  with respect to the variables  $\xi'$  and  $\lambda'$ .

**Proposition 5.4.** *Assume  $(\xi', \lambda') \in \Lambda$ . We have*

$$\|\nabla_{(\xi', \lambda')} T(\zeta)\|_* \leq C \|\zeta\|_{**}.$$

*Proof.* Denote formally  $Z = \partial_{\xi'} \phi$ . We seek for an expression for  $Z$ . Then  $Z$  satisfies the following equation:

$$\begin{aligned} P_g Z - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} Z &= \frac{N+4}{2} \partial_{\xi'} (\tilde{u}_0^{\frac{8}{N-4}}) \phi + \sum_{i=0}^N d_i \chi Z_i \\ &+ \sum_{i=0}^N c_i \partial_{\xi'} (\chi Z_i) \quad \text{in } \mathbb{R}^N, \end{aligned}$$

where  $d_i = \partial_{\xi'} c_i$ . Besides, from differentiating the orthogonality condition  $\int_{\mathbb{R}^N} \phi \chi Z_j = 0$ , we further get

$$\int_{\mathbb{R}^N} \phi \partial_{\xi'} (\chi Z_j) + \int_{\mathbb{R}^N} Z \chi Z_j = 0.$$

Choose  $b_\ell$  such that

$$\sum_{\ell} b_\ell \int_{\mathbb{R}^N} \chi Z_\ell Z_j = \int_{\mathbb{R}^N} \phi \partial_{\xi'} (\chi Z_j).$$

Since this system is diagonal dominant with uniformly bounded coefficients, we see that it is uniquely solvable and that

$$|b_\ell| \leq C \mu \varepsilon^{10} \|\phi\|_*$$

uniformly in  $(\xi', \lambda') \in \Lambda$ .

Let us now set

$$\eta = Z + \sum_{i=0}^N b_i \chi Z_i. \tag{34}$$

Then  $\eta$  satisfies

$$\begin{cases} P_g \eta - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \eta = \tilde{\zeta} + \sum_{i=0}^N d_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \eta \chi Z_j = 0 & \forall j. \end{cases}$$

where

$$\begin{aligned} \tilde{\zeta} &= \sum_i b_i \left( P_g (\chi Z_i) - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} (\chi Z_i) \right) + \frac{N+4}{2} \partial_{\xi'} (\tilde{u}_0^{\frac{8}{N-4}}) \phi \\ &+ \sum_{i=0}^N c_i \partial_{\xi'} (\chi Z_i). \end{aligned}$$

Applying Lemma 5.2,

$$\|\eta\|_* \leq C\|\tilde{\zeta}\|_{**}.$$

It can be directly checked that

$$|b_i| \left\| P_g(\chi Z_i) - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}}(\chi Z_i) \right\|_{**} \leq C\|\phi\|_*,$$

$$\left\| \frac{N+4}{2} \partial_{\xi'}(\tilde{u}_0^{\frac{8}{N-4}})\phi \right\|_{**} \leq C\|\phi\|_*,$$

and

$$\|c_i \partial_{\xi'}(\chi Z_i)\|_{**} \leq C\|\zeta\|_{**}.$$

Thus  $\|\tilde{\zeta}\|_{**} \leq C\|\zeta\|_{**}$ , and then

$$\|\eta\|_* \leq C\|\zeta\|_{**}.$$

Obviously  $\|b_i Z_i\|_* \leq C\|\phi\|_* \leq C\|\zeta\|_{**}$ . Therefore, we get by (34) that

$$\|Z\|_* \leq C\|\zeta\|_{**}.$$

The corresponding result for differentiation with respect to  $\lambda'$  follows similarly. This concludes the proof.  $\square$

## 6 Nonlinear Problem

We recall that our goal is to solve Problem (10). Rather than doing so directly, we shall solve first the intermediate problem

$$\begin{cases} P_g \phi - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \phi = -R + N(\phi) + \sum_{i=0}^N c_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \phi \chi Z_j = 0 & \forall j = 0, 1, \dots, N. \end{cases} \quad (35)$$

**Proposition 6.1.** *There exists a unique solution to (35) such that*

$$\|\phi\|_* \leq \beta$$

where  $\beta > 0$  is a large number independent of  $\alpha$  and  $\varepsilon$ .

*Proof.* In terms of the operator  $T$  defined in Remark 5.3, Problem (35) becomes

$$\phi = T[-R + N(\phi)] := A(\phi).$$

For a given large number  $\beta > 0$ , let us set

$$\mathcal{S} = \{\phi \in \mathcal{H} \cap L_*^\infty(\mathbb{R}^N) : \|\phi\|_* \leq \beta\}.$$

From Proposition 5.1, we get

$$\|A(\phi)\|_* \leq C(\|R\|_{**} + \|N(\phi)\|_{**}).$$

According to Lemma 4.9, direct computation shows

$$\|R\|_{**} \leq C. \quad (36)$$

Here  $C$  is independent of  $\alpha$  and  $\varepsilon$ . By the mean value theorem, we also easily have

$$\|N(\phi)\|_{**} \leq C\varepsilon^{4-\sigma}\|\phi\|_*^2. \quad (37)$$

Thus  $A(\phi) \in \mathcal{S}$ .

Furthermore, it is easy to check that for any  $\phi_1, \phi_2 \in \mathcal{S}$ ,

$$\|N(\phi_1) - N(\phi_2)\|_{**} \leq C\varepsilon^{4-\sigma}\|\phi_1 - \phi_2\|_*.$$

So

$$\|A(\phi_1) - A(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**} \leq C\varepsilon^{4-\sigma}\|\phi_1 - \phi_2\|_*,$$

which implies that  $A$  is a contraction mapping with the norm  $\|\cdot\|_*$  inside  $\mathcal{S}$ . Therefore the contraction mapping theorem yields the proposition.  $\square$

Our purpose in the remains of this section is to analyze the differentiability properties of the function  $\phi$  defined in Proposition 6.1

**Proposition 6.2.** *The function  $(\xi', \lambda') \mapsto \phi(\xi', \lambda')$  provided by Proposition 6.1 is of class  $C^1$  for the norm  $\|\cdot\|_*$ . Moreover*

$$\|\nabla_{(\xi', \lambda')} \phi\|_* \leq C.$$

*Proof.* First, we come to the differentiability of  $\phi_{(\xi', \lambda')}$ . Consider the following map  $H: \Lambda \times \mathcal{H} \cap L_*^\infty(\mathbb{R}^N) \times \mathbb{R}^{N+1} \rightarrow L_{**}^\infty(\mathbb{R}^N) \times \mathbb{R}^{N+1}$  of class  $C^1$ :

$$H((\xi', \lambda'), \phi, \mathbf{c}) = \begin{pmatrix} P_{\tilde{g}}(\tilde{u}_0 + \phi) - \frac{N-4}{2}(\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} - \sum_{i=0}^N c_i \chi Z_i \\ \int_{\mathbb{R}^N} \chi Z_0 \phi \\ \vdots \\ \int_{\mathbb{R}^N} \chi Z_N \phi \end{pmatrix}.$$

Problem (35) is then equivalent to  $H((\xi', \lambda'), \phi, \mathbf{c}) = 0$ . We know that given  $(\xi', \lambda') \in \Lambda$ , there is a unique solution  $\phi_{(\xi', \lambda')}$ . We will prove that the linear operator

$$\frac{\partial H((\xi', \lambda'), \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})} \Big|_{((\xi', \lambda'), \phi_{(\xi', \lambda')}, \mathbf{c}_{(\xi', \lambda')})} : \mathcal{H} \cap L_*^\infty(\mathbb{R}^N) \times \mathbb{R}^{N+1} \longrightarrow L_{**}^\infty(\mathbb{R}^N) \times \mathbb{R}^{N+1}$$

is invertible for small  $\varepsilon$  and  $\alpha$ . Then the  $C^1$  regularity  $(\xi', \lambda') \mapsto \phi_{(\xi', \lambda')}$  follows from the implicit function theorem. Indeed, we have

$$\frac{\partial H((\xi', \lambda'), \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})} \Big|_{((\xi', \lambda'), \phi_{(\xi', \lambda')}, \mathbf{c}_{(\xi', \lambda')})} [\varphi, \mathbf{d}] = \begin{pmatrix} P_{\tilde{g}}\varphi - \frac{N+4}{2}(\tilde{u}_0 + \phi_{(\xi', \lambda')})^{\frac{8}{N-4}}\varphi - \sum_{i=0}^N d_i \chi Z_i \\ \int_{\mathbb{R}^N} \chi Z_0 \varphi \\ \vdots \\ \int_{\mathbb{R}^N} \chi_N Z_N \varphi \end{pmatrix}.$$

Since  $\|\phi_{(\xi', \lambda')}\|_* \leq C$ , the same proof as that of Proposition 5.1 shows that this operator is invertible for  $\varepsilon$  and  $\alpha$  small.

Since now  $\phi = T[-R + N(\phi)]$ , we have

$$\partial_{\xi'} \phi = \partial_{\xi'} T[-R + N(\phi)] + T[-\partial_{\xi'} R + \partial_{\xi'} N(\phi)].$$

This implies, by Proposition 5.1 and 5.4,

$$\|\partial_{\xi'} \phi\|_* \leq C(\|R\|_{**} + \|N(\phi)\|_{**} + \|\partial_{\xi'} R\|_{**} + \|\partial_{\xi'} N(\phi)\|_{**}).$$

Direct calculation shows

$$\begin{aligned} \partial_{\xi'} N(\phi) &= \frac{N+4}{2}(\partial_{\xi'} \tilde{u}_0 + \partial_{\xi'} \phi) \left[ (\tilde{u}_0 + \phi)^{\frac{8}{N-4}} - \tilde{u}_0^{\frac{8}{N-4}} \right] \\ &\quad - \frac{4(N+4)}{N-4} \tilde{u}_0^{-\frac{N-12}{N-4}} (\partial_{\xi'} \tilde{u}_0) \phi. \end{aligned}$$

Hence we can obtain

$$\|\partial_{\xi'} N(\phi)\|_{**} \leq C\varepsilon^{4-\sigma} \|\phi\|_* (1 + \|\partial_{\xi'} \phi\|_*) \leq C\varepsilon^{4-\sigma} (1 + \|\partial_{\xi'} \phi\|_*). \quad (38)$$

It is easy to check

$$\|\partial_{\xi'} R\|_{**} \leq C. \quad (39)$$

Since we already have (36) and (37), we can conclude

$$\|\partial_{\xi'} \phi\|_* \leq C.$$

The corresponding result for differentiation with respect to  $\lambda'$  can be gotten similarly. The proof is concluded.  $\square$

## 7 Variational reduction

As we have said, after Problem (35) has been solved, we find a solution to Problem (7) if  $(\xi', \lambda')$  is such that

$$c_i(\xi', \lambda') = 0 \quad \text{for all } i. \quad (40)$$

This problem is indeed variational: it is equivalent to finding critical points of a function of  $(\xi', \lambda')$ . To see this, we define

$$\mathcal{F}_{\bar{g}}(\xi', \lambda') = E_{\bar{g}}[\tilde{u}_0(\xi', \lambda') + \phi(\xi', \lambda')]$$

where  $\phi$  is the solution given by Proposition 6.1.

**Lemma 7.1.**  *$(\xi', \lambda')$  satisfies System (40) if and only if  $(\xi', \lambda')$  is a critical point of  $\mathcal{F}_{\bar{g}}$ .*

*Proof.* Since

$$P_{\bar{g}}(\tilde{u}_0 + \phi) - \frac{N-4}{2}(\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} = \sum_{i=0}^N c_i \chi Z_i,$$

we have

$$\partial_{(\xi', \lambda')} \mathcal{F}_{\bar{g}}(\xi', \lambda') = \sum_{i=0}^N c_i \int_{\mathbb{R}^N} \chi Z_i [\partial_{(\xi', \lambda')} \tilde{u}_0 + \partial_{(\xi', \lambda')} \phi],$$

from which the necessity follows. In what follows we assume  $\partial_{(\xi', \lambda')} \mathcal{F}_{\bar{g}}(\xi', \lambda') = 0$ . Then

$$\sum_{i=0}^N c_i \int_{\mathbb{R}^N} \chi Z_i [\partial_{(\xi', \lambda')} \tilde{u}_0 + \partial_{(\xi', \lambda')} \phi] = 0.$$

Using Proposition 6.2, we can directly check that

$$\begin{aligned} \partial_{\xi'_i} \tilde{u}_0 + \partial_{\xi'_i} \phi &= Z_i + o(1), \\ \partial_{\lambda'} \tilde{u}_0 + \partial_{\lambda'} \phi &= Z_0 + o(1). \end{aligned}$$

Thus the above system for  $c_i$  is diagonal dominant, which gives  $c_i = 0$  for all  $i = 0, \dots, N$ . This concludes the proof.  $\square$

## 8 Energy Expansion

In this section we obtain an expansion of  $\mathcal{F}_{\bar{g}}$ .

We first need to acquire a more refined estimate for  $\phi$ . Let  $w(y)$  satisfies

$$\begin{cases} P_{\bar{g}} w - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} w = R_1 + \sum_i c_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \chi Z_i w = 0, \end{cases} \quad (41)$$

where

$$R_1(y) = -\mu\varepsilon^{10}\hat{\chi}(y)\left[2\overline{H}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\overline{H}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\overline{H}_{ij})(\partial_i\tilde{u}_0)\right].$$

Here  $\hat{\chi}$  is the cut-off function such that  $\hat{\chi}(y) = 1$  for  $|y| \leq \frac{\rho}{\varepsilon}$  and  $\hat{\chi}(y) = 0$  for  $|y| \geq \frac{2\rho}{\varepsilon}$ . We define

$$\|w\|'_* = \sup_{\mathbb{R}^N} \sum_{i=0}^2 \frac{(1+|y-\xi'|)^{N-14+i}}{\mu\varepsilon^{10}} |\partial^i w(y)|,$$

$$\|R_1\|'_{**} = \sup_{\mathbb{R}^N} \frac{(1+|y-\xi'|)^{N-10}}{\mu\varepsilon^{10}} |R_1(y)|.$$

A similar proof as that of Proposition 5.1 shows that there exists a unique solution  $w$  to (41) such that

$$\|w\|'_* \leq C\|R_1\|'_{**} \leq C. \quad (42)$$

We introduce again that

$$\|\varphi_1\|''_* = \sup_{\mathbb{R}^N} \sum_{i=0}^2 \left\{ \frac{1}{\frac{\mu^2\varepsilon^{20}}{1+(|y-\xi'|)^{N-24+i}} + \alpha(\frac{\varepsilon}{\rho})^{N-4+i}} + \frac{(1+|y-\xi'|)^{N-4+i}}{\alpha} \right\} |\partial^i \varphi_1(y)|,$$

$$\|\varphi_2\|''_{**} = \sup_{\mathbb{R}^N} \left\{ \frac{\chi_{\{|y|\leq\frac{\rho}{\varepsilon}\}}(1+|y-\xi'|)^{N-20}}{\mu^2\varepsilon^{20}} + \frac{\chi_{\{\frac{\rho}{\varepsilon}\leq|y|\leq\frac{1}{\varepsilon}\}}(1+|y-\xi'|)^{N-1}}{\alpha\varepsilon} + \frac{\chi_{\{|y|\geq\frac{1}{\varepsilon}\}}(1+|y-\xi'|)^{N+\sigma}}{\alpha} \right\} |\varphi_2(y)|.$$

**Lemma 8.1.** *The function  $\phi - w$  satisfies the estimate*

$$\|\phi - w\|''_* \leq C.$$

*Proof.* Obviously,

$$\begin{cases} P_{\tilde{g}}(\phi - w) - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}(\phi - w) = -R_2 + N(\phi) + \sum_{i=0}^N c_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \chi Z_i(\phi - w) = 0, \end{cases} \quad (43)$$

where

$$R_2(y) = \Delta_{\tilde{g}}^2 \tilde{u}_0 - \Delta^2 \tilde{u}_0 - \mu\varepsilon^{10}\hat{\chi} \sum_{i,j,s} \left[ 2\overline{H}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\overline{H}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\overline{H}_{ij})(\partial_i\tilde{u}_0) \right].$$

It is easy to check

$$|R_2(y)| \leq \begin{cases} C \frac{\mu^2 \varepsilon^{20}}{(1 + |y - \xi'|)^{N-20}} & \text{for } |y| \leq \frac{\rho}{\varepsilon}, \\ C \frac{\alpha \varepsilon}{(1 + |y - \xi'|)^{N-1}} & \text{for } \frac{\rho}{\varepsilon} \leq |y| \leq \frac{1}{\varepsilon}, \\ 0 & \text{for } |y| \geq \frac{1}{\varepsilon}. \end{cases}$$

On the other hand, since  $|N(\phi)| \leq C(1 + |y - \xi'|)^{N-12}|\phi|^2$  and  $\|\phi\|_* \leq C$ , we have

$$|N(\phi)| \leq \begin{cases} C \frac{\mu^2 \varepsilon^{20}}{(1 + |y - \xi'|)^{N-16}} + \alpha \left(\frac{\varepsilon}{\rho}\right)^{N+4} & \text{for } |y| \leq \frac{\rho}{\varepsilon}, \\ C \frac{\alpha}{(1 + |y - \xi'|)^{N+4}} & \text{for } |y| \geq \frac{\rho}{\varepsilon}. \end{cases}$$

By an analogous process to the proof of Lemma 5.2, we obtain

$$\|\phi - w\|_*'' \leq C\|R_2\|_*'' + C\|N(\phi)\|_*'' \leq C. \quad \square$$

**Proposition 8.2.** *The following expansion holds*

$$\begin{aligned} 2\mathcal{F}_{\tilde{g}}(\xi', \lambda') &= 2E - \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,k,\ell} \tilde{h}_{i\ell} \tilde{h}_{j\ell} (\partial_{ikk} \tilde{u}_0) (\partial_j \tilde{u}_0) + \int_{B_{\frac{\rho}{\varepsilon}}} \left( \sum_{i,j} \tilde{h}_{ij} (\partial_{ij} \tilde{u}_0) \right)^2 \\ &\quad - \frac{a_N}{4} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,k,\ell,m} (\partial_\ell \tilde{h}_{mk})^2 (\partial_i \tilde{u}_0)^2 \\ &\quad - \frac{b_N}{4} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_j \tilde{h}_{ms}) (\partial_i \tilde{h}_{sm}) (\partial_i \tilde{u}_0) (\partial_j \tilde{u}_0) \\ &\quad - \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \left[ \tilde{h}_{ms} (\partial_s \tilde{h}_{ij}) - \tilde{h}_{si} (\partial_s \tilde{h}_{mj}) + \tilde{h}_{sj} (\partial_i \tilde{h}_{ms}) \right. \\ &\quad \left. - \tilde{h}_{ms} (\partial_i \tilde{h}_{sj}) \right] \partial_m (\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) \\ &\quad + \frac{b_N}{2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} \tilde{h}_{is} (\partial_{mm} \tilde{h}_{js}) (\partial_i \tilde{u}_0) (\partial_j \tilde{u}_0) \\ &\quad + \frac{N-4}{8(N-1)} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,k,\ell,m} \left[ (\partial_{i\ell} h_{mk})^2 + (\partial_\ell h_{mk}) (\partial_{i\ell} h_{mk}) \right] \tilde{u}_0^2 \\ &\quad - \frac{N-4}{4(N-2)^2} \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_{mm} \tilde{h}_{ij}) (\partial_{ss} \tilde{h}_{ij}) u_0^2 \\ &\quad + \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,s} \left[ 2\tilde{h}_{ij} (\partial_{ijss} \tilde{u}_0) + 2(\partial_s \tilde{h}_{ij}) (\partial_{ijs} \tilde{u}_0) + (\partial_{ss} \tilde{h}_{ij}) (\partial_{ij} \tilde{u}_0) \right. \\ &\quad \left. - \frac{b_N}{2} (\partial_{ss} \tilde{h}_{ij}) (\partial_{ij} \tilde{u}_0) - \frac{b_N}{2} (\partial_{jss} \tilde{h}_{ij}) (\partial_i \tilde{u}_0) \right] w \end{aligned}$$

$$+ O\left(\mu^3 \varepsilon^{\frac{20N}{N-1}}\right) + O\left(\mu^{\frac{2N}{N-4}} \varepsilon^{\frac{20N}{N-4}}\right) + O\left(\alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).$$

*Proof.* Since  $\phi$  is a solution to (35),

$$\begin{aligned} & \int_{\mathbb{R}^N} \Delta_{\tilde{g}}(\tilde{u}_0 + \phi) \Delta_{\tilde{g}} \phi + \sum_{i,j} (a_N S_{\tilde{g}} g^{ij} + b_N \tilde{\mathcal{R}}^{ij}) \partial_i(\tilde{u}_0 + \phi) \partial_j \phi \\ & \quad + \frac{N-4}{2} Q_{\tilde{g}}(\tilde{u}_0 + \phi) \tilde{u}_0 \\ & = \frac{N-4}{2} \int_{\mathbb{R}^N} (\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} \phi. \end{aligned} \quad (44)$$

While

$$\begin{aligned} & \int_{\mathbb{R}^N} \Delta_{\tilde{g}} \tilde{u}_0 \Delta_{\tilde{g}} \phi + \sum_{i,j} (a_N S_{\tilde{g}} g^{ij} + b_N \tilde{\mathcal{R}}^{ij}) \partial_i \tilde{u}_0 \partial_j \phi + \frac{N-4}{2} Q_{\tilde{g}} \tilde{u}_0 \phi \\ & = \int_{\mathbb{R}^N} \left\{ P_{\tilde{g}} \tilde{u}_0 - \Delta^2 \tilde{u}_0 - \sum_{i,j,s} \left[ 2\tilde{h}_{ij}(\partial_{ijss} \tilde{u}_0) + 2(\partial_s \tilde{h}_{ij})(\partial_{ijs} \tilde{u}_0) + (\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) \right. \right. \\ & \quad \left. \left. - \frac{b_N}{2}(\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) - \frac{b_N}{2}(\partial_{jss} \tilde{h}_{ij})(\partial_i \tilde{u}_0) \right] \right\} \phi \\ & \quad + \int_{\mathbb{R}^N} \sum_{i,j,s} \left[ 2\tilde{h}_{ij}(\partial_{ijss} \tilde{u}_0) + 2(\partial_s \tilde{h}_{ij})(\partial_{ijs} \tilde{u}_0) + (\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) \right. \\ & \quad \left. - \frac{b_N}{2}(\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) - \frac{b_N}{2}(\partial_{jss} \tilde{h}_{ij})(\partial_i \tilde{u}_0) \right] (\phi - w) \\ & \quad + \int_{\mathbb{R}^N} \sum_{i,j,s} \left[ 2\tilde{h}_{ij}(\partial_{ijss} \tilde{u}_0) + 2(\partial_s \tilde{h}_{ij})(\partial_{ijs} \tilde{u}_0) + (\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) \right. \\ & \quad \left. - \frac{b_N}{2}(\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) - \frac{b_N}{2}(\partial_{jss} \tilde{h}_{ij})(\partial_i \tilde{u}_0) \right] w \\ & \quad + \frac{N-4}{2} \int_{\mathbb{R}^N} \tilde{u}_0^{\frac{N+4}{N-4}} \phi \\ & := I_1 + I_2 + I_3 + \frac{N-4}{2} \int_{\mathbb{R}^N} \tilde{u}_0^{\frac{N+4}{N-4}} \phi. \end{aligned} \quad (45)$$

Owing to

$$\begin{aligned} & \left| P_{\tilde{g}} \tilde{u}_0 - \Delta^2 \tilde{u}_0 - \sum_{i,j,s} \left[ 2\tilde{h}_{ij}(\partial_{ijss} \tilde{u}_0) + 2(\partial_s \tilde{h}_{ij})(\partial_{ijs} \tilde{u}_0) + (\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) \right. \right. \\ & \quad \left. \left. - \frac{b_N}{2}(\partial_{ss} \tilde{h}_{ij})(\partial_{ij} \tilde{u}_0) - \frac{b_N}{2}(\partial_{jss} \tilde{h}_{ij})(\partial_i \tilde{u}_0) \right] \right| \\ & \leq \begin{cases} C \frac{\mu^2 \varepsilon^{20}}{(1 + |y - \xi'|)^{N-20}} & \text{for } |y| \leq \frac{\rho}{\varepsilon}, \\ C \frac{\alpha \varepsilon}{(1 + |y - \xi'|)^{N-1}} & \text{for } |y| \geq \frac{\rho}{\varepsilon}, \end{cases} \end{aligned}$$

we have

$$|I_1| \leq C\mu^3\varepsilon^{21}\rho^9|\log\varepsilon| + C\alpha^2\rho\left(\frac{\varepsilon}{\rho}\right)^{N-4}. \quad (46)$$

Also because

$$\begin{aligned} & \left| \sum_{i,j,s} \left[ 2\tilde{h}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) \right. \right. \\ & \quad \left. \left. - \frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0) \right] \right| \\ & \leq \begin{cases} C \frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-10}} & \text{for } |y| \leq \frac{\rho}{\varepsilon}, \\ C \frac{\alpha\varepsilon}{(1+|y-\xi'|)^{N-1}} & \text{for } |y| \geq \frac{\rho}{\varepsilon} \end{cases} \end{aligned}$$

and  $\|\phi - w\|_*'' \leq C$ ,

$$|I_2| \leq C\mu^3\varepsilon^{21}\rho^9|\log\varepsilon| + C\alpha^2\rho\left(\frac{\varepsilon}{\rho}\right)^{N-4}. \quad (47)$$

Since  $\|w\|_*' \leq C$  and  $h = 0$  for  $|y| \geq \frac{1}{\varepsilon}$ .

$$\begin{aligned} I_3 &= \int_{|y| \leq \frac{\rho}{\varepsilon}} \sum_{i,j,s} \left[ 2\tilde{h}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) \right. \\ & \quad \left. - \frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0) \right] w \\ & \quad + O\left(\alpha\mu\rho^{10}\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right). \end{aligned} \quad (48)$$

Thus, combining (44)-(48), we have

$$\begin{aligned} & 2\mathcal{F}_{\tilde{g}}(\xi', \lambda') - 2E_{\tilde{g}}(\tilde{u}_0) \\ &= \int_{|y| \leq \frac{\rho}{\varepsilon}} \sum_{i,j,s} \left[ 2\tilde{h}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) \right. \\ & \quad \left. - \frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0) \right] w \\ & \quad + \frac{N-4}{2} \int_{\mathbb{R}^N} \left\{ \frac{4}{N} \left[ (\tilde{u}_0 + \phi)^{\frac{2N}{N-4}} - \tilde{u}_0^{\frac{2N}{N-4}} \right] \right. \\ & \quad \left. - \left[ (\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} \tilde{u}_0 - \tilde{u}_0^{\frac{N+4}{N-4}} (\tilde{u}_0 + \phi) \right] \right\} \\ & \quad + O\left(\mu^3\varepsilon^{21}\rho^9|\log\varepsilon|\right) + O\left(\alpha^2\rho\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right). \end{aligned} \quad (49)$$

On account of  $\|\phi\|_* \leq C$ , we have the pointwise estimate

$$\left| \frac{4}{N} \left[ (\tilde{u}_0 + \phi)^{\frac{2N}{N-4}} - \tilde{u}_0^{\frac{2N}{N-4}} \right] - \left[ (\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} \tilde{u}_0 - \tilde{u}_0^{\frac{N+4}{N-4}} (\tilde{u}_0 + \phi) \right] \right| \leq C |\phi|^{\frac{2N}{N-4}},$$

which implies

$$\begin{aligned} & \int_{\mathbb{R}^N} \left\{ \frac{4}{N} \left[ (\tilde{u}_0 + \phi)^{\frac{2N}{N-4}} - \tilde{u}_0^{\frac{2N}{N-4}} \right] - \left[ (\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} \tilde{u}_0 - \tilde{u}_0^{\frac{N+4}{N-4}} (\tilde{u}_0 + \phi) \right] \right\} \\ &= O(\mu^{\frac{2N}{N-4}} \varepsilon^{\frac{20N}{N-4}}) + O\left(\alpha \left(\frac{\varepsilon}{\rho}\right)^N\right). \end{aligned}$$

Proposition 4.16 then gives the result.  $\square$

## 9 Finding a critical point of an auxiliary function

We define a function  $F: \mathbb{R}^N \times (0, \infty) \rightarrow \mathbb{R}$  as follows: given an pair  $(\xi', \lambda') \in \mathbb{R}^N \times (0, \infty)$ ,

$$\begin{aligned} F(\xi', \lambda') &= - \int_{\mathbb{R}^N} \sum_{i,j,k,\ell} \bar{H}_{i\ell} \bar{H}_{j\ell} (\partial_{ikk} \tilde{u}_0) (\partial_j \tilde{u}_0) + \int_{\mathbb{R}^N} \left( \sum_{i,j} \bar{H}_{ij} (\partial_{ij} \tilde{u}_0) \right)^2 \\ &\quad - \frac{a_N}{4} \int_{\mathbb{R}^N} \sum_{i,k,\ell,m} (\partial_\ell \bar{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \\ &\quad - \frac{b_N}{4} \int_{\mathbb{R}^N} \sum_{i,j,m,s} (\partial_j \bar{H}_{ms}) (\partial_i \bar{H}_{sm}) (\partial_i \tilde{u}_0) (\partial_j \tilde{u}_0) \\ &\quad - \frac{b_N}{2} \int_{\mathbb{R}^N} \sum_{i,j,m,s} \left[ \bar{H}_{ms} (\partial_s \bar{H}_{ij}) - \bar{H}_{si} (\partial_s \bar{H}_{mj}) + \bar{H}_{sj} (\partial_i \bar{H}_{ms}) \right. \\ &\quad \quad \left. - \bar{H}_{ms} (\partial_i \bar{H}_{sj}) \right] \partial_m (\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) \\ &\quad + \frac{b_N}{2} \int_{\mathbb{R}^N} \sum_{i,j,m,s} \bar{H}_{is} (\partial_{mm} \bar{H}_{js}) (\partial_i \tilde{u}_0) (\partial_j \tilde{u}_0) \\ &\quad + \frac{N-4}{8(N-1)} \int_{\mathbb{R}^N} \sum_{i,k,\ell,m} \left[ (\partial_{i\ell} \bar{H}_{mk})^2 + (\partial_\ell \bar{H}_{mk}) (\partial_{i\ell} \bar{H}_{mk}) \right] \tilde{u}_0^2 \\ &\quad - \frac{N-4}{4(N-2)^2} \int_{\mathbb{R}^N} \sum_{i,j,m,s} (\partial_{mm} \bar{H}_{ij}) (\partial_{ss} \bar{H}_{ij}) \tilde{u}_0^2 \\ &\quad + \int_{\mathbb{R}^N} \sum_{i,j,s} \left[ 2\bar{H}_{ij} (\partial_{ijss} \tilde{u}_0) + 2(\partial_s \bar{H}_{ij}) (\partial_{ijs} \tilde{u}_0) + (\partial_{ss} \bar{H}_{ij}) (\partial_{ij} \tilde{u}_0) \right. \\ &\quad \quad \left. - \frac{b_N}{2} (\partial_{ss} \bar{H}_{ij}) (\partial_{ij} \tilde{u}_0) - \frac{b_N}{2} (\partial_{jss} \bar{H}_{ij}) (\partial_i \tilde{u}_0) \right] \tilde{w} \end{aligned}$$

where  $\bar{w}(y) = w(y)/\mu\varepsilon^{10}$  and  $w(y)$  is defined in (41). Our goal in this section is to show that the function  $F(\xi', \lambda')$  has a strict local minimum point.

**Proposition 9.1.** *The function  $F(\xi', \lambda')$  satisfies  $F(\xi', \lambda') = F(-\xi', \lambda')$  for all  $(\xi', \lambda') \in \mathbb{R}^N \times (0, \infty)$ . Consequently, we have  $\frac{\partial}{\partial \xi'_i} F(0, \lambda') = 0$  and  $\frac{\partial^2}{\partial \xi'_i \partial \lambda'} F(0, \lambda') = 0$  for all  $\lambda' > 0$ .*

*Proof.* This follows immediately from the relation  $\bar{H}_{ik}(-y) = \bar{H}_{ik}(y)$ .  $\square$

To the aim, we list some useful identities, which are all direct results by definitions.

**Lemma 9.2.**

$$\begin{aligned}\partial_i \tilde{u}_0 &= -(N-4)\tilde{u}_0(y) \frac{y_i - \xi'_i}{\lambda'^2 + |y - \xi'|^2}, \\ \partial_{ij} \tilde{u}_0 &= (N-2)(N-4)\tilde{u}_0(y) \frac{(y_i - \xi'_i)(y_j - \xi'_j)}{(\lambda'^2 + |y - \xi'|^2)^2} - (N-4)\tilde{u}_0(y) \frac{\delta_{ij}}{\lambda'^2 + |y - \xi'|^2}, \\ \partial_{ijs} \tilde{u}_0 &= -N(N-2)(N-4)\tilde{u}_0(y) \frac{(y_i - \xi'_i)(y_j - \xi'_j)(y_s - \xi'_s)}{(\lambda'^2 + |y - \xi'|^2)^3} \\ &\quad + (N-2)(N-4)\tilde{u}_0(y) \frac{\delta_{js}(y_i - \xi'_i) + \delta_{is}(y_j - \xi'_j) + \delta_{ij}(y_s - \xi'_s)}{(\lambda'^2 + |y - \xi'|^2)^2}.\end{aligned}$$

**Lemma 9.3.**

$$\begin{aligned}\partial_\ell \bar{H}_{mk} &= 2f'(|y|^2)H_{mk}y_\ell + f(|y|^2)(\partial_\ell H_{mk}), \\ \partial_{i\ell} \bar{H}_{mk} &= 2\delta_{i\ell}f'(|y|^2)H_{mk} + 2f'(|y|^2)(\partial_i H_{mk})y_\ell + 2f'(|y|^2)(\partial_\ell H_{mk})y_i \\ &\quad + f(|y|^2)(\partial_{i\ell} H_{mk}) + 4f''(|y|^2)H_{mk}y_i y_\ell.\end{aligned}$$

**Lemma 9.4.**

$$\begin{aligned}\sum_i y_i \partial_i H_{ms} &= 2H_{ms}, & \sum_j y_j \partial_s H_{tj} &= -H_{st}, \\ \sum_j y_j \partial_{ms} H_{ij} &= -\partial_s H_{im} - \partial_m H_{is}, & \sum_\ell y_\ell (\partial_{i\ell} H_{mk}) &= \partial_i H_{mk}.\end{aligned}$$

**Lemma 9.5.**

$$\begin{aligned}\int_{\partial B_1} \sum_{i,j} H_{ij}^2 &= \frac{|S^{N-1}|}{2N(N+2)} \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2, \\ \int_{\partial B_1} \sum_{i,j,k} (\partial_k H_{ij})^2 &= \frac{|S^{N-1}|}{N} \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2, \\ \int_{\partial B_1} \sum_{i,j,k,\ell} (\partial_{k\ell} H_{ij})^2 &= |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2.\end{aligned}$$

**Lemma 9.6.**

$$\begin{aligned}
\int_{\partial B_1} \sum_{i,j} H_{ij}^2 y_p y_q &= \frac{2|S^{N-1}|}{N(N+2)(N+4)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \\
&\quad + \frac{|S^{N-1}|}{2N(N+2)(N+4)} (W_{ikjl} + W_{iljk})^2 \delta_{pq}, \\
\int_{\partial B_1} \sum_t H_{pt} H_{qt} &= \frac{|S^{N-1}|}{2N(N+2)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}), \\
\int_{\partial B_1} \sum_{i,j,k} (\partial_k H_{ij})^2 y_p y_q &= \frac{2|S^{N-1}|}{N(N+2)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \\
&\quad + \frac{|S^{N-1}|}{N(N+2)} (W_{ikjl} + W_{iljk})^2 \delta_{pq}, \\
\int_{\partial B_1} \sum_{i,j} (\partial_p H_{ij})(\partial_q H_{ij}) &= \frac{|S^{N-1}|}{N} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}), \\
\int_{\partial B_1} \sum_{i,j} H_{ij} (\partial_q H_{ij}) y_p &= \frac{|S^{N-1}|}{N(N+2)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}), \\
\int_{\partial B_1} \sum_{i,j,k,\ell} (\partial_{k\ell} H_{ij})^2 y_p y_q &= \frac{|S^{N-1}|}{N} \sum_{i,j,k,\ell} (W_{ikjl} + W_{iljk})^2 \delta_{pq}.
\end{aligned}$$

We mention that the proof of some identities in Lemma 9.5 and 9.6 can be found in [3, 4]. The others may be proved by the same method.

**Lemma 9.7.** *It holds*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,j,s,t} \bar{H}_{tj} \bar{H}_{it} (\partial_{iss} \tilde{u}_0) (\partial_j \tilde{u}_0) \right\} \Big|_{\xi'=0} = 0,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,s,t} \bar{H}_{tj} \bar{H}_{it} (\partial_{iss} \tilde{u}_0) (\partial_j \tilde{u}_0) \right\} \Big|_{\xi=0} \\
&= |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4} \\
&\quad \cdot \frac{(N-2)(N-4)^2}{N(N+2)} \int_0^\infty f(r^2)^2 \left[ \frac{Nr^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{(N+2)r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} \right] dr.
\end{aligned}$$

*Proof.* Since

$$\sum_{i,j,s,t} \bar{H}_{tj} \bar{H}_{it} (\partial_{iss} \tilde{u}_0) (\partial_j \tilde{u}_0)$$

$$\begin{aligned}
&= f(|y|^2)^2 H_{tj} H_{it} (N-2)(N-4)^2 \tilde{u}_0^2 \left[ \frac{N|y-\xi'|^2}{(\lambda'^2 + |y-\xi'|^2)^4} - \frac{(n+2)}{(\lambda'^2 + |y-\xi'|^2)^3} \right] \\
&\quad \cdot (y_i - \xi'_i)(y_j - \xi'_j) \\
&= N(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4} |y-\xi'|^2}{(\lambda'^2 + |y-\xi'|^2)^N} \sum_{i,j,t} H_{it} H_{jt} \xi'_i \xi'_j \\
&\quad - (N-2)(N+2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y'-\xi'|^2)^{N-1}} \sum_{i,j,t} H_{it} H_{jt} \xi'_i \xi'_j
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,s,t} \bar{H}_{tj} \bar{H}_{it} (\partial_{iss} \tilde{u}_0) (\partial_j \tilde{u}_0) \right\} \Big|_{\xi'=0} \\
&= 2N(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4} |y|^2}{(\lambda'^2 + |y|^2)^N} \sum_t H_{pt} H_{qt} \\
&\quad - 2(N-2)(N+2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \sum_t H_{pt} H_{qt},
\end{aligned}$$

the lemma follows directly by Lemma 9.5 and 9.6.  $\square$

**Lemma 9.8.** *We have*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left\{ \sum_{i,j,s,t} \bar{H}_{ij} \bar{H}_{st} \partial_{ij} \tilde{u}_0 \partial_{st} \tilde{u}_0 \right\} \Big|_{\xi'=0} = 0, \\
&\int_{\mathbb{R}^n} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,s,t} \bar{H}_{ij} \bar{H}_{st} \partial_{ij} \tilde{u}_0 \partial_{st} \tilde{u}_0 \right\} \Big|_{\xi'=0} = 0.
\end{aligned}$$

*Proof.* The lemma easily follows from

$$\begin{aligned}
&\sum_{i,j,s,t} \bar{H}_{ij} \bar{H}_{st} \partial_{ij} \tilde{u}_0 \partial_{st} \tilde{u}_0 \\
&= (N-2)^2 (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^N} \sum_{i,j,s,t} H_{ij} H_{st} \xi'_i \xi'_j \xi'_s \xi'_t. \quad \square
\end{aligned}$$

**Lemma 9.9.** *There hold*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left\{ \sum_{i,\ell,k,m} (\partial_\ell \bar{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \right\} \Big|_{\xi=0} \\
&= |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{iljk})^2 \lambda'^{N-4}
\end{aligned}$$

$$\cdot \left\{ \frac{2(N-4)^2}{N(N+2)} \int_0^\infty [r^2 f'(r^2)^2 + 2f(r^2)f'(r^2)] \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \right. \\ \left. + \frac{(N-4)^2}{N} \int_0^\infty f(r^2)^2 \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr \right\},$$

and

$$\int_{\mathbb{R}^N} \frac{\partial^2}{\partial \xi_p^l \partial \xi_q^l} \left\{ \sum_{i,\ell,k,m} (\partial_\ell \bar{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \right\} \Big|_{\xi'=0} \\ = |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4} \\ \cdot \left\{ \frac{32(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty [r^2 f'(r^2)^2 + 2f(r^2)f'(r^2)] \right. \\ \cdot \left[ \frac{(N-1)r^{N+7}}{(\lambda'^2 + r^2)^N} - \frac{2r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} \right] dr \\ \left. + \frac{8(N-2)(N-4)^2}{N(N+2)} \int_0^\infty f(r^2)^2 \left[ \frac{(N-1)r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{2r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} \right] dr \right\} \\ + |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \delta_{pq} \\ \cdot \left\{ \frac{4(N-4)^2}{N(N+2)(N+4)} \int_0^\infty [r^2 f'(r^2)^2 + 2f(r^2)f'(r^2)] \right. \\ \cdot \left[ \frac{2(N-1)(N-2)r^{N+7}}{(\lambda'^2 + r^2)^N} - \frac{(N-2)(N+8)r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} + \frac{(N+4)r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} \right] dr \\ \left. + \frac{2(N-4)^2}{N(N+2)} \int_0^\infty f(r^2)^2 \right. \\ \left. \cdot \left[ \frac{2(N-1)(N-2)r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{(N-2)(N+6)r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} + \frac{(N+2)r^{N+1}}{(\lambda'^2 + r^2)^{N-2}} \right] dr \right\}.$$

*Proof.* Direct computation shows

$$\sum_{i,\ell,k,m} (\partial_\ell \bar{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \\ = 4(N-4)^2 \left[ |y|^2 f'(|y|^2)^2 + 2f(|y|^2)f'(|y|^2) \right] \lambda'^{N-4} \frac{|y - \xi'|^2}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{m,k} H_{mk}^2 \\ + (N-4)^2 f(|y|^2)^2 \lambda'^{N-4} \frac{|y - \xi'|^2}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2,$$

$$\begin{aligned}
& \left. \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,\ell,k,m} (\partial_\ell \bar{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \right\} \right|_{\xi'=0} \\
= & 16(N-1)(N-2)(N-4)^2 \left[ |y|^2 f'(|y|^2)^2 + 2f(|y|^2) f'(|y|^2) \right] \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2 + |y|^2)^N} \sum_{k,m} H_{km}^2 y_p y_q \\
& - 32(N-2)(N-4)^2 \left[ |y|^2 f'(|y|^2)^2 + 2f(|y|^2) f'(|y|^2) \right] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \sum_{k,m} H_{km}^2 y_p y_q \\
& - 8(N-2)(N-4)^2 \left[ |y|^2 f'(|y|^2)^2 + 2f(|y|^2) f'(|y|^2) \right] \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \sum_{k,m} H_{km}^2 \delta_{pq} \\
& + 8(N-4)^2 \left[ |y|^2 f'(|y|^2)^2 + 2f(|y|^2) f'(|y|^2) \right] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-2}} \sum_{k,m} H_{km}^2 \delta_{pq} \\
& + 4(N-1)(N-2)(N-4)^2 f(|y|^2)^2 \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2 + |y|^2)^N} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 y_p y_q \\
& - 8(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 y_p y_q \\
& - 2(N-2)(N-4)^2 f(|y|^2)^2 \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 \delta_{pq} \\
& + 2(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-2}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 \delta_{pq}.
\end{aligned}$$

Using Lemma 9.5 and 9.6, we finish the proof.  $\square$

**Lemma 9.10.** *We have*

$$\begin{aligned}
& \int_{\mathbb{R}^N} \left\{ \sum_{i,j,m,s} (\partial_i \bar{H}_{ms})(\partial_j \bar{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Big|_{\xi'=0} \\
= & |S^{N-1}| (W_{ikj\ell} + W_{i\elljk})^2 \lambda'^{N-4} \\
& \cdot \frac{2(N-4)^2}{N(N+2)} \int_0^\infty [r^2 f'(r^2) + f(r^2)]^2 \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr, \\
& \int_{\mathbb{R}^n} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} (\partial_i \bar{H}_{ms})(\partial_j \bar{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Big|_{\xi'=0} \\
= & |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{n-4} \\
& \cdot \left\{ \frac{32(N-1)(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty [r^2 f'(r^2) + f(r^2)]^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^N} dr \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{64(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty [r^2 f'(r^2)^2 + f(r^2) f'(r^2)] \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} dr \\
& - \frac{16(N-2)(N-4)^2}{N(N+2)} \int_0^\infty [r^2 f(r^2) f'(r^2) + f(r^2)^2] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} dr \\
& + \frac{16(N-4)^2}{N(N+2)(N+4)} \int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \\
& + \frac{2(N-4)^2}{N} \int_0^\infty f(r^2)^2 \frac{r^{N+1}}{(\lambda'^2 + r^2)^{N-2}} dr \\
& + \left. \frac{8(N-4)^2}{N(N+2)} \int_0^\infty f(r^2) f'(r^2) \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr \right\} \\
& + |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \delta_{pq} \\
& \cdot \left\{ \frac{4(N-2)(N-4)^2}{N(N+2)} \int_0^\infty [r^2 f'(r^2) + f(r^2)]^2 \right. \\
& \quad \cdot \left[ \frac{2(N-1)}{N+4} \frac{r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} \right] dr \\
& - \frac{16(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty [r^2 f'(r^2)^2 + f(r^2) f'(r^2)] \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} dr \\
& \left. + \frac{4(N-4)^2}{N(N+2)(N+4)} \int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \right\}.
\end{aligned}$$

*Proof.* Direct calculation shows

$$\begin{aligned}
& \sum_{i,j,m,s} (\partial_i \bar{H}_{ms})(\partial_j \bar{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \\
& = \sum_{i,j,m,s} [2f'(|y|^2) y_i H_{ms} + f(|y|^2) \partial_i H_{ms}] [2f'(|y|^2) y_j H_{ms} + f(|y|^2) \partial_j H_{ms}] \\
& \quad \cdot (N-4)^2 \tilde{u}_0^2(y) \frac{(y_i - \xi'_i)(y_j - \xi'_j)}{(\lambda'^2 + |y - \xi'|^2)^2} \\
& = 4(N-4)^2 \left[ |y|^2 f'(|y|^2) + f(|y|^2) \right]^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{m,s} H_{ms}^2 \\
& \quad - 8(N-4)^2 \left[ |y|^2 f'(|y|^2)^2 + f(|y|^2) f'(|y|^2) \right] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{j,m,s} H_{ms}^2 y_j \xi'_j \\
& \quad - 4(N-4)^2 \left[ |y|^2 f(|y|^2) f'(|y|^2) + f(|y|^2)^2 \right] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{j,m,s} H_{ms} (\partial_j H_{ms}) \xi'_j \\
& \quad + 4(N-4)^2 f'(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j,m,s} H_{ms}^2 y_i y_j \xi'_i \xi'_j
\end{aligned}$$

$$\begin{aligned}
& + (N-4)^2 f(|y|^2)^2 \frac{\lambda^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j,m,s} (\partial_i H_{ms})(\partial_j H_{ms}) \xi'_i \xi'_j \\
& + 4(N-4)^2 f(|y|^2) f'(|y|^2) \frac{\lambda^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j,m,s} H_{ms} (\partial_j H_{ms}) y_i \xi'_i \xi'_j, \\
& \left. \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} (\partial_i \bar{H}_{ms})(\partial_j \bar{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \right|_{\xi=0} \\
= & 16(N-1)(N-2)(N-4)^2 \left[ |y|^2 f'(|y|^2) + f(|y|^2) \right]^2 \frac{\lambda^{N-4}}{(\lambda'^2 + |y|^2)^N} \sum_{m,s} H_{ms}^2 y_p y_q \\
& - 8(N-2)(N-4)^2 \left[ |y|^2 f'(|y|^2) + f(|y|^2) \right]^2 \frac{\lambda^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \sum_{m,s} H_{ms}^2 \delta_{pq} \\
& - 32(N-2)(N-4)^2 \left[ |y|^2 f'(|y|^2)^2 + f(|y|^2) f'(|y|^2) \right] \frac{\lambda^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \sum_{m,s} H_{ms}^2 y_p y_q \\
& - 8(N-2)(N-4)^2 \left[ |y|^2 f(|y|^2) f'(|y|^2) + f(|y|^2)^2 \right] \frac{\lambda^{N-4}}{(\lambda'^2 + |y|^2)^{N-1}} \\
& \quad \cdot \sum_{m,s} \left[ H_{ms} (\partial_q H_{ms}) y_p + H_{ms} (\partial_p H_{ms}) y_q \right] \\
& + 8(N-4)^2 f'(|y|^2)^2 \frac{\lambda^{N-4}}{(\lambda'^2 + |y|^2)^{N-2}} \sum_{m,s} H_{ms}^2 y_p y_q \\
& + 2(N-4)^2 f(|y|^2)^2 \frac{\lambda^{N-4}}{(\lambda'^2 + |y|^2)^{N-2}} \sum_{i,j,m,s} (\partial_p H_{ms})(\partial_q H_{ms}) \\
& + 4(N-4)^2 f(|y|^2) f'(|y|^2) \frac{\lambda^{N-4}}{(\lambda'^2 + |y|^2)^{N-2}} \sum_{m,s} [H_{ms} (\partial_q H_{ms}) y_p + H_{ms} (\partial_p H_{ms}) y_q].
\end{aligned}$$

Lemma 9.5 and 9.6 then yield the result.  $\square$

**Lemma 9.11.** *It holds*

$$\begin{aligned}
& \sum_{i,j,m,s} \left[ \bar{H}_{ms} (\partial_s \bar{H}_{ij}) - \bar{H}_{si} (\partial_s \bar{H}_{mj}) + \bar{H}_{sj} (\partial_i \bar{H}_{ms}) \right. \\
& \quad \left. - \bar{H}_{ms} (\partial_i \bar{H}_{sj}) \right] \partial_m (\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) = 0.
\end{aligned}$$

*Proof.* Direct computation shows

$$\begin{aligned}
& \sum_{i,j,m,s} \bar{H}_{ms} (\partial_s \bar{H}_{ij}) \partial_m [\partial_i \tilde{u}_0 \partial_j \tilde{u}_0] \\
= & 2 \sum_{i,j,m,s} \bar{H}_{ms} (\partial_s \bar{H}_{ij}) \partial_{im} \tilde{u}_0 \partial_j \tilde{u}_0
\end{aligned}$$

$$\begin{aligned}
&= -2(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2 \\
&\quad - 2(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j,m} H_{im} (\partial_m H_{ij}) \xi'_j \\
&\quad + 4(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-1}} \sum_{i,j,m} H_{im} H_{jm} \xi'_i \xi'_j \\
&\quad + 2(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-1}} \sum_{i,j,m,s} H_{ms} (\partial_s H_{ij}) \xi'_i \xi'_j \xi'_m
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j,m,s} \bar{H}_{is} (\partial_s \bar{H}_{jm}) \partial_m [\partial_i \tilde{u}_0 \partial_j \tilde{u}_0] \\
&= \sum_{i,j,m,s} \bar{H}_{is} (\partial_s \bar{H}_{jm}) [\partial_{im} \tilde{u}_0 \partial_j \tilde{u}_0 + \partial_i \tilde{u}_0 \partial_{jm} \tilde{u}_0] \\
&= -(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2 \\
&\quad - (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j,m} H_{im} (\partial_m H_{ij}) \xi'_j \\
&\quad + 4(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-1}} \sum_{i,j,m} H_{im} H_{jm} \xi'_i \xi'_j \\
&\quad + 2(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-1}} \sum_{i,j,m,s} H_{is} (\partial_s H_{jm}) \xi'_i \xi'_j \xi'_m.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{i,j,m,s} [\bar{H}_{ms} (\partial_s \bar{H}_{ij}) - \bar{H}_{is} (\partial_s \bar{H}_{jm})] \partial_m [\partial_i \tilde{u}_0 \partial_j \tilde{u}_0] \\
&= -(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2 \\
&\quad - (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j,m} H_{im} (\partial_m H_{ij}) \xi'_j.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{i,j,m,s} [\bar{H}_{sj} (\partial_i \bar{H}_{ms}) - \bar{H}_{ms} (\partial_i \bar{H}_{js})] (\partial_m (\partial_i \tilde{u}_0) (\partial_j \tilde{u}_0)) \\
&= \sum_{i,j,m,s} f(|y|^2)^2 [H_{js} (\partial_i H_{ms}) - H_{ms} (\partial_i H_{js})] (\partial_{im} \tilde{u}_0 \partial_j \tilde{u}_0 + \partial_i \tilde{u}_0 \partial_{jm} \tilde{u}_0)
\end{aligned}$$

$$\begin{aligned}
&= (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2 \\
&\quad + (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \sum_{i,j,s} H_{is} (\partial_i H_{js}) \xi'_j.
\end{aligned}$$

The lemma follows immediately.  $\square$

**Lemma 9.12.** *We have*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left\{ \sum_{i,j,m,s} \bar{H}_{si} (\partial_{mm} \bar{H}_{js}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Big|_{\xi'=0} = 0, \\
&\int_{\mathbb{R}^n} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} \bar{H}_{si} (\partial_{mm} \bar{H}_{js}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Big|_{\xi'=0} \\
&= |S^{N-1}| (W_{ikjp} + W_{ipjk}) (W_{ikjq} + W_{iqjk}) \lambda'^{N-4} \\
&\quad \cdot \frac{2(N-4)^2}{N(N+2)} \int_0^\infty \left[ (N+4) f(r^2) f'(r^2) + 2r^2 f(r^2) f''(r^2) \right] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr.
\end{aligned}$$

*Proof.* It is directly checked that

$$\begin{aligned}
&\sum_{i,j,m,s} \bar{H}_{si} (\partial_{mm} \bar{H}_{js}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \\
&= (N-4)^2 \left[ (2N+8) f(|y|^2) f'(|y|^2) + 4|y|^2 f(|y|^2) f''(|y|^2) \right] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \\
&\quad \cdot \sum_{i,j,s} H_{is} H_{js} \xi'_i \xi'_j,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} \bar{H}_{si} (\partial_{mm} \bar{H}_{js}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Big|_{\xi=0} \\
&= 4(N-4)^2 \left[ (N+4) f(|y|^2) f'(|y|^2) + 2|y|^2 f(|y|^2) f''(|y|^2) \right] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y - \xi'|^2)^{N-2}} \\
&\quad \cdot \sum_s H_{ps} H_{qs}.
\end{aligned}$$

we conclude the proof by using Lemma 9.5 and 9.6.  $\square$

**Lemma 9.13.** *The following hold*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,k,\ell,m} [(\partial_{i\ell} \bar{H}_{mk})^2 + (\partial_\ell \bar{H}_{mk})(\partial_{i\ell} \bar{H}_{mk})] \tilde{u}_0^2 \right\} \Big|_{\xi'=0}$$

$$\begin{aligned}
&= |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{iljk})^2 \lambda^{iN-4} \\
&\quad \cdot \left\{ \frac{2}{N(N+2)} \int_0^\infty \left[ 3(N+8)f'(r^2)^2 + 2(N+8)f(r^2)f''(r^2) \right. \right. \\
&\quad \quad \quad + 2(N+18)r^2 f'(r^2)f''(r^2) + 4r^4 f''(r^2)^2 \\
&\quad \quad \quad \left. \left. + 4r^2 f(r^2)f'''(r^2) + 4r^4 f'(r^2)f'''(r^2) \right] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-4}} dr \right. \\
&\quad + \frac{2}{N} \int_0^\infty \left[ 4r^2 f'(r^2)^2 + (N+8)f(r^2)f'(r^2) + 2r^2 f(r^2)f''(r^2) \right] \frac{r^{N+1}}{(\lambda'^2 + r^2)^{N-4}} dr \\
&\quad \left. + \int_0^\infty f(r^2)^2 \frac{r^{N-1}}{(\lambda'^2 + r^2)^{N-4}} dr \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\partial^2}{\partial \xi_p' \partial \xi_q'} \left\{ \sum_{i,k,\ell,m} [(\partial_{i\ell} \bar{H}_{mk})^2 + (\partial_\ell \bar{H}_{mk})(\partial_{i\ell} \bar{H}_{mk})] \bar{u}_0^2 \right\} \Big|_{\xi=0} \\
&= |S^{N-1}| (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda^{iN-4} \\
&\quad \left\{ \frac{8(N-3)(N-4)}{N(N+2)(N+4)} \right. \\
&\quad \cdot \int_0^\infty \left[ 12(N+8)f'(r^2)^2 + 16r^4 f''(r^2)^2 + 8(N+18)r^2 f'(r^2)f''(r^2) \right. \\
&\quad \quad \quad \left. + 8(N+8)f(r^2)f''(r^2) + 16r^2 f(r^2)f'''(r^2) + 16r^4 f'(r^2)f'''(r^2) \right] \\
&\quad \quad \quad \cdot \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} \\
&\quad + \frac{8(N-3)(N-4)}{N(N+2)} \\
&\quad \cdot \int_0^\infty \left[ 8r^2 f'(r^2)^2 + 2(N+8)f(r^2)f'(r^2) + 4r^2 f(r^2)f''(r^2) \right] \frac{r^{N+3}}{(\lambda'^2 + |x|^2)^{N-2}} \Big\} \\
&+ |S^{N-1}| (W_{ikj\ell} + W_{iljk})^2 \lambda^{iN-4} \delta_{pq} \\
&\quad \left\{ \frac{N-4}{N(N+2)} \right. \\
&\quad \cdot \int_0^\infty \left[ 12(N+8)f'(r^2)^2 + 16r^4 f''(r^2)^2 + 8(N+18)r^2 f'(r^2)f''(r^2) \right. \\
&\quad \quad \quad \left. + 8(N+8)f(r^2)f''(r^2) + 16r^2 f(r^2)f'''(r^2) + 16r^4 f'(r^2)f'''(r^2) \right] \\
&\quad \cdot \left[ \frac{2(N-3)}{N+4} \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} - \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-3}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(N-4)}{N} \int_0^\infty \left[ 8r^2 f'(r^2)^2 + 2(N+8) f(r^2) f'(r^2) + 4r^2 f(r^2) f''(r^2) \right] \\
& \cdot \left[ \frac{2(N-3)}{N+2} \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} - \frac{r^{N+1}}{(\lambda'^2 + r^2)^{N-3}} \right] \\
& + 2(N-4) \int_0^\infty f(r^2)^2 \left[ \frac{2(N-3)}{N} \frac{r^{N+1}}{(\lambda'^2 + r^2)^{N-2}} - \frac{r^{N-1}}{(\lambda'^2 + r^2)^{N-3}} \right] \Bigg\}.
\end{aligned}$$

*Proof.* We have

$$\begin{aligned}
& \sum_{i,\ell} (\partial_{i\ell} \bar{H}_{mk})^2 \\
= & \sum_{i,\ell} \left[ 2\delta_{i\ell} f'(|y|^2) H_{mk} + 2f'(|y|^2) y_\ell (\partial_i H_{mk}) + 2f'(|y|^2) y_i (\partial_\ell H_{mk}) \right. \\
& \left. + f(|y|^2) \partial_{i\ell} H_{mk} + 4f''(|y|^2) y_\ell y_i H_{mk} \right]^2 \\
= & 4N f'(|y|^2)^2 H_{mk}^2 + 8|y|^2 f'(|y|^2)^2 \sum_i (\partial_i H_{mk})^2 + f(|y|^2)^2 \sum_{i,\ell} (\partial_{i\ell} H_{mk})^2 \\
& + 16f'(|y|^2)^2 \sum_i y_i H_{mk} (\partial_i H_{mk}) + 4f'(|y|^2) f(|y|^2) \sum_i H_{mk} (\partial_{ii} H_{mk}) \\
& + 8f'(|y|^2)^2 \sum_{i,\ell} y_i y_\ell (\partial_i H_{mk}) (\partial_\ell H_{mk}) + 8f'(|y|^2) f(|y|^2) \sum_{i,\ell} y_\ell (\partial_i H_{mk}) (\partial_{i\ell} H_{mk}) \\
& + 16f''(|y|^2)^2 |y|^2 H_{mk}^2 + 80f'(|y|^2) f''(|y|^2) |y|^2 H_{mk}^2 \\
& + 8f(|y|^2) f''(|y|^2) \sum_{i,\ell} y_i y_\ell H_{mk} (\partial_{i\ell} H_{mk}) \\
= & \left[ (4N + 64) f'(|y|^2)^2 + 16|y|^4 f''(|y|^2)^2 + 80|y|^2 f'(|y|^2) f''(|y|^2) + 16f(|y|^2) f''(|y|^2) \right] H_{mk}^2 \\
& + 8 \left[ |y|^2 f'(|y|^2)^2 + f(|y|^2) f'(|y|^2) \right] \sum_i (\partial_i H_{mk})^2 + f(|y|^2)^2 \sum_{i,\ell} (\partial_{i\ell} H_{mk})^2, \\
& \sum_{i,\ell} (\partial_{i\ell} \bar{H}_{mk}) (\partial_{i\ell} \bar{H}_{mk}) \\
= & \sum_\ell \left\{ \left[ (2N + 8) f'(|y|^2) + 4|y|^2 f''(|y|^2) \right] (\partial_\ell H_{mk}) \right. \\
& \left. + \left[ (4N + 24) f''(|y|^2) + 8|y|^2 f'''(|y|^2) \right] y_\ell H_{mk} \right\} \\
& \cdot \left[ 2f'(|y|^2) y_\ell H_{mk} + f(|y|^2) \partial_\ell H_{mk} \right] \\
= & \left[ 8(N + 4) f'(|y|^2)^2 + 8(N + 8) |y|^2 f'(|y|^2) f''(|y|^2) + 8(N + 6) f(|y|^2) f''(|y|^2) \right. \\
& \left. + 16|y|^2 f(|y|^2) f''(|y|^2) + 16|y|^4 f'(|y|^2) f'''(|y|^2) \right] H_{mk}^2
\end{aligned}$$

$$+ \left[ 2(N+4)f(|y|^2)f'(|y|^2) + 4|y|^2f(|y|^2)f''(|y|^2) \right] \sum_{\ell} (\partial_{\ell} H_{mk})^2,$$

and

$$\begin{aligned} & \left. \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,k,\ell,m} [(\partial_{i\ell} \bar{H}_{mk})^2 + (\partial_{\ell} \bar{H}_{mk})(\partial_{i\ell} \bar{H}_{mk})] \tilde{u}_0^2 \right\} \right|_{\xi'=0} \\ = & 4(N-3)(N-4) \left[ 12(N+8)f'(|y|^2)^2 + 16|y|^4 f''(|y|^2)^2 + 8(N+18)|y|^2 f'(|y|^2) f''(|y|^2) \right. \\ & \left. + 8(N+8)f(|y|^2) f''(|y|^2) + 16|y|^2 f(|y|^2) f'''(|y|^2) + 16|y|^4 f'(|y|^2) f'''(|y|^2) \right] \\ & \cdot \frac{\lambda^{N-4}}{(\lambda^2 + |y|^2)^{N-2}} \sum_{k,m} H_{km}^2 y_p y_q \\ - & 2(N-4) \left[ 12(N+8)f'(|y|^2)^2 + 16|y|^4 f''(|y|^2)^2 + 8(N+18)|y|^2 f'(|y|^2) f''(|y|^2) \right. \\ & \left. + 8(N+8)f(|y|^2) f''(|y|^2) + 16|y|^2 f(|y|^2) f'''(|y|^2) + 16|y|^4 f'(|y|^2) f'''(|y|^2) \right] \\ & \cdot \frac{\lambda^{N-4}}{(\lambda^2 + |y|^2)^{N-3}} \sum_{k,m} H_{km}^2 \delta_{pq} \\ + & 4(N-3)(N-4) \left[ 8|y|^2 f'(|y|^2)^2 + 2(N+8)f(|y|^2) f'(|y|^2) + 4|y|^2 f(|y|^2) f''(|y|^2) \right] \\ & \cdot \frac{\lambda^{N-4}}{(\lambda^2 + |y|^2)^{N-2}} \sum_{k,\ell,m} (\partial_{\ell} H_{km})^2 y_p y_q \\ - & 2(N-4) \left[ 8|y|^2 f'(|y|^2)^2 + 2(N+8)f(|y|^2) f'(|y|^2) + 4|y|^2 f(|y|^2) f''(|y|^2) \right] \\ & \cdot \frac{\lambda^{N-4}}{(\lambda^2 + |y|^2)^{N-3}} \sum_{k,\ell,m} (\partial_{\ell} H_{km})^2 \delta_{pq} \\ + & 4(N-3)(N-4) f(|y|^2)^2 \frac{\lambda^{N-4}}{(\lambda^2 + |y|^2)^{N-2}} \sum_{i,k,\ell,m} (\partial_{i\ell} H_{km})^2 y_p y_q \\ - & 2(N-4) f(|y|^2)^2 \frac{\lambda^{N-4}}{(\lambda^2 + |y|^2)^{N-3}} \sum_{i,k,\ell,m} (\partial_{i\ell} H_{km})^2 \delta_{pq}. \end{aligned}$$

Lemma 9.5 and 9.6 then give the result.  $\square$

**Lemma 9.14.** *We have*

$$\begin{aligned} & \int_{\mathbb{R}^N} \left\{ \sum_{i,j,m,s} (\partial_{mm} \bar{H}_{ij})(\partial_{ss} \bar{H}_{ij}) \tilde{u}_0^2 \right\} \Big|_{\xi'=0} \\ = & |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda^{N-4} \\ & \cdot \frac{1}{2N(N+2)} \int_0^{\infty} \left[ 2(N+4)f'(r^2) + 4r^2 f''(r^2) \right]^2 \frac{r^{N+3}}{(\lambda^2 + r^2)^{N-4}} dr \end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^N} \frac{\partial^2}{\partial \xi_p \partial \xi_q} \left\{ \sum_{i,j,m,s} (\partial_{mm} \bar{H}_{ij})(\partial_{ss} \bar{H}_{ij}) \tilde{u}_0^2 \right\} \Big|_{\xi'=0} \\
&= |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda^{N-4} \\
&\quad \cdot \frac{8(N-3)(N-4)}{N(N+2)(N+4)} \int_0^\infty [2(N+4)f'(r^2) + 4r^2 f''(r^2)]^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \\
&+ |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda^{N-4} \delta_{pq} \\
&\quad \cdot \frac{N-4}{N(N+2)} \int_0^\infty [2(N+4)f'(r^2) + 4r^2 f''(r^2)]^2 \\
&\quad \cdot \left[ \frac{2(N-3)}{N+4} \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} - \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-3}} \right] dr.
\end{aligned}$$

*Proof.* Since

$$\begin{aligned}
\sum_m \partial_{mm} \bar{H}_{ij} &= \sum_m \left( 4f'(|y|^2) y_m \partial_m H_{ij} + 2f'(|y|^2) H_{ij} + 4f''(|y|^2) y_m^2 H_{ij} \right. \\
&\quad \left. + f(|y|^2) \partial_{mm} H_{ij} \right) \\
&= [2(N+4)f'(|y|^2) + 4|y|^2 f''(|y|^2)] H_{ij},
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} (\partial_{mm} \bar{H}_{ij})(\partial_{ss} \bar{H}_{ij}) u_{(\xi',\varepsilon)}^2 \right\} \Big|_{\xi'=0} \\
&= 4(N-3)(N-4) [2(N+4)f'(|y|^2) + 4|y|^2 f''(|y|^2)]^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-2}} \sum_{i,j} H_{ij}^2 y_p y_q \\
&\quad - 2(N-4) [2(N+4)f'(|y|^2) + 4|y|^2 f''(|y|^2)]^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-3}} \sum_{i,j} H_{ij}^2 \delta_{pq},
\end{aligned}$$

we obtain the result by Lemma 9.5 and 9.6.  $\square$

**Lemma 9.15.** *There hold*

$$\begin{aligned}
\sum_{i,j,s} (\partial_{ss} \bar{H}_{ij})(\partial_{ij} \tilde{u}_0) w \Big|_{\xi'=0} &= 0, & \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \sum_{i,j,s} (\partial_{ss} \bar{H}_{ij})(\partial_{ij} \tilde{u}_0) w \Big|_{\xi'=0} &= 0, \\
\sum_{i,j,s} (\partial_s \bar{H}_{ij})(\partial_{sij} \tilde{u}_0) w \Big|_{\xi'=0} &= 0, & \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \sum_{i,j,s} (\partial_s \bar{H}_{ij})(\partial_{sij} \tilde{u}_0) w \Big|_{\xi'=0} &= 0,
\end{aligned}$$

$$\begin{aligned} \sum_{i,j,s} \overline{H}_{ij}(\partial_{ssij}\tilde{u}_0)w \Big|_{\xi'=0} &= 0, & \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \sum_{i,j,s} \overline{H}_{ij}(\partial_{ssij}\tilde{u}_0)w \Big|_{\xi'=0} &= 0, \\ \sum_{i,j,m} (\partial_{jmm}\overline{H}_{ij})(\partial_i u)w \Big|_{\xi'=0} &= 0, & \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \sum_{i,j,m} (\partial_{jmm}\overline{H}_{ij})(\partial_i u)w \Big|_{\xi'=0} &= 0. \end{aligned}$$

*Proof.* Direct computation shows

$$\begin{aligned} & \sum_{i,j,s} (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) \\ &= \left[ 2Nf'(|y|^2) + 8f'(|y|^2) + 4|y|^2 f''(|y|^2) \right] \sum_{i,j} H_{ij}(\partial_{ij}\tilde{u}_0) \\ &= (N-2)(N-4) \left[ (2N+8)f'(|y|^2) + 4|y|^2 f''(|y|^2) \right] \\ & \quad \cdot \frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y - \xi'|^2)^{\frac{N}{2}}} \sum_{i,j} H_{ij}\xi'_i\xi'_j, \\ \\ & \sum_{i,j,s} (\partial_s\overline{H}_{ij})(\partial_{sij}\tilde{u}_0) \\ &= \sum_{i,j,s} \left[ 2f'(|y|^2)y_s H_{ij} + f(|y|^2)(\partial_s H_{ij}) \right] (N-2)(N-4)\tilde{u}_0 \\ & \quad \cdot \left[ \frac{\delta_{is}(y_j - \xi'_j) + \delta_{js}(y_i - \xi'_i) + \delta_{ij}(y_s - \xi'_s)}{(\lambda'^2 + |y - \xi'|^2)^2} - N \frac{(y_s - \xi'_s)(y_i - \xi'_i)(y_j - \xi'_j)}{(\lambda'^2 + |y - \xi'|^2)^3} \right] \\ &= -2N(N-2)(N-4)|y|^2 f'(|y|^2) \frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y - \xi'|^2)^{\frac{N+2}{2}}} \sum_{i,j} H_{ij}\xi'_i\xi'_j \\ & \quad + 2N(N-2)(N-4)f'(|y|^2) \frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y - \xi'|^2)^{\frac{N+2}{2}}} \sum_{i,j,s} H_{ij}y_s \xi'_s \xi'_i \xi'_j \\ & \quad + N(N-2)(N-4)f(|y|^2) \frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y - \xi'|^2)^{\frac{N+2}{2}}} \sum_{i,j,s} (\partial_s H_{ij})\xi'_s \xi'_i \xi'_j, \\ \\ & \sum_{i,j,s} \overline{H}_{ij}(\partial_{ssij}\tilde{u}_0) \\ &= N(N-2)(N+2)(N-4)f(|y|^2) \frac{|y - \xi'|^2 \lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y - \xi'|^2)^{\frac{N+4}{2}}} \sum_{i,j} H_{ij}\xi'_i\xi'_j \\ & \quad - N(N-2)(N-4)(N+4)f(|y|^2) \frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y - \xi'|^2)^{\frac{N+2}{2}}} \sum_{i,j} H_{ij}\xi'_i\xi'_j, \end{aligned}$$

$$\sum_{i,j,m} (\partial_{jmm} \bar{H}_{ij})(\partial_i u) = 0.$$

Recall the equation for  $w$  and note that if  $\xi' = 0$ ,  $R_1(y) = 0$ , so  $w|_{\xi'=0} \equiv 0$ . The Lemma is easily concluded.  $\square$

Combining the above identities, we have the following proposition.

**Proposition 9.16.** *It holds*

$$\begin{aligned} & F(0, \lambda') \\ = & \frac{N-4}{2} |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \\ & \cdot \left\{ -\frac{a_N(N-4)}{N(N+2)} \int_0^\infty [r^2 f'(r^2)^2 + 2f(r^2)f'(r^2)] \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \right. \\ & - \frac{a_N(N-4)}{2N} \int_0^\infty f(r^2)^2 \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr \\ & - \frac{b_N(N-4)}{N(N+2)} \int_0^\infty [r^2 f'(r^2) + f(r^2)]^2 \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr \\ & + \frac{1}{2N(N-1)(N+2)} \int_0^\infty \left[ 3(N+8)f'(r^2)^2 + 2(N+8)f(r^2)f''(r^2) \right. \\ & \quad \left. + 2(N+18)r^2 f'(r^2)f''(r^2) + 4r^4 f''(r^2)^2 \right. \\ & \quad \left. + 4r^2 f(r^2)f'''(r^2) + 4r^4 f'(r^2)f'''(r^2) \right] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-4}} dr \\ & + \frac{1}{2N(N-1)} \int_0^\infty \left[ 4r^2 f'(r^2)^2 + (N+8)f(r^2)f'(r^2) + 2r^2 f(r^2)f''(r^2) \right] \\ & \quad \cdot \frac{r^{N+1}}{(\lambda'^2 + r^2)^{N-4}} dr \\ & + \frac{1}{4(N-1)} \int_0^\infty f(r^2)^2 \frac{r^{N-1}}{(\lambda'^2 + r^2)^{N-4}} dr \\ & \left. - \frac{1}{N(N-2)^2(N+2)} \int_0^\infty \left[ (N+4)f'(r^2) + 2r^2 f''(r^2) \right]^2 \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-4}} dr \right\}. \end{aligned}$$

Our final goal is to choose the auxiliary function  $f$ . Unlike [4], where a quadratic polynomial is chosen, we need to choose a 4th order polynomial

$$f(s) = \tau - 1200s + 2411s^2 - 135s^3 + s^4. \quad (50)$$

Using the software *Mathematica*, we get

**Proposition 9.17.** *Assume  $N \geq 25$ ,*

$$F(0, \lambda')$$

$$= \frac{N-4}{16(N^2-4)} |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \frac{\Gamma[\frac{N}{2}-9] \Gamma[\frac{N}{2}+7]}{\Gamma[N+1]} I(\lambda')$$

where  $\Gamma$  denotes the usual  $\Gamma$ -function and

$$\begin{aligned} & I(\lambda') \\ = & - \frac{(N-4)(N+14)\lambda^{20}}{(N-24)(N-22)(N-20)} \left( N^6 + 42N^5 - 768N^4 - 17248N^3 + 38768N^2 \right. \\ & \qquad \qquad \qquad \left. - 2336N - 38400 \right) \\ & + \frac{270(N-4)\lambda^{18}}{(N-22)(N-20)} \left( N^6 + 32N^5 - 612N^4 - 10768N^3 + 24672N^2 - 640N \right. \\ & \qquad \qquad \qquad \left. - 25600 \right) \\ & - \frac{(N-4)\lambda^{16}}{(N-20)(N+12)} \left( 23047N^6 + 543484N^5 - 10985408N^4 - 146678256N^3 \right. \\ & \qquad \qquad \qquad \left. + 351063488N^2 - 16180224N - 363260160 \right) \\ & + \frac{30(N-4)\lambda^{14}}{(N+10)(N+12)} \left( 22499N^6 + 356784N^5 - 7984044N^4 - 76228592N^3 \right. \\ & \qquad \qquad \qquad \left. + 193344928N^2 - 4902016N - 209193984 \right) \\ & - \frac{(N-18)(N-4)\lambda^{12}}{(N+8)(N+10)(N+12)} \left( 2\tau N^6 + 9052921N^6 + 4\tau N^5 + 84049210N^5 \right. \\ & \qquad \qquad \qquad - 384\tau N^4 - 2265707776N^4 - 1856\tau N^3 - 14204127072N^3 + 4704\tau N^2 \\ & \qquad \qquad \qquad \left. + 40189627120N^2 + 7360\tau N - 1125216800N - 12800\tau - 45033619968 \right) \\ & + \frac{30(N-18)(N-16)(N-4)\lambda^{10}}{(N+8)(N+10)(N+12)} \left( 9\tau N^5 + 1928800N^5 - 54\tau N^4 \right. \\ & \qquad \qquad \qquad - 3857600N^4 - 864\tau N^3 - 293177600N^3 + 1584\tau N^2 + 648076800N^2 \\ & \qquad \qquad \qquad \left. + 2880\tau N + 61721600N - 4608\tau - 740659200 \right) \\ & - \frac{2(N-18)(N-16)(N-14)(N-4)\lambda^8}{(N+6)(N+8)(N+10)(N+12)} \left( 2411\tau N^5 + 72000000N^5 \right. \\ & \qquad \qquad \qquad - 14466\tau N^4 - 288000000N^4 - 135016\tau N^3 - 576000000N^3 \\ & \qquad \qquad \qquad \left. + 308608\tau N^2 + 1497600000N^2 + 347184\tau N - 2304000000N \right. \\ & \qquad \qquad \qquad \left. - 694368\tau - 1382400000 \right) \\ & + \frac{24000(N-18)(N-16)(N-14)(N-12)(N-4)(N-2)\tau\lambda^6}{(N+6)(N+8)(N+10)(N+12)} \left( N^3 - 8N^2 \right. \\ & \qquad \qquad \qquad \left. + 16 \right) \end{aligned}$$

$$- \frac{(N-18)(N-16)(N-14)(N-12)(N-10)(N-4)(N-2)^2 \tau^2 \lambda^4}{(N+4)(N+6)(N+8)(N+10)(N+12)} \left( N^2 - 4N - 4 \right) \Big\}.$$

Next we compute the Hessian of  $F$  at  $(0, \lambda')$ . Because of Lemma 9.15, it is obvious that

$$\begin{aligned} & \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,s} \left[ 2\overline{H}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s \overline{H}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss} \overline{H}_{ij})(\partial_{ij}\tilde{u}_0) \right. \right. \\ & \quad \left. \left. - \frac{b_N}{2}(\partial_{ss} \overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss} \overline{H}_{ij})(\partial_i \tilde{u}_0) \right] \bar{w} \right\} \Big|_{\xi'=0} \\ & = 0. \end{aligned}$$

**Proposition 9.18.** *The second order partial derivatives of  $F(\xi', \lambda')$  at  $(0, \lambda')$  are given by*

$$\begin{aligned} & \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} F(0, \lambda') \\ & = \frac{(N-4)^2}{N(N+2)} |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4} \\ & \quad \cdot \left\{ -(N-2) \int_0^\infty f(r^2)^2 \left[ \frac{Nr^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{(N+2)r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} \right] dr \right. \\ & \quad - \frac{8a_N(N-2)}{N+4} \int_0^\infty [r^2 f'(r^2)^2 + 2f(r^2)f'(r^2)] \\ & \quad \quad \cdot \left[ \frac{(N-1)r^{N+7}}{(\lambda'^2 + r^2)^N} - \frac{2r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} \right] dr \\ & \quad - 2a_N(N-2) \int_0^\infty f(r^2)^2 \left[ \frac{(N-1)r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{2r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} \right] dr \\ & \quad - \frac{8b_N(N-1)(N-2)}{N+4} \int_0^\infty [r^2 f'(r^2) + f(r^2)]^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^N} dr \\ & \quad + \frac{16b_N(N-2)}{N+4} \int_0^\infty [r^2 f'(r^2)^2 + f(r^2)f'(r^2)] \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} dr \\ & \quad + 4b_N(N-2) \int_0^\infty [r^2 f(r^2)f'(r^2) + f(r^2)^2] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} dr \\ & \quad - \frac{4b_N}{N+4} \int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \\ & \quad - \frac{b_N(N+2)}{2} \int_0^\infty f(r^2)^2 \frac{r^{N+1}}{(\lambda'^2 + r^2)^{N-2}} dr \\ & \quad \left. - 2b_N \int_0^\infty f(r^2)f'(r^2) \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr \right\} \end{aligned}$$

$$\begin{aligned}
& + b_N \int_0^\infty \left[ (N+4)f(r^2)f'(r^2) + 2r^2f(r^2)f''(r^2) \right] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr \\
& + \frac{4(N-3)}{(N-1)(N+4)} \\
& \cdot \int_0^\infty \left[ 3(N+8)f'(r^2)^2 + 4r^4f''(r^2)^2 + 2(N+18)r^2f'(r^2)f''(r^2) \right. \\
& \quad \left. + 2(N+8)f(r^2)f''(r^2) + 4r^2f(r^2)f'''(r^2) + 4r^4f'(r^2)f'''(r^2) \right] \\
& \quad \cdot \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \\
& + \frac{2(N-3)}{N-1} \int_0^\infty \left[ 4r^2f'(r^2)^2 + (N+8)f(r^2)f'(r^2) + 2r^2f(r^2)f''(r^2) \right] \\
& \quad \cdot \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} dr \\
& - \frac{8(N-3)}{(N-2)^2(N+4)} \int_0^\infty \left[ (N+4)f'(r^2) + 2r^2f''(r^2) \right]^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \left. \right\} \\
& + \frac{(N-4)^2}{N(N+2)} |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{iljk})^2 \lambda'^{N-4} \delta_{pq} \\
& \cdot \left\{ - \frac{a_N}{N+4} \int_0^\infty \left[ r^2f'(r^2)^2 + 2f(r^2)f'(r^2) \right] \right. \\
& \quad \cdot \left[ \frac{2(N-1)(N-2)r^{N+7}}{(\lambda'^2 + r^2)^N} - \frac{(N-2)(N+8)r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} + \frac{(N+4)r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} \right] dr \\
& - \frac{a_N}{2} \int_0^\infty f(r^2)^2 \\
& \quad \cdot \left[ \frac{2(N-1)(N-2)r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{(N-2)(N+6)r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} + \frac{(N+2)r^{N+1}}{(\lambda'^2 + r^2)^{N-2}} \right] dr \\
& - \frac{b_N(N-2)}{N+4} \int_0^\infty \left[ r^2f'(r^2) + f(r^2) \right]^2 \\
& \quad \cdot \left[ \frac{2(N-1)r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{(N+4)r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} \right] dr \\
& + \frac{4b_N(N-2)}{N+4} \int_0^\infty \left[ r^2f'(r^2)^2 + f(r^2)f'(r^2) \right] \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-1}} dr \\
& - \frac{b_N}{N+4} \int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} dr \\
& + \frac{1}{2(N-1)(N+4)} \\
& \cdot \int_0^\infty \left[ 3(N+8)f'(r^2)^2 + 4r^4f''(r^2)^2 + 2(N+18)r^2f'(r^2)f''(r^2) \right.
\end{aligned}$$

$$\begin{aligned}
& + 2(N+8)f(r^2)f''(r^2) + 4r^2f(r^2)f'''(r^2) + 4r^4f'(r^2)f'''(r^2) \\
& \quad \cdot \left[ \frac{2(N-3)r^{N+5}}{(\lambda'^2+r^2)^{N-2}} - \frac{(N+4)r^{N+3}}{(\lambda'^2+r^2)^{N-3}} \right] dr \\
& + \frac{1}{2(N-1)} \int_0^\infty \left[ 4r^2f'(r^2)^2 + (N+8)f(r^2)f'(r^2) + 2r^2f(r^2)f''(r^2) \right] \\
& \quad \cdot \left[ \frac{2(N-3)r^{N+3}}{(\lambda'^2+r^2)^{N-2}} - \frac{(N+2)r^{N+1}}{(\lambda'^2+r^2)^{N-3}} \right] dr \\
& + \frac{N+2}{4(N-1)} \int_0^\infty f(r^2)^2 \left[ \frac{2(N-3)r^{N+1}}{(\lambda'^2+r^2)^{N-2}} - \frac{Nr^{N-1}}{(\lambda'^2+r^2)^{N-3}} \right] dr \\
& - \frac{1}{(N-2)^2(N+4)} \int_0^\infty [(N+4)f'(r^2) + 2r^2f''(r^2)]^2 \\
& \quad \cdot \left[ \frac{2(N-3)r^{N+5}}{(\lambda'^2+r^2)^{N-2}} - \frac{(N+4)r^{N+3}}{(\lambda'^2+r^2)^{N-3}} \right] dr \Big\}.
\end{aligned}$$

Also by *Mathematica*, the following holds.

**Proposition 9.19.** Assume  $N \geq 25$ ,

$$\begin{aligned}
& \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} F(0, \lambda') \\
& = \frac{(N-4)^2}{32N(N-2)(N-1)(N+2)(N+4)} |S^{N-1}| \frac{\Gamma(\frac{N}{2}-7)\Gamma(\frac{N}{2}+5)}{\Gamma(N+1)} \\
& \quad \cdot \left\{ \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) J_1(\lambda') \right. \\
& \quad \quad \left. + \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \delta_{pq} J_2(\lambda') \right\}
\end{aligned}$$

where

$$\begin{aligned}
& J_1(\lambda') \\
& = \frac{4(N-1)(N+10)(N+12)(N+14)(N+16)\lambda^{18}}{(N-22)(N-20)(N-18)(N-16)} (N^7 + 31N^6 + 3156N^5 \\
& \quad + 42380N^4 - 200000N^3 + 520800N^2 - 788864N + 460800) \\
& - \frac{1080(N-1)(N+10)(N+12)(N+14)\lambda^{16}}{(N-20)(N-18)(N-16)} (N^7 + 27N^6 + 2180N^5 \\
& \quad + 25044N^4 - 118704N^3 + 312096N^2 - 482816N + 286720) \\
& + \frac{4(N-1)(N+10)(N+12)\lambda^{14}}{(N-18)(N-16)} (23047N^7 + 530081N^6 + 33297184N^5 \\
& \quad + 318781420N^4 - 1528367840N^3 + 4088463840N^2 - 6478002432N
\end{aligned}$$

$$\begin{aligned}
& + 3912975360) \\
& - \frac{120(N-1)(N+10)\lambda^{12}}{N-16} \left( 22499N^7 + 427481N^6 + 19767516N^5 \right. \\
& \quad \left. + 151493620N^4 - 734575072N^3 + 2011573728N^2 - 3318136192N \right. \\
& \quad \left. + 2061219840 \right) \\
& + 4(N-1)\lambda^{10} \left( 2\tau N^7 + 9052921N^7 + 30\tau N^6 + 135793815N^6 + 520\tau N^5 \right. \\
& \quad \left. + 4332060500N^5 + 2616\tau N^4 + 25029894268N^4 - 12096\tau N^3 \right. \\
& \quad \left. - 123671649760N^3 + 30144\tau N^2 + 353038701792N^2 - 67840\tau N \right. \\
& \quad \left. - 619870766720N + 51200\tau + 400569799680 \right) \\
& - \frac{120(N-14)(N-1)\lambda^8}{N+8} \left( 9\tau N^7 + 1928800N^7 + 99\tau N^6 + 21216800N^6 \right. \\
& \quad \left. + 1188\tau N^5 + 424336000N^5 + 4068\tau N^4 + 1674198400N^4 - 20304\tau N^3 \right. \\
& \quad \left. - 8486720000N^3 + 55584\tau N^2 + 26478566400N^2 - 140544\tau N \right. \\
& \quad \left. - 51846144000N + 110592\tau + 35551641600 \right) \\
& + \frac{8(N-14)(N-12)(N-1)\lambda^6}{(N+6)(N+8)} \left( 2411\tau N^7 + 72000000N^7 + 16877\tau N^6 \right. \\
& \quad \left. + 504000000N^6 + 106084\tau N^5 + 5184000000N^5 + 202524\tau N^4 \right. \\
& \quad \left. + 11808000000N^4 - 1041552\tau N^3 - 57600000000N^3 + 3780448\tau N^2 \right. \\
& \quad \left. + 24192000000N^2 - 12961536\tau N - 58982400000N + 11109888\tau \right. \\
& \quad \left. + 442368000000 \right) \\
& - \frac{96000(N-14)(N-12)(N-10)(N-2)(N-1)\tau\lambda^4}{(N+6)(N+8)} \left( N^5 + N^4 + 2N^3 \right. \\
& \quad \left. + 16N^2 + 80N - 128 \right) \\
& + \frac{4(N-14)(N-12)(N-10)(N-8)(N-2)(N-1)^2N\tau^2\lambda^2}{(N+6)(N+8)} \left( N^2 - 4N \right. \\
& \quad \left. + 8 \right)
\end{aligned}$$

and

$$\begin{aligned}
& J_2(\lambda') \\
& = \frac{32(N-1)(N+10)(N+12)(N+14)\lambda^{18}}{(N-22)(N-20)(N-18)(N-16)} \left( 31N^6 + 1816N^5 + 15780N^4 \right. \\
& \quad \left. - 79984N^3 + 206368N^2 - 315648N + 188416 \right) \\
& - \frac{1080(N-1)(N+10)(N+12)\lambda^{16}}{(N-20)(N-18)(N-16)} \left( 189N^6 + 8914N^5 + 62004N^4 \right.
\end{aligned}$$

$$\begin{aligned}
& - 323544N^3 + 838688N^2 - 1314112N + 802816) \\
& + \frac{8(N-1)(N+10)\lambda^{14}}{(N-18)(N-16)} \left( 1590243N^6 + 59252435N^5 + 313715178N^4 \right. \\
& \quad \left. - 1724484376N^3 + 4511759584N^2 - 7272932160N + 4564175616 \right) \\
& - \frac{120(N-1)\lambda^{12}}{N-16} \left( 2137405N^6 + 60099118N^5 + 220894524N^4 \right. \\
& \quad \left. - 1323783112N^3 + 3484637248N^2 - 5893950528N + 3867347200 \right) \\
& + \frac{16(N-1)\lambda^{10}}{N+8} \left( 30\tau N^6 + 135793815N^6 + 440\tau N^5 + 2733505510N^5 \right. \\
& \quad + 792\tau N^4 + 5860002912N^4 - 4896\tau N^3 - 42292954688N^3 + 9280\tau N^2 \\
& \quad + 111452683936N^2 - 23552\tau N - 203797809280N + 20480\tau \\
& \quad \quad \quad \left. + 143346417664 \right) \\
& - \frac{120(N-14)(N-1)\lambda^8}{N+8} \left( 297\tau N^5 + 63650400N^5 + 1404\tau N^4 \right. \\
& \quad + 470627200N^4 - 5940\tau N^3 - 2198832000N^3 + 11232\tau N^2 \\
& \quad + 4937728000N^2 - 25056\tau N - 8455859200N + 20736\tau + 5925273600) \\
& + \frac{16(N-14)(N-12)(N-2)(N-1)\lambda^6}{(N+6)(N+8)} \left( 16877\tau N^4 + 504000000N^4 \right. \\
& \quad + 81974\tau N^3 + 2952000000N^3 - 57864\tau N^2 - 4032000000N^2 \\
& \quad + 270032\tau N + 11520000000N - 462912\tau - 16128000000) \\
& - \frac{96000(N-14)(N-12)(N-10)(N-2)(N-1)\tau\lambda^4}{(N+6)(N+8)} \left( 3N^3 - 2N^2 \right. \\
& \quad \quad \quad \left. + 8N - 16 \right).
\end{aligned}$$

**Lemma 9.20.** *Assume that  $N \geq 25$ . Then there exists a  $\tau \in \mathbb{R}$  such that  $I'(1) = 0$ ,  $I''(1) > 0$ ,  $I(1) < 0$ ,  $J_1(1) > 0$  and  $J_2(1) > 0$ .*

*Proof.* By the software *Mathematica*,  $I'(1) = 0$  is equivalent to

$$\begin{aligned}
& - \frac{4(N-18)(N-16)(N-14)(N-12)(N-10)(N-4)(N-2)^2(N^2-4N-4)\tau^2}{(N+4)(N+6)(N+8)(N+10)(N+12)} \\
& + \frac{4(N-18)(N-4)(N-2)}{(N+6)(N+8)(N+10)(N+12)} \left( 27025N^6 - 1481614N^5 + 30312640N^4 \right. \\
& \quad \left. - 273305456N^3 + 893995744N^2 + 378381120N - 1876068864 \right) \tau \\
& - \frac{8(N-4)}{(N-24)(N-22)(N-20)(N+6)(N+8)(N+10)(N+12)} \left( 84113673N^{11} \right. \\
& \quad \left. - 10938700432N^{10} + 598157791200N^9 - 17529195040256N^8 \right)
\end{aligned}$$

$$\begin{aligned}
& + 283729391286736N^7 - 2092231289713280N^6 - 4536333083997056N^5 \\
& + 195286068740668928N^4 - 1181494667636928512N^3 \\
& + 1811151426900393984N^2 + 274396140023021568N \\
& \quad - 1766811639068098560) = 0.
\end{aligned}$$

Since  $N \geq 25$ , the above equation has two real solutions. Let  $\tau$  be one of them, that is

$$\tau = \frac{A_1 + \sqrt{A_2}}{A_3}$$

where

$$\begin{aligned}
A_1 &= 27025N^{11} - 3643614N^{10} + 210999260N^9 - 6810840856N^8 \\
& + 132295854944N^7 - 1534324494976N^6 + 9346495290496N^5 \\
& - 11159485467904N^4 - 183018831796224N^3 + 745142082785280N^2 \\
& + 205924147789824N - 1426412678676480, \\
A_2 &= 57441241N^{22} - 17912057020N^{21} + 2618664424988N^{20} \\
& - 238118535739200N^{19} + 15063089407047536N^{18} \\
& - 702280213198643392N^{17} + 24934025190545250880N^{16} \\
& - 686653269120561219072N^{15} + 14791902504139923647488N^{14} \\
& - 249302232577387990290432N^{13} + 3259352423223475632402432N^{12} \\
& - 32388940524728700064022528N^{11} + 234675169275055587268968448N^{10} \\
& - 1126393736705881095131955200N^9 + 2494958946356596711482261504N^8 \\
& + 7374887457753764509916332032N^7 - 78586427159754019630286372864N^6 \\
& + 278427222229099295548410691584N^5 - 560779470613827428532149551104N^4 \\
& + 764771178562157714441678880768N^3 - 159342150144758078717426663424N^2 \\
& - 1263144326261770762588929392640N \\
& + 879163881274963620673250918400, \\
A_3 &= 2(N-24)(N-22)(N-20)(N-18)(N-16)(N-14)(N-12) \\
& \cdot (N-10)(N-2)(N^2-4N-4).
\end{aligned}$$

For this  $\tau$ , other conclusions of the lemma can be checked by computers.  $\square$

**Proposition 9.21.** *For  $\tau$  chosen in Lemma 9.20, the function  $F(\xi', \lambda')$  has a strict local minimum at  $(0, 1)$ .*

*Proof.* Since  $I'(1) = 0$ , we have  $\frac{\partial}{\partial \lambda'} F(0, 1) = 0$ . In addition Proposition 9.1 shows  $\frac{\partial}{\partial \xi'} F(0, 1) = 0$ . Therefore,  $(0, 1)$  is a critical point of  $F(\xi', \lambda')$ .

Since  $J_1(1) > 0$  and  $J_2(1) > 0$ , it follows from Proposition 9.19 that the matrix  $\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} F(0, 1)$  is positive definite. Lemma 9.20 again tell us  $I''(1) > 0$ ,

which implies  $\frac{\partial^2}{\partial \lambda^2} F(0, 1) > 0$ . Consequently,  $(0, 1)$  is a strict local minimum point.  $\square$

## 10 Proof of the main theorem

In this section we prove the main result of this paper by a gluing method.

**Proposition 10.1.** *Assume  $N \geq 25$ . Moreover, let  $g$  be a smooth metric on  $\mathbb{R}^N$  of the form  $g(x) = e^{h(x)}$ , where  $h(x)$  is a trace-free symmetric two-tensor on  $\mathbb{R}^N$  such that*

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| + |\partial^3 h(x)| + |\partial^4 h(x)| \leq \alpha$$

for all  $x \in \mathbb{R}^N$ ,  $h(0) = 0$ ,  $h(x) = 0$  for  $|x| \geq 1$ , and

$$h_{ik}(x) = \mu \varepsilon^8 f(\varepsilon^{-2}|x|^2) H_{ik}(x)$$

for  $|x| \leq \rho$ . If  $\alpha$  and  $\rho^{4-n} \mu^{-2} \varepsilon^{N-24}$  are sufficiently small, then there exists a positive solution  $u(x)$  to

$$\begin{aligned} P_g u &= \frac{N-4}{2} u^{\frac{N+4}{N-4}} \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^{\frac{2N}{N-4}} &< \int_{\mathbb{R}^N} \left( \frac{1}{1+|y|^2} \right)^N \\ \sup_{|x| \leq \varepsilon} u &\geq C \varepsilon^{\frac{4-N}{2}}. \end{aligned}$$

*Proof.* By Proposition 9.21, the function  $F(\xi', \lambda')$  has a strict local minimum at  $(0, 1)$  and  $F(0, 1) < 0$ . Hence, we can find an open set  $\mathcal{M} \subset \Lambda$  such that  $(0, 1) \in \mathcal{M}$  and

$$F(0, 1) < \inf_{\partial \mathcal{M}} F(\xi', \lambda') < 0.$$

Using Lemma 8.2, we obtain

$$2\mathcal{F}_{\bar{g}}(\xi', \lambda') = 2E + \mu^2 \varepsilon^{20} F(\xi', \lambda') + O\left(\mu^3 \varepsilon^{\frac{20N}{N-4}} + \mu^{\frac{20N}{N-4}} \varepsilon^{\frac{20N}{N-4}}\right) + O\left(\alpha \left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).$$

Hence, if  $\rho^{4-n} \mu^{-2} \varepsilon^{N-24}$  is sufficiently small, we have

$$\mathcal{F}_{\bar{g}}(0, 1) < \inf_{\partial \mathcal{M}} \mathcal{F}_{\bar{g}}(\xi', \lambda') < E.$$

Consequently, there exists a point  $(\bar{\xi}', \bar{\lambda}') \in \mathcal{M}$  such that

$$\mathcal{F}_{\bar{g}}(\bar{\xi}', \bar{\lambda}') < \inf_{\partial \mathcal{M}} \mathcal{F}_{\bar{g}}(\xi', \lambda') < E.$$

Then  $\bar{u}_0(y) + \phi(y)$  is a solution of (8) with  $\|\phi\|_* \leq C$ .

Note that since  $\|\phi\|_* \leq C$  we have

$$|\phi(y)| \leq C \frac{\alpha}{(1 + |y - \xi'|)^{N-4}} \leq C\alpha u_0 \quad (51)$$

which shows that  $u_0 + \phi > 0$  provided  $\alpha$  is small. Thus  $u_0(x) + \varepsilon^{-\frac{N-4}{2}}\phi(x/\varepsilon)$  is the positive solution we need.  $\square$

**Proposition 10.2.** *Let  $N \geq 25$ . Then there exists a smooth metric  $g$  on  $\mathbb{R}^N$  with the following properties:*

- 1)  $g_{ij}(x) = \delta_{ij}$  for  $|x| \geq \frac{1}{2}$ ,
- 2)  $g$  is not conformally flat,
- 3) There exists a sequence of positive function  $u_n$  ( $n \in \mathbb{R}^N$ ) such that

$$\begin{aligned} P_g u_n &= \frac{N-4}{2} u_n^{\frac{N+4}{N-4}}, \\ \int_{\mathbb{R}^N} u_n^{\frac{2N}{N-4}} &< \int_{\mathbb{R}^N} \left( \frac{1}{1+|y|^2} \right)^N, \\ \sup_{|x| \leq 1} u_n &\rightarrow \infty. \end{aligned}$$

*Proof.* Choose a smooth cut-off function  $\eta$  such that  $\eta(r) = 1$  for  $r \leq 1$  and  $\eta(r) = 0$  for  $r \geq 2$ . We define a trace-free symmetric two-tensor on  $\mathbb{R}^N$  by

$$h_{ij}(x) = \sum_{n=N_0}^{\infty} \eta(4n^2|x-x_n|) 2^{-\frac{25}{3}n} f(2^{2n}|x-x_n|) H_{ij}(x-x_n),$$

where  $x_n = (\frac{1}{n}, 0, \dots, 0)$ . Clearly  $h(x)$  is  $C^\infty$ .

Moreover, if  $N_0$  is sufficiently large, then we have  $h(x) = 0$  for  $|x| \geq \frac{1}{2}$  and  $|h| + |\partial h| + |\partial^2 h| + |\partial^3 h| + |\partial^4 h| \leq \alpha$ . Provided that  $n \geq N_0$  and  $|x-x_n| \leq \frac{1}{4n^2}$ , we have

$$h_{ij}(x) = 2^{-\frac{25}{3}n} f(2^{2n}|x-x_n|) H_{ij}(x-x_n).$$

Hence we can apply Proposition 10.1 with  $\mu = 2^{-n/3}$ ,  $\varepsilon = 2^{-n}$ ,  $\rho = \frac{1}{4n^2}$ . From this the assertion follows.  $\square$

## 11 Appendix

In this section we will give the proof of (23) and (26). The proof of (23) can be found in [18] and we repeat it here for the sake of convenience.

**Lemma 11.1.** *Assume that  $0 < s < N$  and  $t > s$ . Then*

$$\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|y|)^t} dy \leq \begin{cases} C(1+|x|)^{s-t} & \text{if } t < N, \\ C(1+|x|)^{s-N} [1 + \log(1+|x|)] & \text{if } t = N, \\ C(1+|x|)^{s-N} & \text{if } t > N. \end{cases}$$

*Proof.* First, observe that the above integral is well defined since  $t > s$ . So we only need to consider the case that  $|x|$  is large. Next we decompose it as follows:

$$\begin{aligned} & \left( \int_{|y-x| \leq \frac{|x|}{2}} + \int_{\frac{|x|}{2} \leq |y-x| \leq 2|x|} + \int_{|y-x| \geq 2|x|} \right) \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|y|)^t} dy \\ & := I_1 + I_2 + I_3. \end{aligned}$$

$I_1$  may be estimated as follows. Since  $|y-x| \leq |x|/2$  implies  $|y| \geq |x|/2$ ,

$$\begin{aligned} I_1 & \leq \int_{|y-x| \leq \frac{|x|}{2}} \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|x|/2)^t} dy \\ & \leq \frac{1}{(1+|x|/2)^t} \int_0^{|x|/2} \frac{1}{r^{N-s}} r^{N-1} dr \\ & \leq C|x|^{s-t}. \end{aligned}$$

$I_3$  may be estimated similarly. Because  $|y-x| \geq 2|x|$ ,  $|y-x| \leq |y| + |x| \leq |y| + |y-x|/2$ . Thus  $|y-x| \leq 2|y|$  and

$$\begin{aligned} I_3 & \leq \int_{|y-x| \geq 2|x|} \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|x-y|/2)^t} dy \\ & \leq \int_{2|x|}^{\infty} \frac{1}{r^{N-s}} \frac{1}{(1+r/2)^t} r^{N-1} dr \\ & \leq C|x|^{s-t}. \end{aligned}$$

Finally, we observe that

$$\begin{aligned} I_2 & \leq \frac{C}{|x|^{N-s}} \int_{\frac{|x|}{2} \leq |y-x| \leq 2|x|} \frac{1}{(1+|y|)^t} dy \\ & \leq \frac{C}{|x|^{N-s}} \left( \int_{|y| \leq 1} + \int_{1 \leq |y| \leq 3|x|} \right) \frac{1}{(1+|y|)^t} dy \\ & \leq C \frac{C}{|x|^{N-s}} \left( C + C \int_1^{3|x|} r^{N-t-1} dr \right) \\ & \leq \begin{cases} C|x|^{s-t} & \text{if } t < N, \\ C|x|^{s-N} \log|x| & \text{if } t = N, \\ C|x|^{s-N} & \text{if } t > N. \end{cases} \end{aligned}$$

Now it is easily seen that the lemma holds.  $\square$

Next we come to the proof of (26).

*Proof of (26).* Now  $t = N - k$ . Let  $L > 0$  is a large number. If  $|y| \leq Lr$ , we have

$$\int_{B_r} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z|)^{N-k}} dz$$

$$\begin{aligned} &\leq \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z|)^{N-k}} dz \\ &\leq C(1+|y|)^{s+k-N} \leq Cr^k(1+|y|)^{s-N}. \end{aligned}$$

For  $|y| \geq Lr$ , obvious  $\frac{(1+|y|)^{N-s}}{|y-z|^{N-s}} \leq C$  since  $|z| \leq r$ . Thus, recalling that  $k > 0$ , we get

$$\begin{aligned} &\int_{B_r} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z|)^{N-k}} dz \\ &\leq C(1+|y|)^{s-N} \int_{B_r} \frac{1}{(1+|z|)^{N-k}} dz \\ &\leq Cr^k(1+|y|)^{s-N}. \end{aligned}$$

This concludes the proof.  $\square$

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